Applied Mathematical Sciences, Vol. 3, 2009, no. 3, 129 - 140

The Martingale Approach for Credit-Risky Exchange Option Pricing

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Abstract

An exchange option allows its holder to exchange one asset for another at maturity. In this short paper, the martingale approach, which is based on Continuous martingale representation theorem and Girsanov's theorem, is used to derive an explicit formula for the valuation of an exchange option with counterparty default. The volatilities of financial market considered here are all non-constant functions, which generalizes the results in [1] (Ammann, 2001).

Mathematics Subject Classification: 60H30; 91B28; 91B30

Keywords: Exchange option, Girsanov's theorem, Continuous martingale representation theorem, Itô's formula

1 The Financial Model

An exchange option allows its holder to exchange one asset for another at maturity. This type of options now is used commonly in foreign exchange markets, bond markets and stock markets amongst others. Black and Scholes in their seminal paper [2] first provided a solution to the option pricing problem. Margrabe in his paper [7] priced an exchange option between two correlated stocks that each satisfy the assumptions in the Black-Scholes model. Ammann in [1] used the probabilistic approach to derive an explicit formula for the valuation of an exchange option with counterparty default, and then applied to pricing the credit derivatives with counterparty default risk.

In this short paper, we apply the martingale approach to derive an explicit formula for the valuation of an exchange option with counterparty default under the assumption that the volatilities of market are non-constant functions. Our result extends the Ammann's result in [1], that the to the case that the volatilities are constants, and we also give a complete proof of our result. Our main tools are Itô's formula, Girsanov's theorem and Continuous martingale representation theorem.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, and let $\mathbf{W} = (W_0, W_1, \ldots, W_4)^*$ be a 5-dimensional correlated standard $\{\mathcal{F}_t\}$-Brownian motion with the correlation matrix;

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ 0 & \rho_{21} & 1 & \rho_{23} & \rho_{24} \\ 0 & \rho_{31} & \rho_{32} & 1 & \rho_{34} \\ 0 & \rho_{41} & \rho_{42} & \rho_{43} & 1 \end{bmatrix}. \tag{1.1}$$

Here all ρ_{ij} satisfy that $|\rho_{ii}| < 1$ and $\rho_{ij} = \rho_{ji}$. That is, each W_i is the standard $\{\mathcal{F}_t\}$-Brownian motion such that W_0 is independent of $(W_1, \ldots, W_4)$, and the cross-variation of W_i and W_j is given by $\langle W_i, W_j\rangle(t) = \rho_{ij}t$. In this paper, * denotes the transpose of a vector or a matrix.

Consider a financial market that consists of two correlated underlying assets whose prices are S_1 and S_2. Meanwhile, there exists a counterparty firm whose value is S_3 and its liability is S_4. They are given by the stochastic differential equations (SDEs):

$$\mathrm{d}S_i(t) = \mu_i(t)S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\mathrm{d}W_i(t), \quad S_i(0) = s_i, \quad i = 1, \ldots, 4, \tag{1.2}$$

where $s_i \geq 0$ and $s_3 > s_4$. The market also has a zero-coupon bond with maturing time $T > 0$ whose price:

$$P(t,T) = e^{-\int_t^T \gamma(t,u)\mathrm{d}u}, \quad 0 \leq t \leq T. \tag{1.3}$$

Here $\gamma(t,u)$ is the instantaneous forward rate, which follows:

$$\gamma(t,u) = \gamma(0,u) + \int_0^t \mu_0(s,u)\mathrm{d}s + \int_0^t \sigma_0(s,u)\mathrm{d}W_0(s), \quad 0 \leq t \leq u \leq T. \tag{1.4}$$

Assume that the coefficients μ_i and σ_i are $\{\mathcal{F}_t\}$-adapted, and such that for $\mu(s) = \mu_0(s,t)$ and $\sigma(s) = \sigma_0(s,t)$, or $\mu(s) = \mu_i(s)$ and $\sigma(s) = \sigma_i(s)$, they satisfy $\mathbb{E}\big[\int_0^t |\mu(s)|\mathrm{d}s\big] < \infty$ and $\mathbb{E}\big[\int_0^t |\sigma(s)|^2\mathrm{d}s\big] < \infty$ for each $t > 0$. Moreover, $\sigma_i(t) \neq 0$ for all $t \geq 0$. Here $\mathbb{E}[\cdot]$ denotes the expected value with respect to $\mathbb{P}$. For notational simplicity, denote $\overline{\mu}_0(s,u) = \int_s^u \mu_0(s,t)\mathrm{d}t$ and $\overline{\sigma}_0(s,u) = \int_s^u \sigma_0(s,t)\mathrm{d}t$ for all $0 \leq s \leq u$. Then, by applying Fubini's theorems, we get

$$\begin{aligned} P(t,T) &= P(0,T)\exp\Big\{\int_0^t \big[\gamma(0,s) - \big(\overline{\mu}_0(s,T) - \overline{\mu}_0(s,t)\big)\big]\mathrm{d}s \\ &\quad - \int_0^t \big(\overline{\sigma}_0(s,T) - \overline{\sigma}_0(s,t)\big)\mathrm{d}W_0(s)\Big\}, \quad 0 \leq t \leq T \end{aligned}$$

An exchange option is an option for which the promised payoff paid from the counterparty firm at maturity T is given by $X = \big(S_1(T) - S_2(T)\big)^+$. Denote the ratios of S_1 to S_2, and S_3 to S_4 respectively by

$$\theta(t) = S_1(t)/S_2(t) \quad \text{and} \quad \delta(t) = S_3(t)/S_4(t), \quad 0 \le t \le T. \tag{1.5}$$

Consider the option with counterparty firm default risk, i.e. the default or bankruptcy of the counterparty firm will occur when $S_3(T) < S_4(T)$. In case of default, the payoff will be not X, but only a fraction thereof $\delta(T)X$, where $\delta(T)$ is the recovery rate, the fraction of the promised payoff paid from the counterparty firm in case of default. Hence, the actual payoff is given by

$$X^d = S_2(T)\big(\theta(T) - 1\big)^+\big(\mathbf{1}_{\{\delta(T)\ge 1\}} + \delta(T)\mathbf{1}_{\{\delta(T)<1\}}\big). \tag{1.6}$$

Here $\mathbf{1}_A$ is the indicator of A. We will introduce some useful results of Stochastic Calculus in the following. Then, in Section 2, we will show that there exists a forward martingale measure for the numéraire $P(\cdot, T)$, and derive some auxiliary results for $\theta(t)$ and $\delta(t)$. Finally, under the assumption that the volatilities are deterministic, we will derive the pricing formula of X^d for the numéraire $P(\cdot, T)$ in Section 3.

Let M be a continuous $\{\mathcal{F}_t\}$-local martingale, and σ be an $\{\mathcal{F}_t\}$-adapted process with $\mathbb{E}\big[\int_0^T \sigma^2(s)\mathrm{d}s\big] < \infty$ such that $\langle M\rangle(t) = \int_0^t \sigma^2(s)\mathrm{d}s$, $t \in [0, T]$.

Theorem 1.1 *If $\sigma(t) \neq 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$, then there exists a standard $\{\mathcal{F}_t\}$-Brownian motion W such that M has the representation:*

$$M(t) = M(0) + \int_0^t \sigma(s)\mathrm{d}W(s), \quad 0 \le t \le T.$$

This result is due to Doob (1953), and its proof can be found in [4] or [5]. Apply Theorem 1.1 and Girsanov's theorem ([5]) we can prove a Girsanov's theorem for the correlated Brownian motion $\mathbf{W}$. Let $\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_4)$ be a 5-dimensional $\{\mathcal{F}_t\}$-adapted process such that, for each i, $\mathbb{E}\big[\int_0^T \lambda_i^2(s)\mathrm{d}s\big] < \infty$. Define $Z_\lambda(t) = \exp\big\{-\int_0^t \Lambda(s)\mathrm{d}\mathbf{W}(s) - \frac{1}{2}\int_0^t \lambda^2(s)\mathrm{d}s\big\}$, $0 \le t \le T$. Where $\lambda(t) = \big(\Lambda(t)C\Lambda(t)^*\big)^{\frac{1}{2}}$. For convenience, we also denote $\rho_{0i} = \rho_{i0} = 0$ for all $i = 1, \cdots, 4$, and $\rho_{ii} = 1$ for all $i = 0, 1, \cdots, 4$.

Theorem 1.2 *If λ satisfies the Novikov's condition that $\mathbb{E}\big[e^{\int_0^T \frac{1}{2}\lambda^2(s)\mathrm{d}s}\big] < \infty$, then, Z_λ is an $\{\mathcal{F}_t\}$-martingale, and $\mathrm{d}\tilde{\mathbb{P}} = Z_\lambda(T)\mathrm{d}\mathbb{P}$ define an equivalent probability on $(\Omega, \mathcal{F}_T)$. Moreover, under $\tilde{\mathbb{P}}$*

$$\tilde{W}_i(t) = W_i(t) + \int_0^t \sum_{j=0}^{4} \rho_{ij}\lambda_j(s)\mathrm{d}s, \quad 0 \le t \le T, \quad i = 0, 1, \ldots, 4, \tag{1.7}$$

are standard $\{\mathcal{F}_t\}$-Brownian motions such that $\langle \tilde{W}_i, \tilde{W}_j\rangle(t) = \rho_{ij}t$.

This theorem implies that the correlation matrix of $\tilde{\mathbf{W}} = (\tilde{W}_0, \tilde{W}_1, \dots, \tilde{W}_4)^*$ is still C. Finally, we recall a well-known fact that, if ϕ is deterministic with $\int_0^T \phi^2(u)\mathrm{d}u < \infty$, and W is a standard $\{\mathcal{F}_t\}$-Brownian motion, then, for any $0 \le s < t \le T$, $\eta = \int_s^t \phi(u)\mathrm{d}W(u)$ is a random variable, which is independent of $\mathcal{F}_s$, having a normal distribution with zero mean and variance $\sigma_\eta^2 = \int_s^t \phi^2(u)\mathrm{d}u$.

2 The Forward Martingale Measure

We assume that there exists a stochastic process Λ satisfies the following system of equations:

$$\begin{cases} \overline{\sigma}_0(t,u)\lambda_0(t) &= -\overline{\mu}_0(t,u) + \frac{1}{2}\overline{\sigma}_0^2(t,u), \\ \sigma_i(t)\sum_{j=1}^4 \rho_{ij}\lambda_j(t) &= \mu_i(t) - \gamma(t,t), \quad i = 1, \cdots, 4. \end{cases} \tag{2.1}$$

for any $0 \le t \le u \le T$. Then, under the assumptions of Theorem 1.2, there is an equivalent probability $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ such that $\tilde{\mathbf{W}} = (\tilde{W}_0, \tilde{W}_1, \dots, \tilde{W}_4)^*$, which is defined in (1.7), is a correlated 5-dimensional standard $(\{\mathcal{F}_t\}, \tilde{\mathbb{P}})$-Brownian motion whose correlation matrix is still C.

Using the relations (1.7) and (2.1), we can verify that, under $\tilde{\mathbb{P}}$, $P(\cdot, T)$ and S_i, $i = 1, \dots, 4$, have the following representation:

$$\begin{cases} \mathrm{d}P(t,T) &= \gamma(t,t)P(t,T)\mathrm{d}t - \overline{\sigma}_0(t,T)P(t,T)\mathrm{d}\tilde{W}_0(t) \\ \mathrm{d}S_i(t) &= \gamma(t,t)S_i(t)\mathrm{d}t + \sigma_i(t)S_i(t)\mathrm{d}\tilde{W}_i(t), \quad i = 1, \dots, 4. \end{cases} \tag{2.2}$$

Define the forward prices, forward value and forward liability respectively:

$$S_i(t,T) = S_i(t)/P(t,T), \quad 0 \le t \le T, \quad i = 1, \dots, 4.$$

Assume that $\overline{\sigma}_0(t,T)$ satisfies the Novikov's condition. According to Girsanov's Theorem, $\mathrm{d}\tilde{\mathbb{P}}^F/\mathrm{d}\tilde{\mathbb{P}} = \exp\big\{ -\int_0^T \overline{\sigma}_0(s,T)\mathrm{d}\tilde{W}_0(s) - \frac{1}{2}\int_0^T \overline{\sigma}_0^2(s,T)\mathrm{d}s \big\}$ defines an equivalent probability on $(\Omega, \mathcal{F}_T)$ such that

$$\tilde{W}_0^F(t) = \tilde{W}_0(t) + \int_0^t \overline{\sigma}_0(s,T)\mathrm{d}s, \quad 0 \le t \le T, \tag{2.3}$$

is a standard $(\{\mathcal{F}_t\}, \tilde{\mathbb{P}}^F)$-Brownian motion. Since $\tilde{W}_0$ is independent of each $\tilde{W}_i$ for $i \ne 0$, each $\tilde{W}_i$ is still a standard $(\{\mathcal{F}_t\}, \tilde{\mathbb{P}}^F)$-Brownian motion such that the correlation matrix of $\tilde{\mathbf{W}}^F = (\tilde{W}_0^F, \tilde{W}_1, \dots, \tilde{W}_4)^*$ is still C. This new probability $\tilde{\mathbb{P}}^F$ is called as the *forward martingale measure*.

Theorem 2.1 *$S_i(t,T)$, $i=1,\dots,4$, are respectively given by*

$$\mathrm{d}S_i(t,T) = S_i(t,T)\big(\sigma_i(t)\mathrm{d}\tilde{W}_i(t) + \overline{\sigma}_0(t,T)\mathrm{d}\tilde{W}_0^F(t)\big). \tag{2.4}$$

Hence, they are all $(\{\mathcal{F}_t\},\tilde{\mathbb{P}}^F)$-martingales, i.e., $\tilde{\mathbb{P}}^F$ is the equivalent martingale measure for $P(\cdot,T)$,.

Proof. From (2.2) and (2.3), we get

$$\begin{aligned} S_1(t,T) &= S_1(0,T)\exp\Big\{-\frac{1}{2}\int_0^t \big(\sigma_1^2(s)+\overline{\sigma}_0^2(s,T)\big)\mathrm{d}s \\ &\quad + \int_0^t \big(\sigma_1(s)\mathrm{d}\tilde{W}_1(s)+\overline{\sigma}_0(s,T)\mathrm{d}\tilde{W}_0^F(s)\big)\Big\}, \quad 0\le t\le T. \end{aligned}$$

This implies that $S_1(t,T)$ satisfies the first SDE in (2.4). Similarly, we can verify other cases. □

For each $i=1,2$, we introduce an $(\{\mathcal{F}_t\},\tilde{\mathbb{P}}^F)$-martingale:

$$M_i(t) = \int_0^t \sigma_i(s)\mathrm{d}\tilde{W}_i(s) + \int_0^t \overline{\sigma}_0(s,T)\mathrm{d}\tilde{W}_0^F(s), \quad 0\le t\le T,$$

and denote

$$\sigma_i(t,T) = \big(\sigma_i^2(t)+\overline{\sigma}_0^2(t,T)\big)^{\frac{1}{2}}, \quad 0\le t\le T. \tag{2.5}$$

Then, since $\tilde{W}_i$ and $\tilde{W}_0^F$ are independent, each M_i is a square integrable martingale with $\langle M_i\rangle(t) = \int_0^t \sigma_i^2(s,T)\mathrm{d}s$, $t\in[0,T]$. Thus,

$$\tilde{W}_i^F(t) = \int_0^t \sigma_i^{-2}(s,T)\mathrm{d}M_i(s), \quad 0\le t\le T, \quad i=1,2,$$

are two $(\{\mathcal{F}_t\},\tilde{\mathbb{P}}^F)$-Brownian motions such that

$$S_i(t,T) = S_i(0,T)\exp\Big\{-\frac{1}{2}\int_0^t \sigma_i^2(s,T)\mathrm{d}s + \int_0^t \sigma_i^2(s,T)\mathrm{d}\tilde{W}_i^F(s)\Big\}, \tag{2.6}$$

for all $t\in[0,T]$ and $i=1,2$. Noting that $\theta(T)=S_1(T,T)/S_2(T,T)$ and $\delta(T)=S_3(T,T)/S_4(T,T)$, from Theorem 2.1 it follows that the price of X^d for the numéraire $P(t,T)$ at time $t\in[0,T]$ is given by

$$X(t) = P(t,T)\tilde{\mathbb{E}}^F\Big[S_2(T,T)\big(\theta(T)-1\big)^+\big(\mathbf{1}_{\{\delta(T)\ge 1\}}+\delta(T)\mathbf{1}_{\{\delta(T)<1\}}\big)\mid \mathcal{F}_t\Big]. \tag{2.7}$$

Here $\tilde{\mathbb{E}}^F[\cdot]$ denotes the expected value with respect to $\tilde{\mathbb{P}}^F$. Now, we derive some representation of θ and δ. Define two $(\{\mathcal{F}_t\}, \tilde{\mathbb{P}}^F)$-martingales:

$$\mathrm{d}M_\theta(t) = \sigma_1(t)\mathrm{d}\tilde{W}_1(t) - \sigma_2(t)\mathrm{d}\tilde{W}_2(t), \quad \mathrm{d}M_\delta(t) = \sigma_3(t)\mathrm{d}\tilde{W}_3(t) - \sigma_4(t)\mathrm{d}\tilde{W}_4(t)$$

for all $t \in [0, T]$, and denote

$$\begin{cases} \sigma_\theta(t) &= \big(\sigma_1^2(t) + \sigma_2^2(t) - 2\rho_{12}\sigma_1(t)\sigma_2(t)\big)^{\frac{1}{2}} \\ \sigma_\delta(t) &= \big(\sigma_3^2(t) + \sigma_4^2(t) - 2\rho_{34}\sigma_3(t)\sigma_4(t)\big)^{\frac{1}{2}} \\ \sigma_{\theta\delta}(t) &= \sigma_{1\delta}(t) - \sigma_{2\delta}(t) \end{cases} \tag{2.8}$$

for each $t \in [0, T]$, where $\sigma_{1\delta}(t)$ and $\sigma_{2\delta}(t)$ are given by

$$\sigma_{i\delta}(t) = \rho_{i3}\sigma_i(t)\sigma_3(t) - \rho_{i4}\sigma_i(t)\sigma_4(t), \quad 0 \le t \le T, \quad i = 1, 2. \tag{2.9}$$

Then, it is clear that, for each $t \in [0, T]$,

$$\langle M_j\rangle(t) = \int_0^t \sigma_j^2(s)\mathrm{d}s, \quad j = \theta, \delta, \quad \text{and} \quad \langle M_\theta, M_\delta\rangle(t) = \int_0^t \sigma_{\theta\delta}(s)\mathrm{d}s.$$

Assume that $\sigma_\theta(t) \neq 0$ and $\sigma_\delta(t) \neq 0$ for all $t \in [0, T]$. Then,

$$\tilde{W}_\theta(t) = \int_0^t \sigma_\theta^{-1}(s)\mathrm{d}M_\theta(s) \quad \text{and} \quad \tilde{W}_\delta(t) = \int_0^t \sigma_\delta^{-1}(s)\mathrm{d}M_\delta(s), \quad 0 \le t \le T,$$

are two standard $(\{\mathcal{F}_t\}, \tilde{\mathbb{P}}^F)$-Brownian motions such that

$$\begin{cases} \mathrm{d}\theta(t)/\theta(t) &= \big(\sigma_2^2(t) - \rho_{12}\sigma_1(t)\sigma_2(t)\big)\mathrm{d}t + \sigma_\theta(t)\mathrm{d}\tilde{W}_\theta(t), \\ \mathrm{d}\delta(t)/\delta(t) &= \big(\sigma_4^2(t) - \rho_{34}\sigma_3(t)\sigma_4(t)\big)\mathrm{d}t + \sigma_\delta(t)\mathrm{d}\tilde{W}_\delta(t), \end{cases} \tag{2.10}$$

for all $t \in [0, T]$. Assume that there are constants $\rho_{2\theta}$, $\rho_{2\delta}$ and $\rho_{\theta\delta}$ such that $\langle \tilde{W}_2^F, \tilde{W}_j\rangle(t) = \rho_{2j}t$ for $j = \theta, \delta$, and $\langle \tilde{W}_\theta, \tilde{W}_\delta\rangle(t) = \rho_{\theta\delta}t$. Then, we have

$$\begin{cases} \rho_{2\theta} = \big(\rho_{12}\sigma_1(t)\sigma_2(t) - \sigma_2^2(t)\big)/\sigma_2(t, T)\sigma_\theta(t), \\ \rho_{2\delta} = \sigma_{2\delta}(t)/\sigma_2(t, T)\sigma_\delta(t), \\ \rho_{\theta\delta} = \sigma_{\theta\delta}(t)/\sigma_\theta(t)\sigma_\delta(t), \end{cases} \quad 0 \le t \le T. \tag{2.11}$$

Define an equivalent probability $\hat{\mathbb{P}}$ by $\mathrm{d}\hat{\mathbb{P}}/\mathrm{d}\tilde{\mathbb{P}}^F = S_2(T, T)/S_2(0, T)$ on $(\Omega, \mathcal{F}_T)$. According to Girsanov's theorem and using the relations in (2.11), under $\hat{\mathbb{P}}$,

$$\hat{W}_2(t) = \tilde{W}_2^F(t) - \int_0^t \sigma_2(s, T)\mathrm{d}s, \quad \hat{W}_j(t) = \tilde{W}_j(t) - \int_0^t \rho_{2j}\sigma_2(s, T)\mathrm{d}s \tag{2.12}$$

for each $j = \theta, \delta$ and $0 \le t \le T$, are standard $\{\mathcal{F}_t\}$-Brownian motions such that

$$\langle \hat{W}_2, \hat{W}_\theta\rangle(t) = \rho_{2\theta}t, \quad \langle \hat{W}_2, \hat{W}_\delta\rangle(t) = \rho_{2\delta}t \quad \text{and} \quad \langle \hat{W}_\theta, \hat{W}_\delta\rangle(t) = \rho_{\theta\delta}t,$$

for all $t \in [0,T]$. From (2.11) and (2.12), under $\hat{\mathbb{P}}$, we obtain

$$\begin{cases} \theta(t) &= \theta(0)\exp\Big\{-\int_0^t \frac{1}{2}\sigma_\theta^2(s)\mathrm{d}s + \int_0^t \sigma_\theta(s)\mathrm{d}\hat{W}_\theta(s)\Big\}, \\ \delta(t) &= \delta(0)\exp\Big\{\int_0^t \big(\frac{1}{2}\sigma_4^2(s) - \frac{1}{2}\sigma_3^2(s) + \sigma_{2\delta}(s)\big)\mathrm{d}s \\ & \quad + \int_0^t \sigma_\delta(s)\mathrm{d}\hat{W}_\delta(s)\Big\}. \end{cases} \tag{2.13}$$

Now, letting $Z_2(t) = S(t,T)/S(0,T)$, from (2.7) it follows that

$$X(t) = S_2(t)Z_2^{-1}(t)\tilde{\mathbb{E}}^F\Big[Z_2(T)\big(\theta(T)-1\big)^+\big(\mathbf{1}_{\{\delta(T)\geq 1\}} + \delta(T)\mathbf{1}_{\{\delta(T)<1\}}\big) \mid \mathcal{F}_t\Big].$$

By the Bayes rule for conditional expectations we get

$$X(t) = S_2(t)\hat{\mathbb{E}}\Big[\big(\theta(T)-1\big)^+\big(\mathbf{1}_{\{\delta(T)\geq 1\}} + \delta(T)\mathbf{1}_{\{\delta(T)<1\}}\big) \mid \mathcal{F}_t\Big]. \tag{2.14}$$

Here $\hat{\mathbb{E}}[\cdot]$ denotes the expected value with respect to $\hat{\mathbb{P}}$.

3 The Pricing Formula

For any fixed time $t \in [0,T]$, denote

$$\sigma_{j,t,T} = \Big(\int_t^T \sigma_j^2(s)\mathrm{d}s\Big)^{\frac{1}{2}}, \quad j = \theta, \delta, \quad \rho_{t,T} = \int_t^T \frac{\sigma_{\theta\delta}(s)}{\sigma_{\theta,t,T}\sigma_{\delta,t,T}}\mathrm{d}s \tag{3.1}$$

and

$$\psi_i(t) = \frac{1}{2}\sigma_4^2(t) - \frac{1}{2}\sigma_3^2(t) + \frac{1}{2}\sigma_\delta^2(t) + \sigma_{i\delta}(t), \quad i = 1, 2, \tag{3.2}$$

where σ_θ, σ_δ and $\sigma_{\theta\delta}$ are given in (2.8), and $\sigma_{i\delta}$, $i = 1, 2$, are given in (2.9). We also denote $N(z_1, z_2, \rho)$ as the bivariate standard normal distribution with the correlation coefficient ρ, i.e.

$$N(z_1, z_2, \rho) = \int_{-\infty}^{z_1}\int_{-\infty}^{z_2} \varphi(x_1, x_2; \rho)\mathrm{d}x_1\mathrm{d}x_2, \quad -\infty < z_1, z_2 < +\infty,$$

where

$$\varphi(x_1, x_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\Big\{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\Big\}, \quad -\infty < x_1, x_2 < +\infty.$$

Then we have the following main result.

Theorem 3.1 *If σ_i, $i = 1, \dots, 4$, are all deterministic, then, under all our assumptions, the price $X(t)$ of claim X^d at time $t \in [0, T]$ is given by*

$$\begin{aligned} X(t) &= S_1(t)N(a_1, a_2, \rho) - S_2(t)N(b_1, b_2, \rho) \\ &\quad + S_1(t)\delta(t)\exp\Big\{\int_t^T \psi_1(s)\mathrm{d}s\Big\}N(c_1, c_2, -\rho) \\ &\quad - S_2(t)\delta(t)\exp\Big\{\int_t^T \psi_2(s)\mathrm{d}s\Big\}N(d_1, d_2, -\rho) \end{aligned}$$

with parameters $\rho = \rho_{t,T}$, and

$$\begin{aligned} &a_1 = \big(\ln\theta(t) + \tfrac{1}{2}\sigma^2_{\theta,t,T}\big)/\sigma_{\theta,t,T}, \quad a_2 = \big(\ln\delta(t) + \textstyle\int_t^T \varphi_1(s)\mathrm{d}s - \tfrac{1}{2}\sigma^2_{\delta,t,T}\big)/\sigma_{\delta,t,T}, \\ &b_1 = \big(\ln\theta(t) - \tfrac{1}{2}\sigma^2_{\theta,t,T}\big)/\sigma_{\theta,t,T}, \quad b_2 = \big(\ln\delta(t) + \textstyle\int_t^T \varphi_2(s)\mathrm{d}s - \tfrac{1}{2}\sigma^2_{\delta,t,T}\big)/\sigma_{\delta,t,T}, \\ &c_1 = \big(\ln\theta(t) + \textstyle\int_t^T \sigma_{\theta\delta}(s)\mathrm{d}s + \tfrac{1}{2}\sigma^2_{\theta,t,T}\big)/\sigma_{\theta,t,T}, \\ &c_2 = -\big(\ln\delta(t) + \textstyle\int_t^T \varphi_1(s)\mathrm{d}s + \tfrac{1}{2}\sigma^2_{\delta,t,T}\big)/\sigma_{\delta,t,T}, \\ &d_1 = \big(\ln\theta(t) + \textstyle\int_t^T \sigma_{\theta\delta}(s)\mathrm{d}s - \tfrac{1}{2}\sigma^2_{\theta,t,T}\big)/\sigma_{\theta,t,T}, \\ &d_2 = -\big(\ln\delta(t) + \textstyle\int_t^T \varphi_2(s)\mathrm{d}s + \tfrac{1}{2}\sigma^2_{\delta,t,T}\big)/\sigma_{\delta,t,T}. \end{aligned}$$

Proof. By the representation (2.14), $X(t)$ is split into four separate terms:

$$\begin{aligned} X(t) &= S_2(t)\hat{\mathbb{E}}\Big[\theta(T)\mathbf{1}_{\{\theta(T)>1\}}\mathbf{1}_{\{\delta(T)\geq 1\}} \mid \mathcal{F}_t\Big] - S_2(t)\hat{\mathbb{E}}\Big[\mathbf{1}_{\{\theta(T)>1\}}\mathbf{1}_{\{\delta(T)\geq 1\}} \mid \mathcal{F}_t\Big] \\ &\quad + S_2(t)\hat{\mathbb{E}}\Big[\theta(T)\delta(T)\mathbf{1}_{\{\theta(T)>1\}}\mathbf{1}_{\{\delta(T)<1\}} \mid \mathcal{F}_t\Big] \\ &\quad - S_2(t)\hat{\mathbb{E}}\Big[\delta(T)\mathbf{1}_{\{\theta(T)>1\}}\mathbf{1}_{\{\delta(T)<1\}} \mid \mathcal{F}_t\Big] \\ &\doteq E_1 - E_2 + E_3 - E_4. \end{aligned}$$

Each of the four terms can be evaluated separately.

Evaluation of term E_1. Define an equivalent probability $\hat{\mathbb{P}}^\theta$ on $(\Omega, \mathcal{F}_T)$ by $\mathrm{d}\hat{\mathbb{P}}^\theta/\mathrm{d}\hat{\mathbb{P}} = \theta(T)/\theta(0)$. Then, according to Girsanov's theorem,

$$\hat{W}^\theta_\theta(t) = \hat{W}_\theta(t) - \int_0^t \sigma_\theta(s)\mathrm{d}s, \quad \hat{W}^\theta_\delta(t) = \hat{W}_\delta(t) - \int_0^t \rho_{\theta\delta}\sigma_\theta(s)\mathrm{d}s, \tag{3.3}$$

for $0 \leq t \leq T$, are two standard $(\{\mathcal{F}_t\}, \hat{\mathbb{P}}^\theta)$-Brownian motions such that $\langle \hat{W}^\theta_\theta, \hat{W}^\theta_\delta\rangle(t) = \rho_{\theta\delta}t$, $t \in [0, T]$. By the Bayes rule for conditional expectations we have

$$E_1 = S_1(t)\hat{\mathbb{E}}^\theta\Big[\mathbf{1}_{\{\theta(T)>1\}}\mathbf{1}_{\{\delta(T)\geq 1\}} \mid \mathcal{F}_t\Big], \tag{3.4}$$

where $\hat{\mathbb{E}}^\theta[\cdot]$ denotes the expected value with respect to $\hat{\mathbb{P}}^\theta$. On the other hand, from (2.8), (2.11), (2.13) and (3.3), under $\hat{\mathbb{P}}^\theta$, we have

$$\begin{aligned}\theta(T) &= \theta(t)\exp\Big\{\int_t^T \frac{1}{2}\sigma_\theta^2(s)\mathrm{d}s + \int_t^T \sigma_\theta(s)\mathrm{d}\hat{W}_\theta^\theta(s)\Big\},\\ \delta(T) &= \delta(t)\exp\Big\{\int_t^T \big(\frac{1}{2}\sigma_4^2(s) - \frac{1}{2}\sigma_3^2(s) + \sigma_{1\delta}(s)\big)\mathrm{d}s + \int_t^T \sigma_\delta(s)\mathrm{d}\hat{W}_\delta^\theta(s)\Big\}.\end{aligned}$$

Thus, we have

$$\theta(T) > 1 \Longleftrightarrow \xi_\theta^\theta \geq -a_1 \quad \text{and} \quad \delta(T) \geq 1 \Longleftrightarrow \xi_\delta^\theta \geq -a_2,$$

where $\xi_j^\theta = \big(\int_t^T \sigma_j(s)\mathrm{d}\hat{W}_j^\theta(s)\big)/\sigma_{j,t,T}$, $j = \theta, \delta$. Since σ_θ and σ_δ are deterministic, under $\hat{\mathbb{P}}^\theta$, ξ_θ^θ and ξ_δ^θ are standard normal distributed with the correlation coefficient $\rho_{t,T}$. Moreover, they are independent of $\mathcal{F}_t$. From (3.4) we get

$$E_1 = S_1(t)\hat{\mathbb{E}}^\theta\Big[\mathbf{1}_{\{\xi_\theta^\theta \geq -a_1\}}\mathbf{1}_{\{\xi_\delta^\theta \geq -a_2\}}\Big] = S_1(t)N(a_1, a_2, \rho).$$

Evaluation of term E_2. From (2.13) we have

$$\theta(T) > 1 \Longleftrightarrow \xi_\theta > -b_1 \quad \text{and} \quad \delta(T) \geq 1 \Longleftrightarrow \xi_\delta \geq -b_2.$$

Here $\xi_j = \big(\int_t^T \sigma_j(s)\mathrm{d}\hat{W}_j(s)\big)/\sigma_{j,t,T}$, $j = \theta, \delta$, are standard normal distributed with the correlation coefficient $\rho_{t,T}$ under $\hat{\mathbb{P}}$, and they all independent of $\mathcal{F}_t$. Thus, we get

$$E_2 = S_2(t)\hat{\mathbb{E}}\Big[\mathbf{1}_{\{\xi_\theta \geq -b_1\}}\mathbf{1}_{\{\xi_\delta \geq -b_2\}}\Big] = S_2(t)N(b_1, b_2, \rho).$$

Evaluation of term E_3. Define two $(\{\mathcal{F}_t\}, \hat{\mathbb{P}})$-martingales:

$$\hat{M}_\theta(t) = \int_0^t \sigma_\theta(s)\mathrm{d}\hat{W}_\theta(s) \quad \text{and} \quad \hat{M}_\delta(t) = \int_0^t \sigma_\delta(s)\mathrm{d}\hat{W}_\delta(s), \quad 0 \leq t \leq T,$$

and let $\hat{M}_\kappa(t) = \hat{M}_\theta(t) + \hat{M}_\delta(t)$ for all $t \in [0, T]$. Denote

$$\sigma_\kappa(t) = \big(\sigma_\theta^2(t) + \sigma_\delta^2(t) + 2\sigma_{\theta\delta}(t)\big)^{\frac{1}{2}}, \quad 0 \leq t \leq T. \tag{3.5}$$

Then, it is clear that $\langle \hat{M}_\kappa\rangle(t) = \int_0^t \sigma_\kappa^2(s)\mathrm{d}s$ for each $t \in [0, T]$. Thus, there exists a standard $(\{\mathcal{F}_t\}, \hat{\mathbb{P}})$-Brownian motion $\hat{W}_\kappa$ such that

$$\int_0^t \sigma_\kappa(s)\mathrm{d}\hat{W}_\kappa(s) = \hat{M}_\kappa(t) = \int_0^t \sigma_\theta(s)\mathrm{d}\hat{W}_\theta(s) + \int_0^t \sigma_\delta(s)\mathrm{d}\hat{W}_\delta(s), \tag{3.6}$$

for all $t \in [0,T]$. Let $\rho_{\kappa\theta}$ and $\rho_{\kappa\delta}$ be constants such that $\langle \hat{W}_\kappa, \hat{W}_j \rangle(t) = \rho_{\kappa j} t$ for all $t \in [0,T]$ and for $j = \theta, \delta$. Then, we have

$$\langle \hat{M}_\kappa, \hat{M}_j \rangle(t) = \int_0^t \rho_{\kappa j} \sigma_\kappa(s) \sigma_j(s) \mathrm{d}s, \quad 0 \le t \le T, \quad j = \theta, \delta.$$

On the other hand, from the definition of $\hat{M}_\kappa$ we also have

$$\langle \hat{M}_\kappa, \hat{M}_j \rangle(t) = \int_0^t \big(\sigma_j^2(s) + \sigma_{\theta\delta}(s)\big) \mathrm{d}s, \quad 0 \le t \le T, \quad j = \theta, \delta.$$

and so that $\rho_{\kappa j} \sigma_\kappa(t) \sigma_j(t) = \sigma_j^2(t) + \sigma_{\theta\delta}(t)$ for all $t \in [0,T]$ and for each $j = \theta, \delta$. Define

$$Z_\kappa(t) = \exp\Big\{ - \int_0^t \frac{1}{2} \sigma_\kappa^2(s) \mathrm{d}s + \int_0^t \sigma_\kappa(s) \mathrm{d}\hat{W}_\kappa(s) \Big\}, \quad 0 \le t \le T.$$

Then, it is clear that

$$\theta(T)\delta(T) = \theta(0)\delta(0) \exp\Big\{ \int_0^T \psi_1(s) \mathrm{d}s \Big\} Z_\kappa(T).$$

Define a probability $\hat{\mathbb{P}}^\kappa$ on $(\Omega, \mathcal{F}_T)$ by $\mathrm{d}\hat{\mathbb{P}}^\kappa / \mathrm{d}\hat{\mathbb{P}} = Z_\kappa(T)$. According to Girsanov's theorem,

$$\hat{W}_\kappa^\kappa(t) = \hat{W}_\kappa(t) - \int_0^t \sigma_\kappa(s) \mathrm{d}s, \qquad \hat{W}_j^\kappa(t) = \hat{W}_j(t) - \int_0^t \rho_{\kappa j} \sigma_\kappa(s) \mathrm{d}s, \tag{3.7}$$

for each $j = \theta, \delta$ and all $t \in [0,T]$, are three standard $\{\mathcal{F}_t\}$-Brownian motions with

$$\langle \hat{W}_\kappa^\kappa, \hat{W}_j^\kappa \rangle(t) = \rho_{\kappa j} t, \quad j = \theta, \delta, \quad \text{and} \quad \langle \hat{W}_\theta^\kappa, \hat{W}_\delta^\kappa \rangle(t) = \rho_{\theta\delta} t, \quad 0 \le t \le T.$$

By the Bayes rule for conditional expectation we get

$$\begin{aligned} E_3 &= S_2(t)\theta(0)\delta(0) \exp\Big\{ \int_0^T \psi_1(s) \mathrm{d}s \Big\} \hat{\mathbb{E}}\Big[Z_\kappa(T) \big(\mathbf{1}_{\{\theta(T)>1\}} \mathbf{1}_{\{\delta(T)<1\}} \big) \mid \mathcal{F}_t \Big] \\ &= S_1(t)\delta(t) \exp\Big\{ \int_t^T \psi_1(s) \mathrm{d}s \Big\} \hat{\mathbb{E}}^\kappa\Big[\big(\mathbf{1}_{\{\delta(T)>1\}} \mathbf{1}_{\{\delta(T)<1\}} \big) \mid \mathcal{F}_t \Big], \end{aligned}$$

where $\hat{\mathbb{E}}^\kappa[\cdot]$ denotes the expected value with respect to $\hat{\mathbb{P}}^\kappa$. On the other hand, from (2.13) and (3.7), under $\hat{\mathbb{P}}^\kappa$ we have

$$\begin{aligned} \theta(T) &= \theta(t) \exp\Big\{ \int_t^T \big(\frac{1}{2} \sigma_\theta^2(s) + \sigma_{\theta\delta}(s) \big) \mathrm{d}s + \int_t^T \sigma_\theta(s) \mathrm{d}\hat{W}_\theta^\kappa(s) \Big\}, \\ \delta(T) &= \delta(t) \exp\Big\{ \int_t^T \big(\varphi_1(s) + \frac{1}{2} \sigma_\delta^2(s) \big) \mathrm{d}s + \int_t^T \sigma_\delta(s) \mathrm{d}\hat{W}_\delta^\kappa(s) \Big\}. \end{aligned}$$

Thus, we get

$$\delta(T) > 1 \Longleftrightarrow \xi_\theta^\kappa > -c_1 \quad \text{and} \quad \delta(T) < 1 \Longleftrightarrow \xi_\delta^\kappa < c_2,$$

where $\xi_j^\kappa = \left(\int_t^T \sigma_j(s)\mathrm{d}\hat{W}_j^\kappa(s)\right)/\sigma_{j,t,T}$, $j = \theta, \delta$. Under $\hat{\mathbb{P}}^\kappa$, ξ_j^κ, $j = \theta, \delta$, are standard normal distributed with correlation coefficient $\rho_{t,T}$. Moreover, they are all independent of $\mathcal{F}_t$. Hence, E_3 can be expressed by

$$\begin{aligned} E_3 &= S_1(t)\delta(t)\exp\Big\{\int_t^T \psi_1(s)\mathrm{d}s\Big\}\hat{\mathbb{E}}^\kappa\Big[\mathbf{1}_{\{\xi_\theta^\kappa > -c_1\}}\mathbf{1}_{\{\xi_\delta^\kappa < c_2\}}\Big] \\ &= S_1(t)\delta(t)\exp\Big\{\int_t^T \psi_1(s)\mathrm{d}s\Big\}N(c_1, c_2, -\rho). \end{aligned}$$

Evaluation of term E_4. Denote

$$Z_\delta(t) = \exp\Big\{-\int_0^t \frac{1}{2}\sigma_\delta^2(s)\mathrm{d}s + \int_0^t \sigma_\delta(s)\mathrm{d}\hat{W}_\delta(s)\Big\}, \quad 0 \le t \le T.$$

Then, it is clear that

$$\delta(t) = \delta(0)\exp\Big\{\int_0^t \psi_2(s)\mathrm{d}s\Big\}Z_\delta(t), \quad 0 \le t \le T.$$

Define a probability $\hat{\mathbb{P}}^\delta$ on $(\Omega, \mathcal{F}_T)$ by $\mathrm{d}\hat{\mathbb{P}}^\delta/\mathrm{d}\hat{\mathbb{P}} = Z_\delta(T)$. According to Girsanov theorem, under $\hat{\mathbb{P}}^\delta$,

$$\hat{W}_\theta^\delta(t) = \hat{W}_\theta(t) - \int_0^t \rho_{\theta\delta}\sigma_\delta(s)\mathrm{d}s, \quad \hat{W}_\delta^\delta(t) = \hat{W}_\delta(t) - \int_0^t \sigma_\delta(s)\mathrm{d}s, \tag{3.8}$$

for $0 \le t \le T$, are two standard $\{\mathcal{F}_t\}$-Brownian motions with $\langle \hat{W}_\theta^\delta, \hat{W}_\delta^\delta\rangle(t) = \rho_{\theta\delta}t$. By the Bayes rule for conditional expectation we get

$$\begin{aligned} E_4 &= S_2(t)\delta(0)\exp\Big\{\int_0^T \psi_2(s)\mathrm{d}s\Big\}\hat{\mathbb{E}}\Big[Z_\delta(T)\big(\mathbf{1}_{\{\theta(T)>1\}}\mathbf{1}_{\{\delta(T)<1\}}\big) \mid \mathcal{F}_t\Big] \\ &= S_2(t)\delta(t)\exp\Big\{\int_t^T \psi_2(s)\mathrm{d}s\Big\}\hat{\mathbb{E}}^\delta\Big[\big(\mathbf{1}_{\{\theta(T)>1\}}\mathbf{1}_{\{\delta(T)<1\}}\big) \mid \mathcal{F}_t\Big], \end{aligned}$$

where $\hat{\mathbb{E}}^\delta[\cdot]$ denotes the expected value with respect to $\hat{\mathbb{P}}^\delta$. Now, from (2.8), (2.11), (2.13) and (3.8), we have

$$\begin{aligned} \theta(T) &= \theta(t)\exp\Big\{\int_t^T \big(-\frac{1}{2}\sigma_\theta^2(s) + \sigma_{\theta\delta}(s)\big)\mathrm{d}s + \int_t^T \sigma_\theta(s)\mathrm{d}\hat{W}_\theta^\delta(s)\Big\}, \\ \delta(T) &= \delta(t)\exp\Big\{\int_t^T \big(\varphi_2(s) + \frac{1}{2}\sigma_\delta^2(s)\big)\mathrm{d}s + \int_t^T \sigma_\delta(s)\mathrm{d}\hat{W}_\delta^\delta(s)\Big\}, \end{aligned}$$

for all $t \in [0,T]$. Therefore, we get

$$\theta(T) > 1 \Longleftrightarrow \xi_\theta^\delta > -d_1 \quad \text{and} \quad \delta(T) < 1 \Longleftrightarrow \xi_\delta^\delta < d_2.$$

Here, under $\hat{\mathbb{P}}^\delta$, $\xi_j^\delta = \left(\int_t^T \sigma_j(s)\mathrm{d}\hat{W}_j^\delta(s)\right)/\sigma_{j,t,T}$, $j = \theta, \delta$, are standard normal distributed with the correlation coefficient $\rho_{t,T}$. Moreover, they are all independent of $\mathcal{F}_t$. Hence, E_4 can be expressed by

$$\begin{aligned} E_4 &= S_2(t)\delta(t)\exp\Big\{\int_t^T \psi_2(s)\mathrm{d}s\Big\}\hat{\mathbb{E}}^\delta\Big[\mathbf{1}_{\{\xi_\theta^\delta > -d_1\}}\mathbf{1}_{\{\xi_\delta^\delta < d_2\}}\Big] \\ &= S_2(t)\delta(t)\exp\Big\{\int_t^T \psi_2(s)\mathrm{d}s\Big\}N(d_1, d_2, -\rho). \end{aligned}$$

Combining the evaluation from the first term E_1 to the fourth term E_4, we finish the proof. □

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Received: July 1, 2008