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Common Fixed Points for Noncommuting Generalized (f,g)-Nonexpansive Maps

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Abstract. The results presented in this paper extend and complement the corresponding ones of Hussain-Jungck[Common fixed point and invariant approximation results for noncommuting generalized (f,g)-nonexpansive maps, J. Math. Anal. Appl. 321(2006), 851-861], and drop some restricts such as the continuity of f, g and demiclsedness of $f - T$.

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Let M be a nonempty subset of a normed space E, and f and g and T three selfmaps of M, and $F(T)$ the set of fixed points of T ($F(T) = \{x \in M; x = Tx\}$). We shall denote the closed ball of M by $\overline{M}$ and all positive integer by $\mathbb{N}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$(respectively, $x_n \rightharpoonup x$)will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x.

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The set M is called $q-$*starshaped* with $q \in M$ if $kx + (1-k)q \in M$ for all $x \in M$ and all $k \in [0,1]$; *convex* if $kx + (1-k)y \in M$ for all $x, y \in M$ and all $k \in [0,1]$. The selfmap f on M is called *affine* if M is convex and $f(kx + (1-k)y) = kfx + (1-k)fy$ for all $x, y \in M$ and all $k \in [0,1]$; The selfmap T on M is called $(f,g)-$*contraction* if, there exists $0 \le k < 1$ such that $d(Tx, Ty) \le kd(fx, gy)$ for any $x, y \in M$. If $k = 1$, then T is called $(f,g)-$*nonexpansive*. If $f = g$, then T is called $f-$contraction (or $f-$nonexpansive). If $f = g = I$, an identity operator, then T is called contraction (or nonexpansive). A mapping $T : M \to M$ is called *demiclosed* at 0 if for every sequence $\{x_n\} \subset M$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to 0; we have $Tx = 0$.

The map pair (f,T) is called *commuting* if $Tfx = fTx$ for all $x \in M$; *R-weakly commuting* if for all $x \in M$ there exists $R > 0$ such that $d(fTx, Tfx) \le Rd(fx, Tx)$. If $R = 1$, then the map pair are called weakly commuting; *compatible* [3] if $\lim_{n\to\infty} d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = t$ for some t in M; *weakly compatible* [1] if they commute at their coincidence points, i.e.,if $fTx = Tfx$ whenever $fx = Tx$. Suppose that E is compact metric space and both T and f are continuous, then (f,T) compatible equivalent to (f,T) weakly compatible [3, Theorem2.2,Corollary 2.3].

Suppose that M is $q-$starshaped with $q \in F(f)$ and is both $T-$ and $f-$invariant. Then (T,f) are called *R-subweakly commuting* on M (see [1, 6, 7]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx, Tfx) \le R\delta(fx, [Tx, q])$, where $[Tx, q] = \{kTx + (1-k)q; x \in M, k \in (0,1]\}$ and $\delta(p, M) = \inf_{z\in M} d(z, p)$ for $p \in E$; *R-subcommuting* on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx, Tfx) \le \frac{R}{k} d(kTx + (1-k)q, fx)$ for all $k \in (0,1]$. R-subcommuting and R-subweakly commuting maps are weakly compatible but the converse does not hold in general [5, 6, 7].

Recently, Hussain and Jungck[2, Theorem 2.2] proved the following theorem.

Theorem HJ *Let M be a nonempty $q-$starshaped subset of a normed space X and T, f, and g be self-maps of M. Suppose that f and g are affine and continuous with $q \in F(f) \cap F(g)$, and $T(M) \subset f(M) \cap g(M)$. If the pairs (T,f) and (T,g) are R-subweakly commuting and satisfy, for all $x, y \in M$,*

$$\begin{aligned} \|Tx - Ty\| \le \max\{&\|fx - gy\|, d(fx, [Tx, q]), d(gy, [Ty, q]), \\ &\frac{1}{2}[d(fx, [Ty, q]) + d(gy, [Tx, q])]\}, \end{aligned} \tag{1}$$

then $F(T) \cap F(f) \cap F(g) \neq \emptyset$ *provided one of the following conditions holds:*
(i) M *is complete,* $\overline{T(M)}$ *is compact and* T *is continuous;*
(ii) M *is weakly compact,* $(f - T)$ *is demiclosed at 0 and* X *is complete.*

A mapping $T : M \to M$ is called *continuous* if for all $\{x_n\} \subset M$ such that $\{x_n\} \subset M$ converges to x implies that $\{Tx_n\}$ converges strongly to Tx; *strongly continuous* if for all $\{x_n\} \subset M$ such that $\{x_n\} \subset M$ converges weakly to x implies that $\{Tx_n\}$ converges strongly to Tx; *weakly continuous* if for all $\{x_n\} \subset M$ such that $\{x_n\} \subset M$ converges weakly to x implies that $\{Tx_n\}$ converges weakly to Tx[8]. Clearly the strong continuity of T implies both continuity and weakly continuity of T but not conversely. The continuity of T doesn't imply weakly continuity, but also weakly continuity of T doesn't imply weakly continuity.

The aim of this paper drops the restricts of the continuity for f, g and demiclsedness for $f - T$ in Theorem HJ, and so extends and complements the other corresponding ones.

Now, we show our main result.

Theorem 1 *Let* M *be a nonempty* $q-$*starshaped subset of a normed space* X *and* $T, f,$ *and* g *be self-maps of* M*. Suppose that* f *and* g *are affine with* $q \in F(f) \cap F(g)$*, and* $\overline{T(M)} \subset f(M) \cap g(M)$*. If the pairs* (T, f) *and* (T, g) *are R-subweakly commuting and satisfy Eq.(1), then* $F(T) \cap F(f) \cap F(g) \neq \emptyset$ *provided one of the following conditions holds:*
(i) $\overline{T(M)}$ *is compact, and* T *and* f *and* g *are continuous;*
(ii) M *is weakly compact, and* T *and* f *and* g *is weakly continuous and* X *is complete.*

Proof. It follows from the similar argumentation of Theorem HJ(see [2, Theorem 2.2]) that for each n, there exists $x_n \in M$ such that

$$x_n = fx_n = gx_n = T_n x_n = k_n T x_n + (1 - k_n)q. \tag{2}$$

(i) It follows from the compactness of $\overline{T(M)}$ that there exists $\{x_{n_i}\} \subset \{x_n\}$ and $z \in M$ such that

$$Tx_{n_i} \to z \in \overline{T(M)}.$$

Thus, noticing Eq.(2),

$$x_{n_i} = fx_{n_i} = gx_{n_i} = k_{n_i} T x_{n_i} + (1 - k_{n_i})q \to z (i \to \infty). \tag{3}$$

And also $z \in f(M) \cap g(M)$ by $\overline{T(M)} \subset f(M) \cap g(M)$. It follows from continuity of T and f and g that $Tx_{n_i} \to Tz$ and $fx_{n_i} \to fz$ and $gx_{n_i} \to gz$, respectively.

Hence, noticing Eq.(3), we get

$$z = Tz = fz = gz.$$

(ii) The weak compactness of M implies that there exists $z \in M$ and $\{x_{n_i}\} \subset \{x_n\}$ such that

$$x_{n_i} = fx_{n_i} = gx_{n_i} = k_{n_i}Tx_{n_i} + (1 - k_{n_i})q \rightharpoonup z.$$

As $k_n \to 1$, then

$$Tx_{n_i} = \frac{x_{n_i} - (1 - k_{n_i})q}{k_{n_i}} \rightharpoonup z. \tag{4}$$

It follows from weak continuity of T and f and g that $Tx_{n_i} \rightharpoonup Tz$ and $fx_{n_i} \rightharpoonup fz$ and $gx_{n_i} \rightharpoonup gz$, respectively. Hence, noticing Eq.(3), we have

$$z = Tz = fz = gz.$$

This complete the proof. □

Theorem 2 *Let M be a nonempty $q-$starshaped subset of a normed space X and T, f, and g be self-maps of M. Suppose that f and g are affine and $\overline{T(M)} \subset f(M) \cap g(M)$. If the pairs (T, f) and (T, g) are R-subweakly commuting and T is $(f, g)-$nonexpansive, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$ provided one of the following conditions holds:*
(i) $\overline{T(M)}$ is compact and either T or f or g is continuous;
(ii) M is weakly compact, either T or f or g is strongly continuous and X is complete.

Proof. Since $(f, g)-$nonexpansive mapping satisfies Eq.(1), then it follows from Theorem 1 that for each n, there exists $x_n \in M$ such that

$$x_n = fx_n = gx_n = T_nx_n = k_nTx_n + (1 - k_n)q. \tag{5}$$

(i) It follows from Theorem 1 (i) that there exists $\{x_{n_i}\} \subset \{x_n\}$ and $z \in M$ such that

$$Tx_{n_i} \to z \in \overline{T(M)},$$

and

$$fx_{n_i} = gx_{n_i} \to z(i \to \infty).$$

And also $z \in f(M) \cap g(M)$ by $\overline{T(M)} \subset f(M) \cap g(M)$. Hence there exists $u \in M$ such that $z = fu = gu$. As $i \to \infty$,

$$\|Tu - Tx_{n_i}\| \leq \|fu - gx_{n_i}\| = \|z - gx_{n_i}\| \to 0,$$

therefore, $Tx_{n_i} \to Tu = z$, i.e. $z = Tu = fu = gu$. As R-subweakly commuting of (T, f) and (T, g) implies weakly compatible, then

$$fz = fTu = Tfu = Tz = Tgu = gTu = gz.$$

It follows from continuity of either T or f or g that $Tx_{n_i} \to Tz$ or $fx_{n_i} \to fz$ or $gx_{n_i} \to gz$. Hence

$$z = Tz = fz = gz.$$

(ii) It follows from Theorem 1 (ii) that there exists $z \in M$ and $\{x_{n_i}\} \subset \{x_n\}$ such that

$$x_{n_i} = fx_{n_i} = gx_{n_i} \rightharpoonup z,$$

and

$$Tx_{n_i} \rightharpoonup z.$$

Since in Banach space X, weak closedness of subset implies closedness [8, 4], thus $z \in f(M) \cap g(M)$ by $\overline{T(M)} \subset f(M) \cap g(M)$. Hence there exists $u \in M$ such that

$$z = fu = gu.$$

As R-subweakly commuting of (T, f) and (T, g) implies weakly compatible, then

$$fz = fTu = Tfu = Tz = Tgu = gTu = gz.$$

Assumed that f is strongly continuous, then $gx_{n_k} = fx_{n_k} \to fz$. Since as $k \to \infty$

$$\|Ty - Tx_{n_k}\| \leq \|fy - gx_{n_k}\| = \|fy - fx_{n_k}\| \to 0,$$

then $Tx_{n_k} \to Tz$, that is, T is strongly continuous at z. Thus we only need show that the result holds if T is strongly continuous.

As T is strongly continuous together with $x_{n_i} \rightharpoonup z$, then $Tx_{n_i} \to Tz$. By $Tx_{n_i} \rightharpoonup z$, we have

$$z = Tz = fz = gz.$$

which completes the proof. □

The other results in Hussain and Jungck[2] (such as Theorem 2.3, 2.8, 2.11, 2.12 and Corollary 2.4, 2.5,2.6, 2.7, 2.9, 2.10 and so on) similarly modify to get the corresponding new results. Since this is a repeated work, we omit it.

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