On Solving Nonlinear Modified Boussinesq Equations

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Abstract

In this letter, He's Variational iteration method (VIM) is implemented for solving the special case of nonlinear Whitham-Broer-Kaup partial differential equation, is named modified Boussinesq (MB). This method is based on Lagrange multipliers for identification of optimal values of parameters in a functional. Using this method creates a sequence which tends to the exact solution of the problem. The results are compared with those obtained by the Adomian decomposition method. The work confirms that the VIM method is superior and very faster to the ADM

Keywords: Whitham-Broer-Kaup(WBK) equations, modified Boussinesq equations(MB), Variational iteration method (VIM), Adomian decomposition method (ADM)

1   Introduction

Most scientific problems such as Whitham-Broer-Kaup(WBK) equations are inherently of nonlinearity. We know that except a limited number of these problems, most of them do not have analytical solution. Therefore, these nonlinear equations should be solved using other methods. In this article, VIM is used to solve nonlinear equations of Whitham-Broer-Kaup with the initial conditions $u(x,0) = F(x)$, $v(x,0) = G(x)$ for the first time:

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ \\ v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0. \end{cases} \tag{1}$$

If $\alpha = 0$, $\beta \neq 0$, then the system of Eqs. (1) becomes classical long wave equation that describe shallow water wave with dispersive [13]. If $\alpha = 1$, $\beta = 0$, then the system of Eqs. (1) becomes variant Boussinesq equations [13]. In this paper we will focus on finding analytical approximate and exact traveling wave solution of the systems of Eq. (1) using the variational iteration method. Also we compare the result with given solutions using ADM [3, 11].

2   Variational iteration method

In this section, we use the following non-homogeneous system of partial differential equations

$$\begin{cases} L_1 u(x,t) + N_1(u(x,t), v(x,t)) = f(x,t), \\ \\ L_2 u(x,t) + N_2(u(x,t), v(x,t)) = g(x,t), \end{cases} \quad (2)$$

to illustrate the basic idea of the variational iteration method [1, 2, 4, 5, 6, 9, 10, 12]. In the above system of equations L_1 and L_2 are linear differential operators with respect to t and N_1 and N_2 are nonlinear operators and $f(x,t)$, $g(x,t)$ are some given functions. According to the variational iteration method, we can construct a correct functional as follows [1, 2, 4, 5, 6, 9, 10, 12]:

$$\begin{cases} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau)\{L_1 u_n(x,\tau) + N_1(\widetilde{u}_n(x,\tau), \widetilde{v}_n(x,\tau)) - f(x,\tau)\}d\tau, \\ \\ v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\tau)\{L_2 u_n(x,\tau) + N_2(\widetilde{u}_n(x,\tau), \widetilde{v}_n(x,\tau)) - g(x,\tau)\}d\tau, \end{cases}$$
$$(3)$$

where λ_1 and λ_2 are general lagrange multipliers, which can be identified optimally via variational theory [1, 2, 4, 5, 6, 7, 8, 9, 10, 12]. The second terms on the right-hand side in (3) are called the correction and the subscript n denotes the nth order approximation. Under a suitable restricted variational assumptions (i.e. $\widetilde{u}_n$ and $\widetilde{v}_n$ are considered as a restricted variation), we may assume that the above correctional functional are stationary (i.e. $\delta\widetilde{u}_n = 0$ and $\delta\widetilde{v}_n = 0$), then the Lagrange multipliers are identified. Now we may start the procedures with the given initial approximation and using the above iteration formulas to obtain the approximate solutions.

3   Implementation of the method

We first consider the application of VIM to WBK Eqs. (1) with the initial conditions of

$$u(x,0) = \omega - 2Bk \ \coth(k\xi), \qquad v(x,0) = -2B(B+\beta)k^2 \ \text{csch}^2(k\xi), \quad (4)$$

where $B = \sqrt{\alpha + \beta^2}$ and $\xi = x + x_0$ and x_0, k, ω are arbitrary constants.

To solve Eqs. (1) with the initial conditions (4) by means of VIM, we consider the correction functional in t-direction in the following form:

$$u_{n+1} = u_n + \int_0^t \lambda_1(\tau)\{u_{n_\tau} + \widetilde{u}_n\widetilde{u}_{n_x} + \widetilde{v}_{n_x} + \beta\widetilde{u}_{n_{xx}}\}d\tau,$$

$$v_{n+1} = v_n + \int_0^t \lambda_2(\tau)\{v_{n_\tau} + (\widetilde{u}_n\widetilde{v}_n)_x + \alpha\widetilde{u}_{n_{xxx}} - \beta\widetilde{v}_{n_{xx}}\}d\tau,$$

(5)

where λ_1 and λ_2 are general lagrange multipliers which its optimal value is found by using variational theory. The value of u_0 and v_0 are initial approximations and must be chosen suitably and $\widetilde{u}_n$ and $\widetilde{v}_n$ are the restricted values i.e. $\delta\widetilde{u}_n = 0$ and $\delta\widetilde{v}_n = 0$. Now we have

$$\delta u_{n+1} = \delta u_n + \delta \int_0^t \lambda_1(\tau)\{u_{n_\tau} + \widetilde{u}_n\widetilde{u}_{n_x} + \widetilde{v}_{n_x} + \beta\widetilde{u}_{n_{xx}}\}d\tau,$$

$$\delta v_{n+1} = \delta v_n + \delta \int_0^t \lambda_2(\tau)\{v_{n_\tau} + (\widetilde{u}_n\widetilde{v}_n)_x + \alpha\widetilde{u}_{n_{xxx}} - \beta\widetilde{v}_{n_{xx}}\}d\tau,$$

(6)

to find the optimal value of λ_i, we have $(\delta u_n(0) = 0)$

$$\delta u_{n+1} = \delta u_n + \int_0^t \lambda_1(\tau)\{\delta u_{n_\tau}\}d\tau = (1 + \lambda_1(\tau))|_{\tau=t}\ \delta u_n - \int_0^t \lambda_1'(\tau)\delta u_n\ d\tau = 0,$$

$$\delta v_{n+1} = \delta v_n + \int_0^t \lambda_2(\tau)\{\delta v_{n_\tau}\}d\tau = (1 + \lambda_2(\tau))|_{\tau=t}\ \delta v_n - \int_0^t \lambda_2'(\tau)\delta v_n\ d\tau = 0,$$

(7)

Therefore, the stationary conditions are obtained in the following form:

$$\lambda_1'(\tau) = 0, \qquad 1 + \lambda_1(\tau) = 0|_{\tau=t},$$

$$\lambda_2'(\tau) = 0, \qquad 1 + \lambda_2(\tau) = 0|_{\tau=t},$$

(8)

Therefore the Lagrange multipliers is defined as the following form:

$$\lambda_1(\tau) = -1, \qquad \lambda_2(\tau) = -1.$$

(9)

Substituting Eq. (9) into the correctional functional Eq. (5) we get the following iteration formulas:

$$u_{n+1} = u_n - \int_0^t \{u_{n_\tau} + u_n u_{n_x} + v_{n_x} + \beta u_{n_{xx}}\}d\tau,$$

$$v_{n+1} = v_n - \int_0^t \{v_{n_\tau} + (u_n v_n)_x + \alpha u_{n_{xxx}} - \beta v_{n_{xx}}\}d\tau.$$

(10)

We start with initial approximations $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$ and by the above formulas, we can obtain the $u_1(x,t)$ and $v_1(x,t)$ as follows:

$$u_0(x,t) \quad = \quad \omega - 2Bk\ \coth(k(x + x_0)),$$

$$v_0(x,t) = -2Bk^2(B+\beta) \operatorname{csch}^2(k(x+x_0)),$$

$$
\begin{aligned}
u_1(x,t) = \ & \omega - 2Bk \coth(k(x+x_0)) \\
+ \ & 2Bk^2t \left[\omega - 2Bk \coth(k(x+x_0))\right] \left[1 - \coth^2(k(x+x_0))\right] \\
- \ & 4Bk^3t(B+\beta) \operatorname{csch}^2(k(x+x_0)) \coth(k(x+x_0)) \\
- \ & 4\beta Bk^3t \coth(k(x+x_0)) \left[1 - \coth^2(k(x+x_0))\right],
\end{aligned}
$$

$$
\begin{aligned}
v_1(x,t) = \ & -2k^2(B+\beta) \coth^2(k(x+x_0)) \\
- \ & 4B^2k^4t(B+\beta) \, A \, \coth^2(k(x+x_0)) \\
- \ & 4Bk^3t(B+\beta) \operatorname{csch}^2(k(x+x_0)) \coth(k(x+x_0)) \\
\times \ & [\omega - 2Bk \coth(k(x+x_0))] \\
- \ & \alpha t \left[4Bk^4 \, A^2 - 8Bk^4 \, A \coth^2(k(x+x_0))\right] \\
- \ & 8\beta Bk^4t(B+\beta) \operatorname{csch}^2(k(x+x_0)) \coth^2(k(x+x_0)) \\
+ \ & 4\beta Bk^4t(B+\beta) \, A \operatorname{csch}^2(k(x+x_0)),
\end{aligned}
$$

$$\vdots$$

where

$$A = 1 - \coth^2(k(x+x_0)),$$

in the same manner the rest of components of the iteration formulas (10) can be obtained using MATLAB software.

Exact solutions of WBK Eqs. (1) are as follows:

$$u(x,t) = \omega - 2Bk \, \coth(k(\xi - \omega t)), \tag{11}$$

$$v(x,t) = -2B(B+\beta)k^2 \, \operatorname{csch}^2(k(\xi - \omega t)). \tag{12}$$

Where $B = \sqrt{\alpha + \beta^2}$ and $\xi = x + x_0$, these solutions are constructed by Xie et al. [13].

The behavior of the two solutions obtained by VIM with the exact solutions of WBK equation for a different values of t are shown in Fig. 1(a-d), we see that figures are similar to each other. For comparing results between VIM and exact solution, $\mathrm{Error}(u(x,t)) = |u_{Exact} - u_{VIM}|$ and $\mathrm{Error}(v(x,t)) = |v_{Exact} - v_{VIM}|$ are calculated and shown in Table 1(a and b).

Table 1.a
Compare results of $u(x,t)$ for WBK equations by VIM($u_{VIM} = u_1$) with exact result.

$x_i \setminus t_i$	$|u_{Exact} - u_{VIM}|$				
	0.2	0.4	0.6	0.8	1
0.2	3.4575e-09	1.3832e-08	3.1124e-08	5.5338e-08	8.6474e-08
0.4	3.2495e-09	1.2999e-08	2.9251e-08	5.2008e-08	8.1271e-08
0.6	3.0569e-09	1.2229e-08	2.7517e-08	4.8924e-08	7.6452e-08
0.8	2.8782e-09	1.1514e-08	2.5909e-08	4.6065e-08	7.1984e-08
1	2.7123e-09	1.0850e-08	2.4416e-08	4.3410e-08	6.7835e-08

Table 1.b
Compare results of $v(x,t)$ for WBK equations by VIM($v_{VIM} = v_1$) with exact result.

$x_i \setminus t_i$	$|v_{Exact} - v_{VIM}|$				
	0.2	0.4	0.6	0.8	1
0.2	3.7154e-09	1.4864e-08	3.3447e-08	5.9470e-08	9.2934e-08
0.4	3.4380e-09	1.3754e-08	3.0950e-08	5.5030e-08	3.5995e-08
0.6	3.1859e-09	1.2745e-08	2.8681e-08	5.0994e-08	7.9689e-08
0.8	2.9564e-09	1.1827e-08	2.6614e-08	4.7320e-08	7.3947e-08
1	2.7471e-09	1.0990e-08	2.4730e-08	4.3969e-08	6.8711e-08

3.1 Modified Boussinesq equations

To examine the accuracy and reliability of the VIM for Whitham-Broer-Kaup equations we can also consider the different initial values. As a special case, if $\alpha = 1$ and $\beta = 0$, then WBK Eqs. (1) can be reduced to the modified Boussinesq (MB) Eqs. (13).

$$\begin{cases} u_t + uu_x + v_x = 0, \\ v_t + (uv)_x + u_{xxx} = 0. \end{cases} \tag{13}$$

We shall second consider the initial conditions of the MB equations:

$$u(x,0) = \omega - 2k \ \coth(k\xi), \qquad v(x,0) = -2k^2 \ \text{csch}^2(k\xi). \tag{14}$$

If we substitute $\alpha = 1$ and $\beta = 0$ the in obtained solutions for WBK Eqs. (1), or directly solve the system of Eqs. (13) with above initial conditions (Eqs. (14)) by VIM, we can obtain $u_1(x,t)$ and $v_1(x,t)$ as follows:

$$u_0(x,t) \ = \ \omega - 2k \ \coth(k(x+x_0)),$$

$$v_0(x,t) \;=\; -2k^2 \; \mathrm{csch}^2(k(x+x_0)),$$

$$
\begin{aligned}
u_1(x,t) \;=\;& \omega - 2k \; \coth(k(x+x_0)) \\
& + \; 2k^2 t \; [\omega - 2k \; \coth(k(x+x_0))] \; [1 - \coth^2(k(x+x_0))] \\
& - \; 4k^3 t \; \mathrm{csch}^2(k(x+x_0)) \; \coth(k(x+x_0)),
\end{aligned}
$$

$$
\begin{aligned}
v_1(x,t) \;=\;& -2k^2 \; \mathrm{csch}^2(k(x+x_0)) \\
& - \; 4k^4 t \; [1 - \coth^2(k(x+x_0))] \; \mathrm{csch}^2(k(x+x_0)) \\
& - \; 4k^3 t \; \mathrm{csch}^2(k(x+x_0)) \; \coth(k(x+x_0)) \\
& \times \; [\omega - 2k \; \coth(k(x+x_0))] \\
& - \; 4k^4 t \; [1 - \coth^2(k(x+x_0))]^2 \\
& + \; 8k^4 t \; [1 - \coth^2(k(x+x_0))] \; \coth^2(k(x+x_0)),
\end{aligned}
$$

$$\vdots$$

in the same manner the rest of components of the iteration formulas (10) can be obtained using MATLAB software.

Exact solutions of MB Eqs. (13) are as follows:

$$u(x,t) = \omega - 2k \; \coth(k(\xi - \omega t)), \qquad (15)$$

$$v(x,t) = -2k^2 \; \mathrm{csch}^2(k(\xi - \omega t)). \qquad (16)$$

The behavior of the two solutions obtained by VIM with the exact solutions of MB equations for a different values of t are shown in Fig. 2(a-d), we see that figures are similar to each other. For comparing results between VIM and exact solution, $\mathrm{Error}(u(x,t)) = |u_{Exact} - u_{VIM}|$ and $\mathrm{Error}(v(x,t)) = |v_{Exact} - v_{VIM}|$ are calculated and shown in Table 2(a and b).

Table 2.a

Compare results of $u(x,t)$ for MB equations by VIM($u_{VIM} = u_1$) with exact result.

$x_i \backslash t_i$	$|u_{Exact} - u_{VIM}|$				
	0.2	0.4	0.6	0.8	1
0.2	1.7855e-09	7.1426e-09	1.6072e-08	2.8576e-08	4.4655e-08
0.4	1.6780e-09	6.7128e-09	1.5105e-08	2.6857e-08	4.1968e-08
0.6	1.5786e-09	6.3148e-09	1.4210e-08	2.5264e-08	3.9480e-08
0.8	1.4863e-09	5.9458e-09	1.3379e-08	2.3788e-08	3.7172e-08
1	1.4006e-09	5.6031e-09	1.2608e-08	2.2417e-08	3.5030e-08

Table 2.b
Compare results of $v(x,t)$ for MB equations by VIM($v_{VIM} = v_1$) with exact result.

	$\left\lvert v_{Exact} - v_{VIM} \right\rvert$				
$x_i \setminus t_i$	0.2	0.4	0.6	0.8	1
0.2	5.5831e-10	2.2335e-09	5.0261e-09	8.9365e-09	1.3965e-08
0.4	5.1663e-10	2.0668e-09	4.6509e-09	8.2692e-09	1.2922e-08
0.6	4.7875e-10	1.9152e-09	4.3098e-09	7.6629e-09	1.1975e-08
0.8	4.4426e-10	1.7773e-09	3.9993e-09	7.1108e-09	1.1112e-08
1	4.1280e-10	1.6514e-09	3.7161e-09	6.6072e-09	1.0325e-08

4   Conclusions

In this paper, He's variational iteration method has been successfully applied to find the solutions of nonlinear WBK and MB equations. All the examples show that the results of the present method are in excellent agreement with those of exact solutions and the obtained solutions are shown graphically. In our work, we use the MATLAB software to calculate the functions obtained from the variational iteration method. The results show that this method provides excellent approximations to the solution of this nonlinear system with high accuracy. This new method accelerated the convergence to the solutions. Finally, it has been attempted to show the capabilities and wide-range applications of the variational iteration method.

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