Applied Mathematical Sciences, Vol. 2, 2008, no. 50, 2459 - 2471

Stability of Limit Cycle in a Delayed IS-LM Business Cycle model

A. Abta, A. Kaddar and H. Talibi Alaoui

Université Chouaib Doukkali Faculté des Sciences
Département de Mathématiques et Informatique
B.P:20, El Jadida, Morocco
a.kaddar@yahoo.fr, talibi_1@hotmail.fr

Abstract

In this paper, we extend the work of J. Cai [1] in the nonlinear case using the investment function of the Kaldor-type. Actually we investigate the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions using the methods presented by O. Diekmann et al. in [3]. In the end we show numerical application.

Mathematics Subject Classification: 37G15, 91B62

Keywords: IS-LM business cycle, delayed differential equations, Hopf bifurcation, periodic solutions

1 Introduction and mathematical models

Kaldor was probably the first economist to realize the importance of the nonlinear mechanism of the economy and to introduce his first model (in 1940, [7]) by an ordinary differential equations as follows

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \frac{dK}{dt} = I(Y(t), K(t)), \end{cases}$$

where Y is the gross product, K is the capital stock, α is the adjustment coefficient in the goods market, $I(Y, K)$ is the investment function and $S(Y, K)$ is the saving function. In this model the nonlinearity of investment and saving function leads to limit cycle solution (see also [2, 5, 13] for more information). In (1977, [12]) Torre revised and updated this model by replacing the capital stock $K(t)$ with the interest rate $R(t)$ to formulate the following standard IS-LM business cycle model

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), R(t)) - S(Y(t), R(t))], \\ \frac{dR}{dt} = \beta[L(Y(t), R(t)) - \widetilde{M}], \end{cases}$$

where $\widetilde{M}$ is the constant money supply, β is the adjustment coefficient in money market and L is the demand for money.
In (1989, [4]), Gabisch and Lorenz considered an augmented IS-LM business cycle model as follows

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), K(t), R(t)) - S(Y(t), R(t))], \\ \frac{dK}{dt} = I(Y(t), K(t), R(t)) - \delta K(t), \\ \frac{dR}{dt} = \beta[L(Y(t), R(t)) - \widetilde{M}], \end{cases}$$

where δ is the depreciation rate of capital stock.
Based on the Kalecki's idea of time delay (see [8, 10, 11] for more information), Cai (in 2005, [1]) presented the following delayed IS-LM model:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), K(t), R(t)) - S(Y(t), R(t))], \\ \frac{dK}{dt} = I(Y(t-\tau), K(t), R(t)) - \delta K(t), \\ \frac{dR}{dt} = \beta[L(Y(t), R(t)) - \widetilde{M}], \end{cases} \tag{1}$$

with τ is the time delay needed for new capital to be installed, and he investigated the local stability and the local Hopf bifurcation for (1) in the linear case (the functions I, S, and L are linear).
In the following analysis we will consider the nonlinear case by using the investment function of the Kaldor-type. The dynamics are studied in terms of local stability and of the description of the Hopf bifurcation, that is proven to exists as the delay (taken as a parameter of bifurcation) cross some critical value. Additionally we establish an explicit algorithm for determining the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions using the methods presented by O. Diekmann et al. in [3]. In the end, we give some numerical simulations which show the existence and the nature of the periodic solutions.

2 Steady state and local stability analysis

As in [9], we assume that the investment function I, is given by

$$I(Y, K, R) = I(Y) - \delta_1 K - \beta_1 R.$$

The saving function S, and the demand for money L are given by

$$S(Y, R) = l_1 Y + \beta_2 R,$$

and

$$L(Y,R) = l_2 Y - \beta_3 R,$$

with $\delta_1, l_1, l_2, \beta_1, \beta_2, \beta_3$ are positive constants. Then system (1) becomes:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t)) - \delta_1 K - (\beta_1+\beta_2)R(t)) - l_1 Y], \\ \frac{dK}{dt} = I(Y(t-\tau)) - (\delta+\delta_1)K(t)) - \beta_1 R(t), \\ \frac{dR}{dt} = \beta[l_2 Y(t) - \beta_3 R(t) - \widetilde{M}]. \end{cases} \tag{2}$$

2.1 Steady state

In the following proposition, we give a sufficient conditions for the existence and uniqueness of positive equilibrium E^* of the system (2).

Proposition 2.1 *Suppose that*
(i): There exists a constant $L > 0$ such that $|I(Y)| \leq L$ for all $Y \in \boldsymbol{R}$.
(ii): $I'(Y) - \rho < 0$ for all $Y \in \boldsymbol{R}$,
where ρ is given by

$$\rho = \frac{((\delta+\delta_1)\beta_2 + \delta\beta_1)l_2}{\delta\beta_3} + \frac{(\delta+\delta_1)l_1}{\delta}. \tag{3}$$

Then there exists a unique equilibrium $E^* = (Y^*, K^*, R^*)$ of system (2), where Y^* is the positive solution of

$$I(Y) - \rho Y + \frac{[(\delta+\delta_1)\beta_2 + \delta\beta_1]\widetilde{M}}{\delta\beta_3} = 0 \tag{4}$$

and K^*, R^* are given by

$$K^* = \frac{\beta_2 l_2 + l_1\beta_3}{\delta\beta_3} Y^* - \frac{\beta_2}{\delta\beta_3}\widetilde{M}, \tag{5}$$

$$R^* = \frac{l_2 Y^* - \widetilde{M}}{\beta_3}. \tag{6}$$

proof. (Y, K, R) is a steady-state of (2) if

$$\frac{dY}{dt} = \frac{dK}{dt} = \frac{dR}{dt} = 0,$$

that is

$$\begin{cases} I(Y) - \delta_1 K - (\beta_1+\beta_2)R - l_1 Y = 0, \\ I(Y) - (\delta+\delta_1)K - \beta_1 R = 0, \\ l_2 Y - \beta_3 R - \widetilde{M} = 0, \end{cases} \tag{7}$$

Let us assume that $Y > 0$, $K > 0$, and $R > 0$ satisfy (5). Then

$$K = \frac{\beta_2 l_2 + l_1 \beta_3}{\delta\beta_3} Y - \frac{\beta_2}{\delta\beta_3}\widetilde{M}, \tag{8}$$

$$R = \frac{l_2 Y - \widetilde{M}}{\beta_3}, \tag{9}$$

and

$$I(Y) - \rho Y + \frac{[(\delta + \delta_1)\beta_2 + \delta\beta_1]\widetilde{M}}{\delta\beta_3} = 0 \tag{10}$$

where ρ is defined in (3).
In view of hypotheses i) and ii) of proposition 2.1 it's clear that equation (10) has a unique solution $Y^* > 0$. This concludes the proof.

2.2 Local stability analysis of E^*

We will study the stability of the positive equilibrium E^*. Firstly we recall that E^* is asymptotically stable if all roots of the characteristic equation associated to the linearized system of (2) have negative real parts, and the stability is lost only if characteristic roots cross the imaginary axis, that is if purely imaginary roots appear. Let $y = Y - Y^*$, $k = K - K^*$ and $r = R - R^*$. By linearizing system (2) around $E^* = (Y^*, K^*, R^*)$, we obtain

$$\begin{cases} \frac{dy}{dt} = \alpha(I'(Y^*) - l_1)y(t) - \alpha\delta_1 k(t) - \alpha(\beta_1 + \beta_2)r(t), \\ \frac{dk}{dt} = I'(Y^*)y(t-\tau) - (\delta + \delta_1)k(t) - \beta_1 r(t), \\ \frac{dr}{dt} = \beta l_2 y(t) - \beta\beta_3 r(t), \end{cases} \tag{11}$$

The characteristic equation associated to system (11) takes the general forme

$$P(\lambda) + Q(\lambda)exp(-\lambda\tau) = 0 \tag{12}$$

with

$$P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C$$

and

$$Q(\lambda) = D\lambda + E,$$

where

$$A = \delta + \delta_1 + \beta\beta_3 - \alpha(I'(Y^*) - l_1),$$

$$B = (\delta + \delta_1)[\beta\beta_3 - \alpha(I'(Y^*) - l_1)] + \alpha\beta[(\beta_1 + \beta_2)l_2 - \beta_3(I'(Y^*) - l_1)],$$

$$C = \alpha\beta[\beta_1\delta l_2 + \beta_2(\delta + \delta_1)l_2 - \beta_3(\delta + \delta_1)(I'(Y^*) - l_1)],$$

$$D = \alpha\delta_1 I'(Y^*),$$

and

$$E = \alpha\beta\beta_3\delta_1 I'(Y^*).$$

Recall that the equilibrium of (2) is asymptotically stable if all roots of (12) have negative real parts, and the stability is lost only if characteristic roots cross the imaginary axis, that is if pure imaginary roots appear. In order to investigate the local stability of the steady state, we begin by considering the case without delay $\tau = 0$. This case is of importance, because it can be necessary that the nontrivial positive equilibrium of (2) is stable when $\tau = 0$ to be able to obtain the local stability for all nonnegative values of the delay, or to find a critical values which could destabilize the equilibrium.
When $\tau = 0$ the characteristic equation (9) reads as

$$\lambda^3 + A\lambda^2 + (B + D)\lambda + (C + E) = 0, \tag{13}$$

hence, according to the Routh-Hurwitz criterion, we have the following lemma.

Lemma 2.1 *For $\tau = 0$, the equilibrium E^* is locally asymptotically stable if and only if*
(H1): $A > 0$;
(H2): $C + E > 0$;
(H3): $A(B + D) - (C + E) > 0$;
where A, B, C, D, E, are defined in (12).

We assume in the sequel, that hypotheses (H1), (H2), (H3) are true, and we return to the study of equation (12) with $\tau > 0$.
Let $\lambda = i\omega$, $\omega \in \mathbf{R}$, and rewrite (12) in terms of its real and imaginary parts as

$$A\omega^2 - C = E\cos(\omega\tau) + D\omega\sin(\omega\tau) \tag{14}$$

$$\omega^3 - B\omega = D\omega\cos(\omega\tau) - E\sin(\omega\tau) \tag{15}$$

It follows by taking the sum of squares that

$$\omega^6 + a\omega^4 + b\omega^2 + c = 0, \tag{16}$$

with $a = A^2 - 2B$, $b = B^2 - 2AC - D^2$, $c = C^2 - E^2$,
where $A; B; C; D; E$ are given in (12).
Then equation (16) becomes

$$h(z) := z^3 + az^2 + bz + c = 0, \tag{17}$$

with $z = \omega^2$.
Suppose that equation (17) has simple positive roots. Without loss of generality, assume that it has three positive roots, denoted by z_1, z_2 and z_3, respectively. Then equation (13) has three positive roots, say

$$\omega_1 = \sqrt{z_1}; \omega_2 = \sqrt{z_2}; \omega_3 = \sqrt{z_3}$$

Let

$$\tau_l = \frac{1}{\omega_l}[\arccos(\frac{(A\omega_l^2 - C)(F - D\omega_l^2) + (\omega_l^3 - B\omega_l)E\omega_l}{(D\omega_l - F)^2 + E^2\omega_l^2})], l = 1, 2, 3.$$

Then $\pm i\omega_l$ is a pair of purely imaginary roots of equation (13) with $\tau = \tau_l$, l=1,2,3.
Define

$$\tau_0 = \tau_{l_0} = \min_{l=1,2,3}(\tau_l), \omega_0 = \omega_{l_0}. \tag{18}$$

We have the following theorem.

Theorem 2.2 *Assume that (H1), (H2) and (H3) hold (see lemma 2.1). If one of the following hypotheses is true:*
(H4) $c < 0$ *and* $h'(\omega_0) \neq 0$; *(H5)* $a < 0$, $b \geq 0$, $c > 0$, $a^2 > 3b$, *and* $\Delta < 0$;
(H6) $b < 0$, $c > 0$, *and* $\Delta < 0$;
where Δ *is defined by*

$$\Delta = \frac{4}{27}b^3 - \frac{1}{27}a^2b^2 + \frac{4}{27}a^3c - \frac{2}{3}abc + c^2. \tag{19}$$

and ω_0 *is defined by (18).*
Then $\tau_0 > 0$ *and when* $\tau \in [0, \tau_0)$ *the steady state* E^* *is locally asymptotically stable, and when* $\tau = \tau_0$, *system (2) will undergo a Hopf bifurcation.*
Moreover, we have

$$Re(\frac{d\lambda}{d\tau})(\tau_0) > 0.$$

Proof. Similar to the proof in [1].

3 Direction of Hopf Bifurcation

In the last section we have a Hopf bifurcation at a critical value τ_0 of the delay. Thus (see [6]), there exists $\epsilon_0 > 0$ such that for each $0 \leq \epsilon < \epsilon_0$, system (2) has a family of periodic solutions $p(\epsilon)$ with period $T_0 = T_0(\epsilon)$, for the parameter values $\tau = \tau(\epsilon)$ such that $p(0) = 0$, and $T_0(0) = \frac{2\pi}{\omega_0}$ and $\tau_0 = \tau(0)$, where $\tau_0 = \frac{1}{\omega_0}\arccos\frac{(A\omega_0^2 - C) + (\omega_0^3 - B\omega_0)D\omega_0}{E^2\omega_0^2}$, with $A; B; C; D; E$ are given in (12) and ω_0 is the least simple positive root of (16).
In this section we use a formula on the direction of the Hopf bifurcation given by Diekmann in [3] to formulate an explicit algorithm about the direction and the stability of the bifurcating branch of periodic solutions of (4).
Normalizing the delay τ by scaling $t \to \frac{t}{\tau}$ and effecting the change $U(t) = Y(\tau t)$, $V(t) = K(\tau t)$, and $W(t) = R(\tau t)$, the system (4) is transformed into

$$\begin{cases} \frac{dU}{dt} = \alpha\tau[I(U(t)) - \delta_1 V(t) - (\beta_1 + \beta_2)W(t) - l_1 U(t)], \\ \frac{dV}{dt} = \tau(I(U(t-1)) - (\delta + \delta_1)V(t)) - \beta_1 W(t), \\ \frac{dW}{dt} = \beta\tau(l_2 U(t) - \beta_3 W(t) - \widetilde{M}). \end{cases} \tag{20}$$

By the translation $Z(t) = (U, V, W) - (Y^*, K^*, R^*)$, system (20) is written as a functional differential equation in $C := C([-1, 0], \mathbf{R}^3)$,

$$\dot{Z(t)} = L(\tau)Z_t + h(Z_t, \tau), \tag{21}$$

where $L(\tau) : C \to \mathbf{R}^3$ the linear operator and h: $C \times \mathbf{R} \to \mathbf{R}^3$ the nonlinear part of (21) are given respectively by:

$$L(\tau)\varphi = \tau \begin{pmatrix} \alpha[I'(Y^*) - l_1)\varphi_1(0) - \delta_1\varphi_2(0) - (\beta_1 + \beta_2)\varphi_3(0)] \\ I'(Y^*)\varphi_1(-1) - (\delta + \delta_1)\varphi_2(0) - \beta_1\varphi_3(0) \\ \beta[l_2\varphi_1(0) - \beta_3\varphi_3(0)] \end{pmatrix}$$

$$h(\varphi, \tau) = \tau \begin{pmatrix} \alpha[I(\varphi_1(0) + Y^*) - I'(Y^*)\varphi_1(0) - \delta_1 K^* - (\beta_1 + \beta_2)R^* - l_1 Y^*)] \\ I(\varphi_1(-1) + Y^*) - I'(Y^*)\varphi_1(0) - (\delta + \delta_1)K^* - \beta_1 R^* \\ \beta l_2 Y^* - \beta\beta_3 R^* - \beta\widetilde{M}) \end{pmatrix}$$

Let

$$L := L(\tau_0) : C \longrightarrow \mathbf{R}^3.$$

Using the Riesz representation theorem (see [6]), we obtain

$$L\varphi = \int_{-1}^{0} d\eta(\theta)\varphi(\theta) \tag{22}$$

where,

$$d\eta(\theta) = \tau_0 \begin{pmatrix} \alpha(I'(Y^*) - l_1)\delta(\theta) & -\alpha\delta_1\delta(\theta) & -\alpha(\beta_1 + \beta_2)\delta(\theta) \\ I'(Y^*)\delta(\theta + 1) & -(\delta + \delta_1)\delta(\theta) & -\beta_1\delta(\theta) \\ \beta l_2\delta(\theta) & 0 & -\beta\beta_3\delta(\theta) \end{pmatrix} \tag{23}$$

$\delta(.)$ denotes the Dirac function.
Let $A(\tau)$ denotes the generator of semigroup generated by the linear part of (21) and $A = A(\tau_0)$.
Then,

$$A\varphi(\theta) = \begin{cases} \frac{d\varphi}{d\theta}(\theta) \text{ for } \theta \in [-1, 0) \\ L\varphi \text{ for } \theta = 0 \end{cases} \tag{24}$$

for $\varphi = (\varphi_1, \varphi_2) \in C$.
From Theorem 2.1, a Hopf bifurcation occurs at the critical value $\tau = \tau_0$. By the Taylor expansion of the time delay function $\tau(\varepsilon)$ near the critical value τ_0, we have

$$\tau(\varepsilon) = \tau_0 + \tau_2\varepsilon^2 + o(\varepsilon^2). \tag{25}$$

The sign of τ_2 determines either the bifurcation is supercritical (if $\tau_2 > 0$) and periodic orbits exist for $\tau > \tau_0$, or it is subcritical (if $\tau_2 < 0$) and periodic orbits exist for $\tau < \tau_0$. The term τ_2 may be calculated (see [3]) using the formula,

$$\tau_2 = \frac{Re(c)}{Re(qD_2M_0(i\omega_0, \tau_0)p)}, \tag{26}$$

where M_0 is the characteristic matrix of the linear part of (21),

$$M_0(\lambda, \tau) = \begin{pmatrix} \lambda - \tau\alpha(I'(Y^*) - l_1) & \tau\alpha\delta_1 & \tau\alpha(\beta_1 + \beta_2) \\ -\tau I'(Y^*)\exp(-\lambda) & \lambda + \tau(\delta + \delta_1) & \tau\beta_1 \\ -\tau\beta l_2 & 0 & \lambda + \tau\beta\beta_3 \end{pmatrix}, \tag{27}$$

$D_2M_0(i\omega_0, \tau_0)$ denotes the derivative of M_0 with respect to τ at $\tau = \tau_0$, the constant c is defined as follows

$$c = \frac{1}{2}qD_1^3h(0, \tau_0)(P^2(\theta), \overline{P}(\theta))$$

$$+qD_1^2h(0, \tau_0)(e^{0\cdot}M_0^{-1}(0, \tau_0)D_1^2h(0, \tau_0)(P(\theta), \overline{P}(\theta)), P(\theta))$$

$$+\frac{1}{2}qD_1^2h(0, \tau_0)(e^{2i\omega_0\cdot}M_0^{-1}(2i\omega_0, \tau_0)D_1^2h(0, \tau_0)(P(\theta), P(\theta)), \overline{P}(\theta))$$

where $D_1^ih, i = 2, 3$, denotes the $i - th$ derivative of h with respect to φ, $P(\theta)$ denotes the eigenvector of A, $\overline{P}(\theta)$ denotes it conjugate eigenvector and p, q are defined later.

To study the direction of Hopf bifurcation, one needs to calculate the second and third derivatives of nonlinear part of (21)with respect to φ,

$$D_1^2h(\varphi, \tau)\psi\chi = \tau \begin{pmatrix} \alpha I''(\varphi_1(0) + Y^*)\psi_1(0)\chi_1(0) \\ I''(\varphi_1(-1) + Y^*)\psi_1(-1)\chi_1(-1) \\ 0 \end{pmatrix} \tag{28}$$

and

$$D_1^3h(\varphi, \tau)\psi\chi\upsilon = \tau \begin{pmatrix} \alpha I'''(\varphi_1(0) + Y^*)\psi_1(0)\chi_1(0)\upsilon_1(0) \\ I'''(\varphi_1(-1) + Y^*)\psi_1(-1)\chi_1(-1)\upsilon_1(-1) \\ 0 \end{pmatrix} \tag{29}$$

Then

$$D_1^2h(0, \tau_0)\psi\chi = \left[\tau_0\alpha I''(Y^*)\psi_1(0)\chi_1(0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \tau_0 I''(Y^*)\psi_1(-1)\chi_1(-1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \tag{30}$$

and

$$D_1^3h_0(0, \tau_0)\psi\chi\upsilon = \left[\tau_0\alpha I'''(Y^*)\psi_1(0)\chi_1(0)\upsilon_1(0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \tau_0 I'''(Y^*)\psi_1(-1)\chi_1(-1)\upsilon_1(-1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]. \tag{31}$$

$\psi = (\psi_1, \psi_2, \psi_3), \chi = (\chi_1, \chi_2, \chi_3), \upsilon = (\upsilon_1, \upsilon_2, \upsilon_3) \in C([-1, 0], \mathbf{R}^3)$.

As $i\omega_0$ is a solution of (12) at $\tau = \tau_0$, then $i\omega_0$ is an eigenvalue of A and there exist a corresponding eigenvector of the form $P(\theta) = pe^{i\omega_0\theta}$ where $p = (p_1, p_2, p_3) \in \mathbf{C}^2$, satisfy the equations:

$$Mp = 0$$

with

$$M = M_0(i\omega_0, \tau_0). \tag{32}$$

Then one may assume

$$p_1 = 1,$$

and calculate

$$p_2 = \frac{-1}{\tau_0\alpha\delta_1}\Big[\frac{\tau_0^2\alpha\beta l_2(\beta_1 + \beta_2)(-i\omega_0 + \tau_0\beta\beta_3)}{\omega_0^2 + \tau_0^2\beta^2\beta_3^2} + i\omega_0 - \tau_0\alpha(I'(Y^*) - l_1)\Big],$$

and

$$p_3 = \frac{\tau_0\beta l_2(-i\omega_0 + \tau_0\beta\beta_3))}{\omega_0^2 + \tau_0^2\beta^2\beta_3^2}.$$

So, from (30) and (31), we have

$$D_1^2 h(0, \tau_0)(P(\theta), \overline{P}(\theta)) = \tau_0 I''(Y^*)\begin{pmatrix}\alpha\\ 1\\ 0\end{pmatrix} \tag{33}$$

$$D_1^2 h(0, \tau_0)(P(\theta), P(\theta)) = \tau_0 I''(Y^*)\begin{pmatrix}\alpha\\ \exp(-2i\omega_0)\\ 0\end{pmatrix} \tag{34}$$

and

$$D_1^3 h(0, \tau_0)(P^2(\theta), \overline{P}(\theta)) = \tau_0 I'''(Y^*)\begin{pmatrix}\alpha\\ \exp(-i\omega_0)\\ 0\end{pmatrix}. \tag{35}$$

Now, consider A^*, a conjugate operator of A, $A^* : C([0, 1], \mathbf{R}^2) \longrightarrow \mathbf{R}^2$, defined by,

$$A^*\psi(s) = \begin{cases} -\frac{d\psi}{ds}(s), \text{ for } s \in (0, 1] \\ -\int_{-1}^0 \psi(-s)d\eta(s), \text{ for } s = 0 \end{cases} \tag{36}$$

$\psi = (\psi_1, \psi_2) \in C([0, 1], \mathbf{R}^2)$.

Let $Q(s) = qe^{i\omega_0 s}$ be the eigenvector for A^* associated to the eigenvalue $i\omega_0$, $q = (q_1, q_2, q_3)^T$. One needs to choose q such that the inner product (see [6]),

$$< Q, P >= 1,$$

where

$$< Q, P >= Q(0)\overline{P}(0) - \int_{-1}^{0}\int_{0}^{\theta} Q(\xi-\theta)d\eta(\theta)\overline{P}(\xi)d\xi.$$

If we take $q_2 = 0$, and $q_3 = 0$, then

$$q_1 = 1$$

and from (35), we have

$$\frac{1}{2}qD_1^3h(0,\tau_0)(P^2(\theta),\overline{P}(\theta)) = \frac{\alpha\tau_0 I'''(Y^*)}{2} \tag{37}$$

From (27), (30),(33), we deduce,

$$qD_1^2h(0,\tau_0)(e^{0\cdot}M_0^{-1}(0,\tau_0)D_1^2h(0,\tau_0)(P(\theta),\overline{P}(\theta)),P(\theta)) = \frac{\tau_0^4\alpha^2\beta\beta_3\delta I''(Y^*)^2}{\det M_0(0,\tau_0)}, \tag{38}$$

where

$$\det M_0(0,\tau_0) = \tau_0^3\alpha\beta[-\beta_3(I'(Y^*)-l_1)(\delta+\delta_1)+\beta_3\delta_1 I'(Y^*)-\beta_1\delta_1 l_2+l_2(\beta_1+\beta_2)(\delta+\delta_1)]$$

and

$$\frac{1}{2}qD_1^2h(0,\tau_0)(e^{2i\omega_0\cdot}M_0^{-1}(2i\omega_0,\tau_0)D_1^2h(0,\tau_0)(P(\theta),P(\theta)),\overline{P}(\theta)) =$$

$$\frac{\alpha^2\tau_0^2 I''(Y^*)^2}{2(C_1^2+C_2^2)}[(C_1C_3+C_2C_4)+i(C_1C_4-C_2C_3)], \tag{39}$$

where

$$C_1 = \tau_0\alpha(I'(Y^*)-l_1)(4\omega_0^2-\tau_0^2(\delta+\delta_1)\beta\beta_3) - 4\omega_0^2\tau_0(\beta\beta_3+\delta+\delta_1)$$

$$+\tau_0^2\alpha\delta_1 I'(Y^*)[\tau_0\beta\beta_3\cos(2\omega_0)+2\omega_0\sin(2\omega_0)]+\tau_0^3\alpha\beta(\beta_1+\beta_2)l_2(\delta+\delta_1)-\tau_0^3\beta l_2\alpha\delta_1\beta_1,$$

$$C_2 = 2\omega_0(\tau_0^2\beta\beta_3(\delta+\delta_1)-4\omega_0^2) - 2\omega_0\tau_0^2\alpha(\beta\beta_3+\delta+\delta_1)(I'(Y^*)-l_1)$$

$$+\tau_0^2\alpha\delta_1 I'(Y^*)[2\omega_0\cos(2\omega_0)-\tau_0\beta\beta_3\sin(2\omega_0)] + 2\tau_0^2\alpha\beta(\beta_1+\beta_2)l_2\omega_0,$$

$$C_3 = -4\omega_0^2+\tau_0^2\beta\beta_3(\delta+\delta_1)-\tau_0^2\beta\beta_3\delta_1\cos(2\omega_0)-2\tau_0\delta_1\omega_0\sin(2\omega_0),$$

$$C_4 = 2\omega_0\tau_0(\beta\beta_3+\delta+\delta_1)-2\tau_0\delta_1\omega_0\cos(2\omega_0)+\tau_0^2\delta_1\beta\beta_3\sin(2\omega_0).$$

Then

$$Re(c) = \frac{\alpha\tau_0 I'''(Y^*)}{2} + \frac{\tau_0^2\alpha^2 I''(Y^*)^2}{2(C_1^2+C_2^2)}(C_1C_3+C_2C_4) + \frac{\alpha^2\tau_0^4\beta\beta_3\delta I''(Y^*)^2}{\det M_0(0,\tau_0)}. \quad (40)$$

Now, from (27) we have

$$Re(qD_2M_0(i\omega_0,\tau_0)p) = Re(\frac{d\lambda}{d\tau})(\tau_0),$$

and from theorem 2.1, we have

$$Re(\frac{d\lambda}{d\tau})(\tau_0) > 0.$$

Consequently we deduce the following result :

Theorem 3.1 *Assume* $(H1)$, $(H2)$, *and* $(H3)$. *If one of the previous conditions* $(H4)$, *or* $(H5)$ *or* $(H6)$ *(see theorem 2.2)holds, then,*
(a) the Hopf bifurcation occurs as τ *crosses* τ_0 *to the right (supercritical Hopf bifurcation) if* $Re(c) > 0$ *and to the left (subcritical Hopf bifurcation) if* $Re(c) < 0$*; and*
(b) the bifurcating periodic solutions is stable if $Re(c) > 0$ *and unstable if* $Re(c) < 0$; *where* $Re(c)$ *is given by (40).*

Note that, Theorem 3.1 provides an explicit algorithm for detecting the direction and stability for the Hopf bifurcated branch of periodic solution given by theorem 2.1.

4 Application

From section 3, we can determine the direction of a Hopf bifurcation and the stability of the bifurcating periodic solutions through the formula (40). In this section, we give a numerical simulation supporting the theoretical analysis.
Consider the following Kaldor-type investment function:

$$I(Y) = \frac{\exp(Y)}{1+\exp(Y)}.$$

When

$$\alpha = 3; \beta = 2; \delta = 0.1; \delta_1 = 0.5; \widetilde{M} = 0.05$$
$$l_1 = 0.2; l_2 = 0.1; \beta_1 = \beta_2 = \beta_3 = 0.2,$$

the system (2) has the positive equilibrium $E^* = (0.5134, 1.0404, 0.0067)$. It follows from section 3, that $\tau_0 = 1.797542549$ and $Re(c) = -0.3505703676$. Thus from theorem 2.1 we know that when $0 \leq \tau < \tau_0$, E^* is asymptotically stable, this property is depicted in the numerical simulation by Fig.1. When τ passes through the critical value τ_0, E^* loses its stability and a family of unstable periodic solutions bifurcating from E^* occurs (see Fig.2).

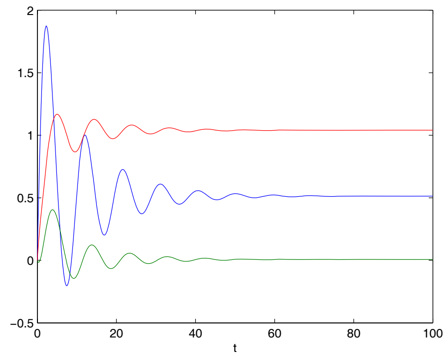

Figure 1: The steady state E^* of (2) is stable when $\tau = 1$.

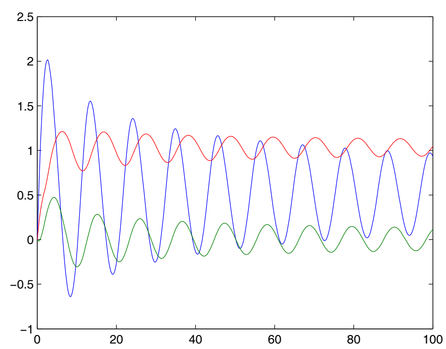

Figure 2: A family of periodic solution bifurcating from E^* occurs when $\tau_0 = 1.7975$.

References

[1] Cai J.P.; Hopf bifurcation in the IS-LM business cycle model with time delay, Electronic Journal of Differential Equations, 2005(15):1-6.

[2] Chang W.W. and Smyth D.J.; The existence and persistence of cycles in a nonlinear model: Kaldor's 1940 model re-examined, Review of Economic Studies, 1971 Vol. 38, 37-44.

[3] Diekmann O., Van Giles S., Verduyn Lunel S. and Walter H., Delay equations. Springer-Verlag, New-York (1995).

[4] Gabisch G. and Lorenz H.W. (1987) Business Cycle Theory: A survey of methods and concepts. 1989 edition Berlin:Springer-Verlag.

[5] Grasman J. Wentzel J.J.; Co-Existence of a limit cycle and an equilibrium in Kaldor's business cycle model and it's consequences. Journal of Economic Behavior and Organisation. 24. 1994. P. 369-377.

[6] Hale J. K. and Verduyn Lunel S M; Introduction to Functional Differential Equations, Springer- Verlag, New York, 1993.

[7] Kaldor N.; A Model of the Trade Cycle, Economic Journal, 1940, V.50, 78-92.

[8] Kalecki M.; A Macrodynamic Theory of Business Cycles, Econometrica, 1935, V.3, 327-344.

[9] Krawiec A. and SzydBlowski M.; The Kaldor-Kalecki Business Cycle Model, Ann. of Operat. Research, 1999, V.89, 89-100.

[10] Krawiec A. and Szydlowski M.; The Hopf Bifurcation in the Kaldor-Kalecki Model, in Computation in Economics, Finance and Engineering: Economic Systems, Editors S Holly and S Greenblatt, Elsevier, 2000 (forthcoming).

[11] Krawiec A. and Szydlowski M.; The stability problem in the Kaldor-Kalecki business cycle model, Chaos,Solitons and Fractals, 2005(25):299-305.

[12] Torre V.; Existence of limit cycles and control in complete Kynesian systems by theory of bifurcations, Econometrica, 45(1977), 1457-1466.

[13] Varian HR.; Catastrophe theory and the business cycle. Econ Inamiry 1979, 17: 14-28.

Received: March 19, 2008