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Radii of Starlikeness for Certain Subclasses of Logharmonic Mappings

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Abstract. In this paper, we determine the radius of univalence and starlikeness for subclasses of logharmonic mappings which satisfy certain conditions. Moreover, we obtain the radius of univalence and starlikeness for the set of all close-to-starlike logharmonic mappings of order α.

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1. Introduction

Let $H(U)$ be the linear space of all analytic functions defined on the unit disk $U = \{z : |z| < 1\}$ and let B be the set of all functions $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$. A logharmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = a\frac{f_z}{f}, \tag{1.1}$$

where the second dilatation function $a \in B$. It has been shown that if f is a nonvanishing logharmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

where h and g are analytic functions in U. On the other hand, if f vanishes at $z = 0$ but is not identically zero, then f admits the following representation

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$$

where $Re\ \beta > -1/2$, and h and g are analytic functions in U, $g(0) = 1$ and $h(0) \neq 0$ (see [4]). Univalent logharmonic mappings have been studied extensively (for details see [1, 2, 3, 4, 5, 6]).

Let $f = z|z|^{2\beta}h\overline{g}$ be a univalent logharmonic mapping. We say that f is starlike logharmonic mapping of order α if

$$\frac{\partial \arg f(re^{i\theta})}{\partial\theta} = \text{Re}\ \frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha, \quad 0 \leq \alpha < 1 \tag{1.2}$$

for all $z \in U$. Denote by $ST_{Lh}(\alpha)$ the set of all starlike logharmonic mappings of order α. If $\alpha = 0$, we get the class of starlike logharmonic mappings. Also, let $ST(\alpha) = \{f \in ST_{Lh}(\alpha)$ and $f \in H(U)\}$. If $f \in ST_{Lh}(0)$ then $F(\zeta) = \log(f(e^{\zeta}))$ is univalent and harmonic on the half plane $\{\zeta : \text{Re}\{\zeta\} < 0\}$. Univalent harmonic mappings have interesting links with geometric function theory, minimal surfaces and locally quasiconformal mappings, (see [8]).

In Section 2 we obtain the radii of starlikeness for subclasses of logharmonic mappings which satisfy certain conditions. In Section 3 we determine the radius of starlikeness for the set of all close to starlike logharmonic mappings of order α.

2. Radii of starlikeness

In the first result, we determine the radius of starlikness for logharmonic mappings F which satisfy the condition $Re\dfrac{F(z)}{z|z|^{2\beta}} > 0$.

Theorem 1. *Suppose that* $F(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ *be a logharmonic mapping on* U *with* $h(0) = g(0) = 1$ *and such that* $\phi(z) = \dfrac{zh(z)}{g(z)} = z + a_{n+1}z^{n+1} + \ldots.$ *Suppose also* $Re\dfrac{F(z)}{z|z|^{2\beta}} > 0$ *for* $|z| < 1$. *Then* $F(z)$ *is univalent and starlike in* $|z| < \left(\sqrt{n^2+1} - n\right)^{1/n}$.

Proof. Let $p(z) = \dfrac{\phi(z)}{z} = 1 + a_{n+1}z^n + \ldots.$ Since $Re\dfrac{F(z)}{z|z|^{2\beta}} = \text{Re}\ \dfrac{z|z|^{2\beta}h(z)\overline{g(z)}}{z|z|^{2\beta}} = |g(z)|^2\,\text{Re}(p(z)) > 0$, it follows that $Re(p(z)) > 0$. It is known (see[7]) that

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2n|z|^n}{1-|z|^{2n}}. \tag{2.1}$$

But since

$$\operatorname{Re}\ \frac{zF_z - \overline{z}F_{\overline{z}}}{F} = \operatorname{Re}\frac{z\phi'(z)}{\phi(z)} = 1 + \operatorname{Re}\frac{zp'(z)}{p(z)},$$

we deduce that $F(z)$ will be univalent and starlike if

$$\left|\frac{z\phi'(z)}{\phi(z)} - 1\right| < \left|\frac{zp'(z)}{p(z)}\right| < 1. \tag{2.2}$$

Using (2.1) we deduce that (2.2) will be satisfied if $\dfrac{2n|z|^n}{1-|z|^{2n}} < 1$ and this implies that $|z| < \left(\sqrt{n^2+1} - n\right)^{1/n}$. The function

$$F(z) = \varphi(z)|z|^{2\beta}\exp\left(2\operatorname{Re}\int_0^z \frac{a(s)\varphi'(s)}{\varphi(s)(1-a(s))}ds\right),$$

where $\varphi(z) = \dfrac{z+z^{n+1}}{1-z^n} = z + 2z^{n+1} + ...$ and $a(z) = z$, satisfies $Re\dfrac{F(z)}{z|z|^{2\beta}} > 0$ for $|z| < 1$, but is not univalent in $|z| < r$ for $r > r_n = \left((n^2+1)^{1/2} - n\right)^{1/n}$, since $J_f(r_n e^{\frac{i\pi}{n}}) = 0$. Hence the upper bound is best possible. ∎

Remark 1. *Note that if* $n = 1$*, the radius of starlikeness is equal to* $\sqrt{2} - 1 = 0.4142....$ *and the extremal function is given by*

$$F(z) = z|z|^{2\beta}\left(\frac{1+z}{1-z}\right)\left|\frac{1+z}{1-z}\right|\exp\left(\operatorname{Re}\frac{2z}{1-z}\right).$$

In the next result we find the radius of starlikeness for logharmonic mappings $F = z|z|^{2\beta}h(z)\overline{g(z)}$ which satisfy the condition $\left|\dfrac{h(z)g^*(z)}{h^*(z)g(z)} - 1\right| < 1$ where $f = z|z|^{2\beta}h^*(z)\overline{g^*(z)} \in ST_{Lh}(\alpha)$.

Theorem 2. *Suppose that* $F = z|z|^{2\beta}h(z)\overline{g(z)}$ *be a logharmonic mapping on* U *and suppose that* $f = z|z|^{2\beta}h^*(z)\overline{g^*(z)} \in ST_{Lh}(\alpha)$*. If* $\left|\dfrac{h(z)g^*(z)}{h^*(z)g(z)} - 1\right| < 1$ *for* $|z| < 1$ *then* $F(z)$ *is univalent and starlike for* $|z| < \dfrac{2\alpha - 3 + \sqrt{4\alpha^2 - 4\alpha + 9}}{4\alpha}$.

Proof. Since $\dfrac{h(0)g^*(0)}{h^*(0)g(0)} - 1 = 0$, we can write

$$\frac{h(z)/g(z)}{h^*(z)/g^*(z)} - 1 = z\omega(z),$$

where $\omega(z)$ is analytic and satisfies $|\omega(z)| \le 1$ for $|z| < 1$. Let $\varphi(z) = zh(z)/g(z)$ and $\psi(z) = zh^*(z)/g^*(z)$. Now expressing φ and φ' in terms of ψ and ω, we get

$$\text{Re } \frac{z\varphi'(z)}{\varphi(z)} = \text{Re} \frac{z\psi'(z)}{\psi(z)} + \text{Re} \left[z \frac{\omega(z) + z\omega'(z)}{1 + z\omega(z)} \right]. \tag{2.3}$$

From theorem 2.1 in [3], we deduce that $\psi \in ST(\alpha)$ and therefore,

$$\text{Re} \frac{z\psi'(z)}{\psi(z)} = \text{Re}[(1-\alpha)p(z) + \alpha] \geq (1-\alpha)\frac{1-r}{1+r} + \alpha. \tag{2.4}$$

Also, it is known that,

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2}. \tag{2.5}$$

Making use of inequality (2.5) we get

$$\left| z \frac{\omega(z) + z\omega'(z)}{1 + z\omega(z)} \right| \leq |z| \frac{|\omega(z)| + |z| \frac{1 - |\omega(z)|^2}{1 - |z|^2}}{1 - |z|\,|\omega(z)|} = |z| \frac{|\omega(z)| + |z|}{1 - |z|^2}$$
$$\leq |z| \frac{1 + |z|}{1 - |z|} \leq \frac{|z|}{1 - |z|}.$$

Thus

$$\text{Re} \left[z \frac{\omega(z) + z\omega'(z)}{1 + z\omega(z)} \right] \geq \frac{-r}{1 - r}. \tag{2.6}$$

But

$$Re \frac{zF_z - \overline{z}F_{\overline{z}}}{F} = \text{Re } \frac{z\varphi'(z)}{\varphi(z)}. \tag{2.7}$$

Now combining (2.3), (2.4), (2.6) and (2.7) we obtain

$$Re \frac{zF_z - \overline{z}F_{\overline{z}}}{F} \geq \frac{1 + (2\alpha - 3)r - 2\alpha r^2}{1 - r^2}. \tag{2.8}$$

Therefore, if $|z| < \dfrac{2\alpha - 3 + \sqrt{4\alpha^2 - 4\alpha + 9}}{4\alpha}$ then $Re \dfrac{zF_z - \overline{z}F_{\overline{z}}}{F} > 0$, and hence F is univalent and starlike. The function $F(z) = \dfrac{z(1+z)}{(1-z)^{2-2\alpha}}$ is an extremal function for this theorem since $f(z) = \dfrac{z}{(1-z)^{2-2\alpha}}$ is starlike of order α. ∎

Remark 2. *If $\alpha = 0$, the radius of starlikeness is equal to $\frac{1}{3} = 0.3333...$ and the extremal function is given by $F(z) = \dfrac{z(1+z)}{(1-z)^2}$.*

3. The class $CST_{Lh}(\alpha)$

Let $F(z)=z|z|^{2\beta}h(z)\overline{g(z)}$ be logharmonic mapping with respect to $a\in B$. We say that F is a close-to-starlike of order α, $0\le\alpha<1$, if there is a logharmonic mapping $f(z)=z|z|^{2\beta}h^{*}(z)\overline{g^{*}(z)}\in ST_{Lh}(\alpha)$ with respect to the same a such that $Re\dfrac{F(z)}{f(z)}>0$. The geometric interpretation is the following: under a close-to-starlike logharmonic mapping $F(z)$, the radius vector of the image of $|z|=r<1$, never turns back by an amount more than $(1-\alpha)\pi$. Let $CST_{Lh}(\alpha)$ be the set of all close-to-starlike logharmonic mappings of order α, defined on the unit disc U. It contains the set $CST(\alpha)$ of all analytic close-to-starlike functions of order α. A mapping $F\in CST_{Lh}(\alpha)$ need not necessarily to be univalent. For example, take $F(z)=z(1+z)$.

In the next result we determine the radius of univalence and starlikeness for the mappings in the set $CST_{Lh}(\alpha)$.

Theorem 3. *Let $F=z|z|^{2\beta}h(z)\overline{g(z)}\in CST_{Lh}(\alpha)$. Then F maps the disk $|z|<R$, $R\le\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}$ onto a starlike domain. The upper bound is best possible for all $a\in B$.*

Proof. Let $F=z|z|^{2\beta}h(z)\overline{g(z)}\in CST_{Lh}(\alpha)$ with respect to a given $a\in B$. Then there exists $f=z|z|^{2\beta}h^{*}(z)\overline{g^{*}(z)}\in ST_{Lh}(\alpha)$ with respect to the same a and such that $Re\dfrac{F(z)}{f(z)}>0$. It follows that F(z) can be written as $F(z)=f(z)R(z)$, where $f=z|z|^{2\beta}h^{*}(z)\overline{g^{*}(z)}\in ST_{Lh}(\alpha)$ with respect to $a\in B$ and $R(z)=H\overline{G}$ *is* positive real part logharmonic mapping with respect to the same a.

Using simple calculations we get

$$Re\frac{zF_z-\overline{z}F_{\overline{z}}}{F}=Re\frac{zf_z-\overline{z}f_{\overline{z}}}{f}+Re\frac{zR_z-\overline{z}R_{\overline{z}}}{R}. \tag{3.1}$$

Now it follows from [3] that
$Re\dfrac{zf_z-\overline{z}f_{\overline{z}}}{f}=Re\dfrac{z\varphi'z)}{\varphi(z)}$, where $\varphi(z)=\dfrac{zh^{*}(z)}{g^{*}(z)}\in ST(\alpha)$.
Thus

$$Re\frac{z\varphi'z)}{\varphi(z)}=Re[(1-\alpha)p(z)+\alpha]\ \ge(1-\alpha)\frac{1-r}{1+r}+\alpha. \tag{3.2}$$

Also,

$$Re\frac{zR_z-\overline{z}R_{\overline{z}}}{R}=Re\frac{zp'z)}{p(z)}, \tag{3.3}$$

where $Re(p(z)) = Re\dfrac{H(z)}{G(z)} > 0$, since $Re(H\overline{G}) = |G|^2 \operatorname{Re} \dfrac{H}{G} > 0$.
Hence

$$Re\frac{zp'z)}{p(z)} \geq \frac{-2r}{1-r^2}. \tag{3.4}$$

Combining (3.1), (3.2) (3.3) and (3.4), we obtain

$$Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F} \geq (1-\alpha)\frac{1-r}{1+r} + \alpha - \frac{2r}{1-r^2} = \frac{1+(2\alpha-4)r+(1-2\alpha)r^2}{1-r^2}.$$

Hence,
$Re\dfrac{zF_z - \overline{z}F_{\overline{z}}}{F} > 0$ if $1+(2\alpha-4)r+(1-2\alpha)r^2 > 0$. The radius of starlikness ρ is the smallest positive root (less than 1) of $1+(2\alpha-4)\rho+(1-2\alpha)\rho^2 = 0$ which is $\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}$. We deduce that F is univalent on $|z| < \dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}$ and maps $\left\{z; |z| < \dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}\right\}$ onto a starlike domain. The analytic function $F(z) = \dfrac{z(1+z)}{(1-z)^{3-2\alpha}}$ belong to the set $CST(\alpha)$ and we have $F'\ (\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}) = 0$. Hence, the upper bound $\dfrac{2-\alpha-\sqrt{\alpha^2-2\alpha+3}}{1-2\alpha}$ is best possible for $CST(\alpha)$. Since $f = z|z|^{2\beta}h^*(z)\overline{g^*(z)} \in S^*_{Lh}(\alpha)$ if and only if $\varphi(z) = \dfrac{zh^*(z)}{g^*(z)}$ is starlike of order α, the same bound is best possible for all $a \in B$. ∎

Remark 3. *The minimum of the first term on the right hand side of equation (3.1) is attained for the function* $f(z) = \overline{\eta}f_0(\eta z), |\eta| = 1,$*where*
$$f_0(z) = z\frac{(1-\overline{z})}{1-z}\frac{1}{(1-\overline{z})^{2\alpha}}\exp\left((1-\alpha)\operatorname{Re}\frac{4z}{1-z}\right).$$

Remark 4. *If* $\alpha = 0$, *then the radius of starlikeness is equal to* $2-\sqrt{3} = 0.26795...$ *and the extermal function is given by* $F(z) = \dfrac{z(1+z)}{(1-z)^3}$.

Finally, we prove the following theorem.

Theorem 4. *Let* $S = z|z|^{2\beta}h(z)\overline{g(z)}$ *be logharmonic mapping with respect to* a *and let* $F = z|z|^{2\beta}h_1(z)\overline{g_1(z)} \in CST_{Lh}(\alpha)$ *with respect to the same* a *such that* $Re\dfrac{S(z)}{F(z)} > 0$. *Then* F *maps the disk* $|z| < R$, $R \leq \dfrac{3-\alpha-\sqrt{\alpha^2-4\alpha+8}}{1-2\alpha}$ *onto a starlike domain. The upper bound is best possible for all* $a \in B$.

Proof. Let $S = z|z|^{2\beta}h(z)\overline{g(z)}$ be logharmonic mapping with respect to $a \in B$ and let $F = z|z|^{2\beta}h_1(z)\overline{g_1(z)} \in CST_{Lh}(\alpha)$ with respect to the same a such that

$Re\dfrac{S(z)}{F(z)} > 0$. Then $S(z)$ can be written as

$$S(z) = f(z)R_1(z)R_2(z),$$

where $f = z|z|^{2\beta}h^*(z)\overline{g^*(z)} \in S^*_{Lh}(\alpha)$ with respect to a and where R_1 and R_2 are positive real part logharmonic mappings with respect to the same a. Direct calculations lead to

$$Re\frac{zS_z - \overline{z}S_{\overline{z}}}{F} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} + Re\frac{zR_{1z} - \overline{z}R_{1\overline{z}}}{R_1} + Re\frac{zR_{2z} - \overline{z}R_{2\overline{z}}}{R_2}. \tag{3.5}$$

Using similar argument as in Theorem 3 we get

$$Re\frac{zR_{1z} - \overline{z}R_{1\overline{z}}}{R_1} \geq \frac{-2r}{1-r^2} \quad \textit{and} \quad Re\frac{zR_{2z} - \overline{z}R_{2\overline{z}}}{R_2} \geq \frac{-2r}{1-r^2}. \tag{3.6}$$

Substituting (3.2) and (3.6) in (3.5) we obtain

$$Re\frac{zS_z - \overline{z}S_{\overline{z}}}{F} \geq (1-\alpha)\frac{1-r}{1+r} + \alpha - \frac{4r}{1-r^2} = \frac{1+(2\alpha-6)r+(1-2\alpha)r^2}{1-r^2}.$$

Thus

$Re\dfrac{zS_z - \overline{z}S_{\overline{z}}}{F} > 0$ if $1+(2\alpha-4)r+(1-2\alpha)r^2 > 0$. The radius of starlikness ρ is the smallest positive root (less than 1) of $1+(2\alpha-6)\rho+(1-2\alpha)\rho^2 = 0$ which is $\dfrac{3-\alpha-\sqrt{\alpha^2-4\alpha+8}}{1-2\alpha}$.We deduce that S is univalent on

$|z| < \dfrac{3-\alpha-\sqrt{\alpha^2-4\alpha+8}}{1-2\alpha}$ and maps $\left\{z; |z| < \dfrac{3-\alpha-\sqrt{\alpha^2-4\alpha+8}}{1-2\alpha}\right\}$ onto a starlike domain. The analytic function $S(z) = \dfrac{z(1+z)^2}{(1-z)^{4-2\alpha}}$ satisfies

$S'(\dfrac{3-\alpha-\sqrt{\alpha^2-4\alpha+8}}{1-2\alpha}) = 0$ and $F(z) = \dfrac{z(1+z)}{(1-z)^{4-2\alpha}} \in CST(\alpha)$ such that $Re\dfrac{S(z)}{F(z)} > 0$. Hence the upper bound is best possible. ∎

Remark 5. *If* $\alpha = 0$, *the radius of starlikeness is equal to* $3-\sqrt{8} = 0.17157...$*and the extremal function is given by* $f(z) = \dfrac{z(1+z)}{(1-z)^4}$.

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