Applied Mathematical Sciences, Vol. 2, 2008, no. 51, 2543 - 2548

Relationship between Pareto Optimality in MOILP and DEA

Sh. Razavyan [a], Gh. Tohidi [b, 1] and H. Chitsaz [b]

[a] Department of Mathematics, Islamic Azad University
Tehran-South Branch, Tehran, Iran

[b] Department of Mathematics, Islamic Azad University
Tehran-Central Branch, Tehran, Iran

Abstract

In this paper we present a relationship between efficient solution in Multi-Objective Integer Linear Programming (MOILP) and Data Envelopment Analysis (DEA). The presented paper, develops the J.W. Yougbare and J. Teghem in EJOR by development the Definitions and Theorems from 0-1 Multi-Objective Linear Programming (MOLP) to the MOILP.

Keywords: DEA; Multi-objective 0-1 linear program; Multi-objective integer linear program; BCC efficient; Pareto optimality

1 Introduction

Data Envelopment Analysis (DEA), was first put forward by Charnes, Cooper and Rhodes, [2]. It is a performance measurement technique which, can be used for evaluating the relative efficiency of decision-making units (DMUs) by using ratio of the weighted some of outputs to the weighted some of inputs.

Here, we index the different DMUs by $j \in \{1, \ldots, n\}$, a DMU is denoted by W_j (see [4]). W_j is characterized by $X_j = (x_{ij}, i = 1, \ldots, m)$ an input variables vector of dimension m an $Y_j = (y_{rj}, r = 1, \ldots, s)$ an output variables vector of dimension s. We present by W_d ($d \in \{1, \ldots, n\}$), the DMU under

[1]Corresponding author, P.O. Box 14515-459, e-mail: ghatohidi@yahoo.com

evaluation with the BCC method. Depending if the emphasis is on input reduction or output augmentation , two orientation can be considered. For evaluating relative efficiency of DMU, BCC [1] method is used which is as follows in the input and output oriented, respectively.

a) Input orientation

$$\begin{array}{rl} \min h_d = & \omega_d - \epsilon(\sum_{i=1}^{m} s_i^{(d)} + \sum_{r=1}^{s} s_r^{'(d)}) \\ \text{s.t.} & \omega_d x_{id} - \sum_{j=1}^{n} \lambda_j^{(d)} x_{ij} - s_i^{(d)} = 0, i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j^{(d)} y_{rj} - s_r^{'(d)} = y_{rd}, r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j^{(d)} = 1, \\ & s_i^{(d)}, s_r^{'(d)}, \lambda_j^{(d)} \geq 0, \forall i, r, j, \end{array} \tag{1}$$

with an arbitrary smallest positive number ϵ and ω_d is an unrestricted variable.

b) Output orientation

By using previous notations we have:

$$\begin{array}{rl} \max z_d = & \phi_d + \epsilon(\sum_{i=1}^{m} s_i^{(d)} + \sum_{r=1}^{s} s_r^{'(d)}) \\ \text{s.t.} & \phi_d y_{rd} - \sum_{j=1}^{n} \lambda_j^{(d)} y_{rj} + s_r^{'(d)} = 0, r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \lambda_j^{(d)} x_{ij} + s_i^{(d)} = x_{id}, i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j^{(d)} = 1, \\ & s_i^{(d)}, s_r^{'(d)}, \lambda_j^{(d)} \geq 0, \forall i, r, j, \end{array} \tag{2}$$

with an arbitrary smallest positive number ϵ and ϕ_d is an unrestricted variable. A Multi-Objective programming problem is defined as follows:

$$\begin{array}{rl} \max & (f_1(W), f_2(W), \ldots, f_k(W)) \\ \text{s.t.} & W \in X, \end{array} \tag{3}$$

where $f_1, f_2, \ldots, f_k$ are objective functions, $W = (w_1, w_2, \ldots, w_n)$ and X is the feasible region. If all objective functions are linear and X is a convex polyhedron, the problem (3) is called a Multi-Objective Linear Programming (MOLP) problem. There are different methods available in the literature to solve problem (3). If in the problem (3) all variables are integer and all objective functions and constraints are linear, then problem (3) is called the MOILP problem and is defined as follows:

$$\begin{array}{rl} \max & (f_1(W), f_2(W), \ldots, f_k(W)) \\ \text{s.t.} & W \in X, \\ & w_j \in \{0, 1, 2, 3, \ldots\}, j = 1, \ldots, n. \end{array} \tag{4}$$

2 Optimal solutions of the MOILP problem

Consider the following MOILP problem:

$$\begin{array}{rl} \max & \sum_{q=1}^{Q} c_{rq}w_q, r = 1, \dots, s \\ \text{s.t.} & \sum_{q=1}^{Q} a_{iq}w_q \leq b_i, i = 1, \dots, m \\ & w_q \in \{0, 1, 2, 3, \dots\}, q = 1, \dots, Q. \end{array} \tag{5}$$

We suppose that (5) has a finite number n of feasible solutions. Additional notations are introduced to describe the feasible solutions $j(j = 1, \dots, n)$:
• These solutions are denoted by $W_j = (w_{qj}, q = 1, \dots, Q)$.
• $X_j = (x_{ij} = \sum_{q=1}^{Q} a_{iq}w_q, i = 1, \dots, m)$ denotes the vector of value taken by the constraints left-hand sides when computed on W_j.
• $Y_j = (y_{ij} = \sum_{q=1}^{Q} c_{rq}w_q, r = 1, \dots, s)$ denotes the vector value of the objectives corresponding to W_j.

The following Definitions and Theorems have been stated already about 0-1 MOLP (see [4]). Now, we state and develop them about MOILP.

Definition 1. $W_d(d = 1, \dots, n)$ is a Pareto optimal solution of problem (5), if there doesn't exist any other feasible solution W_j such that:

$$Y_j \geq Y_d \& Y_j \neq Y_d. \tag{6}$$

Definition 2. A solution W_d is a Supported Pareto optimal solution of problem (5) if there does not exist any $\lambda \in \Lambda = \{\lambda \in R^n | \lambda_j \geq 0, \forall j; \sum_{j=1}^{n} \lambda_j = 1\}$ such that

$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y_d \& \sum_{j=1}^{n} \lambda_j Y_j \neq Y_d. \tag{7}$$

Definition 3. A solution W_d is an extreme supported Pareto optimal solution of problem (5) if there does not exist any $\lambda \in \Lambda \backslash \overline{\Lambda}$ such that, $\sum_{j=1}^{n} \lambda_j Y_j \geq Y_d$, where $\overline{\Lambda}$ defined such as $\overline{\Lambda} = \{\lambda | \lambda \in \{0, 1\}^n, \sum_{j=1}^{n} \lambda_j = 1\}$.

Definition 4. The non-supported Pareto optimal solutions are those which are not supported i.e. condition (6) is satisfied but not condition (7).

Definition 5. W_d is a BCC efficient (input orientation) DMU if the optimal solution of problem (1) is such that, $h_d^* = 1 (= \omega_d^*)$ and $s_d^{*(d)} = 0 \forall i, s_r^{*(d)} = 0 \forall r$.

Theorem 1. W_d is a BCC efficient, if there does not exist any $\lambda \in \Lambda$ such that:

$$\sum_{j=1}^{n} \lambda_j (Y_j, -X_j)^T \geq (Y_d, -X_d)^T \text{ and } \sum_{j=1}^{n} \lambda_j (Y_j, -X_j)^T \neq (Y_d, -X_d)^T.$$

Proof. It is clear that $\omega_d = 1; \lambda_d^{(d)} = 1; \lambda_j^{(d)} = 0, j \neq d; s_i^{(d)} = 0, \forall i; s_r^{'(d)} =$

$0, \forall r$; is a feasible solution of problem (1).So if W_d is BCC efficient there does not exist any feasible solution of problem (1) such that $\omega_d \leq 1$ and at least a slack variable, $s_i^{(d)}$ or $s_r^{\prime(d)}$, strictly positive. So, $X_d \geq \omega_d X_d \geq \sum_{j=1}^n \lambda_j X_j$ then $-X_d \leq -\sum_{j=1}^n \lambda_j X_j$ and $Y_d \leq \sum_{j=1}^n \lambda_j Y_j$. By adding two relations we have: $\sum_{j=1}^n \lambda_j (Y_j, -X_j)^T \geq (Y_d, -X_d)^T$. $\square$

Definition 6. W_d is a weakly BCC efficient if there does not exist $\lambda \in \Lambda$ such that: $\sum_{j=1}^n \lambda_j (Y_j, -X_j)^T > (Y_d, -X_d)^T$.

3 Example

Consider the following MOILP problem:

$$\begin{array}{rl} \text{Max} & 3w_1 + 9w_2 \\ \text{Max} & 4w_1 + 2w_2 \\ \text{s.t.} & 2w_1 + 3w_2 \leq 6 \\ & w_1 - 2w_2 \geq -2 \\ & w_1, w_2 \in \{0, 1, 2, \ldots\}. \end{array} \tag{8}$$

It is easy to see that there are only six feasible solutions $W_1 = (0, 0)$, $W_2 = (0, 1)$, $W_3 = (1, 0)$, $W_4 = (2, 0)$, $W_5 = (3, 0)$ and $W_6 = (1, 1)$ for problem (8) which can be represented in the objectives space by the points $Y_1 = (0, 0)$, $Y_2 = (9, 2)$, $Y_3 = (3, 4)$, $Y_4 = (6, 8)$, $Y_5 = (9, 12)$ and $Y_6 = (12, 6)$. Clearly,

- W_5 and W_6 is an extreme supported Pareto optimal solutions.
- W_1 , W_2 , W_3 and W_4 are not Pareto optimal solutions.

4 Comparison between BCC efficiency and Pareto optimality of MOILP

The following Theorems already have been stated about comparison between BCC efficient and Pareto optimality of 0-1 MOLP. So, we restate them about MOILP.

Theorem 2. If W_d is an extreme supported Pareto optimal solution of (5) and if there exists no other W_j such that $Y_j = Y_d$ and $-X_j \geq -X_d$, then $(X_d, Y_d)^T$ is a BCC efficient DMU.

Proof. To not be a BCC efficient DMU, it is necessary to find $\lambda \in \Lambda$ such that:

$\sum_{j=1}^n \lambda_j (Y_j, -X_j)^T \geq (Y_d, -X_d)^T$. Obviously,the vector λ_d with $(\lambda_j = 0, j \neq$

$d, \lambda_d = 1$) does not answer to this requirement. So it cannot be satisfied at all since,by assumption of theorem there exists no $\lambda \in \Lambda^d$ such that $\sum_{j=1}^{n} \lambda_j Y_j \geq Y_d$. □

Theorem 3. If W_d is a supported Pareto optimal solution of problem (5), and if there exist no other W_j such that $Y_j = Y_d$ and $-X_j \geq -X_d$, then $(X_d, Y_d)^T$ is a weakly BCC efficient DMU.

Proof. To not be a weakly BCC efficient DMU, it is necessary to find $\lambda \in \Lambda$ such that $\sum_{j=1}^{n} \lambda_j (Y_j, -X_j)^T > (Y_d, -X_d)^T$. Obviously, this requirement can not be satisfied at all since, by assumption of theorem, there exists no $\lambda \in \Lambda$ such that, $\sum_{j=1}^{n} \lambda_j Y_j \geq Y_d$.□

Due to Theorem 2, in case where the supported Pareto optimal solution is an extreme one, we have in addition $s_i^{*(d)} = 0$, which is not necessarily the case for a non-extreme one.

5 Conclusion

In this paper, by development the Definitions and Theorems from relationship DEA and 0-1 MOLP, we present the relationship between Pareto optimality in MOILP and DEA. The presented relationship, can be used to design an algorithm to generate the Pareto optimal solutions of a MOILP in the future.

References

[1] R.D. Banker, A. Charnes, W.W. Cooper, (1984), Some models for estimating technical and scale inefficiencies in data envelopment analysis, Management Science, 30, 1078?1092.

[2] Charnes A., W.W. Cooper, E. Rhodes, (1978), Measuring the efficiency of decision making units, European Journal of Operational Research, Operations Research, 2, 429-444.

[3] F. H. F. Liu, C.C. Huang and Y.L. Yen, (2000), Using DEA to Obtain Efficient Solutions for Multi-Objective 0-1 Linear Programs: European Journal of Operationl Research, 126, 51-68.

[4] J.W. Yougbare and J. Teghem, (2007), Ralationships between Pateto optimality in Multi-Objective 0-1 Linear Programming and DEA efficiency, European Journal of Operationl Research.

Received: April 14, 2008