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Canonical Reduction of the Self-Dual Yang-Mills Equations to Inhomogeneous Nonlinear Schrödinger Equation and Exact Solutions

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Abstract

The (constrained) canonical reduction of four-dimensional self-dual Yang-Mills theory to two-dimensional inhomogeneous nonlinear Schrödinger equation are considered. On the other hand, other methods and transformations are developed to obtain exact solutions for the original two-dimensional inhomogeneous nonlinear Schrödinger equation. The corresponding gauge potential A_μ and the gauge field strengths $F_{\mu\nu}$ are also obtained. New traveling wave solution for inhomogeneous nonlinear Schrödinger equation are obtained by using the Bäcklund transformations with the aid of Mathematica.

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1. Introduction

One of the multidimensional integrable equations, important both in physics and mathematics, is the self-dual Yang-Mills (SDYM) equation [1,2]. It arises in relativity [3,4] and in field theory [5]. The SDYM equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space. Integrability for a SDYM connection means that its curvature vanishes on certain two-planes in the tangent space of the Grassmannian.

As shown in [6,7], This allows one to characterize SDYM connections in terms of the splitting problem for a transition function in a holomorphic bundle over the Riemann sphere, i.e. the trivialization of the bundle [8,9].

The theory of integrable systems has been an active area of mathematics for the past thirty years. Different aspects of the subject have fundamental relations with mechanics and dynamics, applied mathematics, algebraic structures, theoretical physics, analysis including spectral theory and geometry. In recent decades, a class of transformations having their origin in the work by Bäcklund in the late nineteenth century has provided a basis for remarkable advances in the study of nonlinear partial differential equations (NLPDEs)[1-11]. The importance of Bäcklund transformations (BTs) and their generalizations is basically twofold. Thus, on one hand, invariance under a BT may be used to generate an infinite sequence of solutions for certain NLPDEs by purely algebraic superposition principles. On the other hand, BTs may also be used to link certain NLPDEs [12-20]

(particularly nonlinear evolution equations (NLEEs) modelling nonlinear waves) to canonical forms whose properties are well known [21-23].

Nonlinear wave phenomena have attracted the attention of physicists for a long time. Investigation of a certain kind of NLPDEs has made great progress in the last decades. These equations have a wide range of physical applications and share several remarkable properties [24-27]: (i) the initial value problem can be solved exactly in terms of linear procedures, the so-called " inverse scattering method (ISM)"; (ii) they have an infinite number of " conservation laws "; (iii) they have "BTs"; (iv) they describe pseudo-spherical surfaces (pss), and hence one may interpret the other properties (i-iii) from a geometrical point of view; (v) they are completely integrable [3,24,26].

Non-Abelian gauge theories first appeared in the seminal work of Yang and Mills [28] as a non-Abelian generalization of Maxwell's equations. Let G be a Lie group (referred to as the gauge group) with Lie algebra (LG) and let $\{x_\mu\}_{\mu=1,2,3,4}$ be coordinates on a four- dimensional manifold M which can be R^4, $R^{1,3}$ or $R^{2,2}$. Given the gauge potential $A_\mu(x) \in LG$, we introduce the covariant derivatives

$$D_\mu = \partial_\mu - A_\mu, \tag{1}$$

and their commutators

$$F_{\mu\nu} = -[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \tag{2}$$

where $F_{\mu\nu}$ are the gauge field strengths.

The Yang- Mills equations are a set of coupled, second-order NLPDEs in four dimensions for the LG-valued gauge potential functions A_μ's, and are extremely difficult to solve in general. It is however possible to obtain a special class of first-order reductions of the full Yang-Mills equations by noting that any $F_{\mu\nu}$ that satisfies

$$\lambda F_{\mu\nu} =^{*} F_{\mu\nu}, \quad \lambda = \begin{cases} \pm 1 & \text{on} \quad R^4, R^{2,2}; \\ \\ \pm i & \text{on} \quad R^{3,1}. \end{cases} \tag{3}$$

All real solutions of the equations ${}^{*}F_{\mu\nu} = \pm i F_{\mu\nu}$ are trivial. On R^4 and $R^{2,2}$, the equations ${}^{*}F_{\mu\nu} = (-)F_{\mu\nu}$ are called the (anti) SDYM equations. Now consider four complex variables y, $\bar{y}$, z and $\bar{z}$ defined in [29]

$$\begin{aligned} \sqrt{2}y = x_1 + ix_2, \quad \sqrt{2}\bar{y} = x_1 - ix_2, \\ \sqrt{2}z = x_3 - ix_4, \quad \sqrt{2}\bar{z} = x_3 + ix_4, \end{aligned} \tag{4}$$

it is simple to check that the self-duality equations $F_{\mu\nu} =^{*} F_{\mu\nu}$ reduces to

$$F_{yz} = 0, \quad F_{\bar{y}\bar{z}} = 0, \quad F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{5}$$

Equations (5) are the compatibility condition of the linear problem [28]

$$(\psi_y + i\zeta\psi_{\bar{z}}) = (A_y + i\zeta A_{\bar{z}})\psi, \tag{6}$$

$$(\psi_z + i\zeta\psi_{\bar{y}}) = (A_z + i\zeta A_{\bar{y}})\psi, \tag{7}$$

where ζ is a parameter, independent of y, $\bar{y}$, z and $\bar{z}$.
The compatibility condition is simply

$$(\partial_z - i\zeta\partial_{\bar{y}})(\partial_y + i\zeta\partial_{\bar{z}})\psi = (\partial_y + i\zeta\partial_{\bar{z}})(\partial_z - i\zeta\partial_{\bar{y}})\psi. \tag{8}$$

On using equations (6) and (7), this gives

$$[F_{yz} - i\zeta(F_{y\bar{y}} + F_{z\bar{z}}) - \zeta^2 F_{\bar{y}\bar{z}}]\psi = 0. \tag{9}$$

Equations (5) can be immediately integrated, since they are pure gauge, to give

$$A_y = D^{-1}D_y, \quad A_z = D^{-1}D_z, \quad A_{\bar{y}} = \bar{D}^{-1}\bar{D}_{\bar{y}}, \quad A_{\bar{z}} = \bar{D}^{-1}\bar{D}_{\bar{z}}, \tag{10}$$

where D and $\bar{D}$ are arbitrary 2×2 complex matrix functions of y, $\bar{y}$, z and $\bar{z}$ with determinant $= 1$ (for SU(2) gauge group) and $D_y = \partial_y D$, etc. For real gauge fields $A_\mu \doteq -A_\mu^+$ (the symbol $\doteq$ is used for equations valid only for real values of x_1, x_2, x_3 and x_4), we require

$$\bar{D} \doteq (D^+)^{-1}. \tag{11}$$

Gauge transformations are the transformations

$$D \to DU, \quad \bar{D} \to \bar{D}U, \quad U^{+}U \doteq I, \tag{12}$$

where U is a 2×2 matrix function of $y, \bar{y}, z, \bar{z}$ with determined $= 1$. Under transformation (12), equation (11) remains unchanged. We now define the hermitian matrix J as

$$J \equiv D\bar{D}^{-1} \doteq DD^{+}. \tag{13}$$

J has the very important property of being invariant under the gauge transformation (12). The only nonvanishing field strengths in terms of J becomes

$$F_{u\bar{v}} = -\bar{D}^{-1}(J^{-1}J_u)_{\bar{v}}\bar{D}, \tag{14}$$

$(u, v = y, z)$ and the remaining self-duality equation (5) takes the form :

$$(J^{-1}J_y)_{\bar{y}} + (J^{-1}J_z)_{\bar{z}} = 0. \tag{15}$$

The action density in terms of J is

$$\begin{aligned} \phi(J) &\equiv -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F_{\mu\nu} \\ &= -2\mathrm{Tr}(F_{y\bar{y}}F_{z\bar{z}} + F_{y\bar{z}}F_{\bar{y}z}) \\ &= -2\mathrm{Tr}\left\{(J^{-1}J_y)_{\bar{y}}(J^{-1}J_z)_{\bar{z}} - (J^{-1}J_y)_{\bar{z}}(J^{-1}J_z)_{\bar{y}}\right\}. \end{aligned} \tag{16}$$

In this paper, the canonical reduction of four dimensional self-dual Yang-Mills theory to two dimensional inhomogeneous nonlinear Schrödinger (IHNLS) equation are considered. We give a new of exact solution for the IHNLS equation by applying the BTs method with the aid of Mathematica [17-29]. Consequently we find exact solutions for self-dual Yang Mills equations. In addition the corresponding gauge potential A_μ and the gauge field strengths $F_{\mu\nu}$ are also obtained.

The paper is organized as follows: On one hand the reduction of Yang-Mills theory to IHNLS equation and exact solutions are presented in sections 2 and 3 respectively. Moreover the gauge potential A_μ and the gauge field strengths $F_{\mu\nu}$ are also obtained. Sections 4 contains the conclusion.

2. The canonical reduction of four-dimensional SDYM theory to two -dimensional IHNLS equation

Suppose that $A'_\mu s$ depend on $x = y + \overline{y}$ and $t = z$ only. If we use a gauge in which $A_{\overline{y}} = 0$, in terms of the matrix-valued functions $P := A_y,\ Q := A_z,\ R := A_{\overline{z}}$, the SDYM equations (5) are

$$R_x = 0, \tag{17}$$

$$Q_x - P_t - [P, Q] = 0, \tag{18}$$

$$R_t - P_x - [Q, R] = 0. \tag{19}$$

Let R take the canonical form

$$R = \begin{pmatrix} \frac{-i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}. \tag{20}$$

We then find that

$$P = \begin{pmatrix} 0 & u\ e^f \\ -u^\star e^{-f} & 0 \end{pmatrix}, \tag{21}$$

$$Q = \begin{pmatrix} 2i \mid u \mid^2 & (2iu_x + 4btu)\ e^f \\ (2iu^\star_x - 4btu^\star)e^{-f} & -2i \mid u \mid^2 \end{pmatrix}, \tag{22}$$

from Eq. (18), we obtain the IHNLS equation

$$iu_t + u_{xx} + 2(|u|^2 - bx)u = 0, \tag{23}$$

where $f = 2ibxt + \frac{4}{3}b^2t^3$ and b is a complex constant.

3. The BTs and exact solution for IHNLS equation

We recall the definition [15,22] of a differential equation (DE) that describes a pss. Let M^2 be a two-dimensional differentiable manifold with coordinates (x, t). A DE for a real function $u(x, t)$ describes a pss if it is a necessary and sufficient condition for the existence of differentiable functions

$$f_{ij} \quad 1 \le i \le 3, \quad 1 \le j \le 2, \tag{24}$$

depending on u and its derivatives such that the one-forms

$$\omega_1 = f_{11}dx + f_{12}dt, \quad \omega_2 = f_{21}dx + f_{22}dt, \quad \omega_3 = f_{31}dx + f_{32}dt, \tag{25}$$

satisfy the structure equations of a pss, i.e.,

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \omega_2 \wedge \omega_1. \tag{26}$$

As a consequence, each solution of the DE provides a metric on M^2, whose Gaussian curvature is constant, equal to -1. Moreover, the above definition is equivalent to saying that DE for u is the integrability condition for the problem [14,26]:

$$d\phi = \Omega\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{27}$$

where d denotes exterior differentiation, ϕ is a column vector and the 2×2 matrix Ω (Ω_{ij}, $i, j = 1, 2$) is traceless

$$\Omega = \frac{1}{2} \begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}.$$

Take

$$\Omega = \begin{pmatrix} \frac{\eta}{2}dx + Adt & qdx + Bdt \\ rdx + Cdt & -\frac{\eta}{2}dx - Adt \end{pmatrix} = Sdx + Tdt, \tag{28}$$

from Eqs. (27) and (28), we obtain

$$\phi_x = S\phi, \quad \phi_t = T\phi, \tag{29}$$

where S and T are two 2×2 null-trace matrices

$$S = \begin{pmatrix} \frac{\eta}{2} & q \\ r & -\frac{\eta}{2} \end{pmatrix}, \tag{30}$$

$$T = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{31}$$

Here η is a parameter, independent of x and t, while q and r are functions of x and t. Now

$$0 = d^2\phi = d\Omega\phi - \Omega \wedge d\phi = (d\Omega - \Omega \wedge \Omega)\phi,$$

which requires the vanishing of the two form

$$\Theta \equiv d\Omega - \Omega \wedge \Omega = 0, \tag{32}$$

or in component form

$$\begin{gathered} A_x = qC - rB, \\ q_t - 2Aq - B_x + \eta B = 0, \\ C_x = r_t + 2Ar - \eta C, \end{gathered} \tag{33}$$

Chern and Tenenblat [10] obtained Eqs. (33) directly from the structure Eqs. (26). By suitably choosing r, A, B and C in (33), we shall obtain various IHNLS equation which q must satisfy. Konno and Wadati introduced the function [30]

$$\Gamma = \frac{\phi_1}{\phi_2}, \tag{34}$$

this function first appeared used and explained in the geometric context of pss equations [11-13], and also the classical papers by Sasaki [31] and Chern-Tenenblat [10]. Then Eq. (29) is reduced to the Riccati equations:

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma - r\Gamma^2 + q, \tag{35}$$

$$\frac{\partial \Gamma}{\partial t} = 2A\Gamma - C\Gamma^2 + B. \tag{36}$$

Our procedure in the following is that we construct a transformation Γ' satisfying the same equation as (35) and (36) with a potential u' where

$$u' = u + f(\Gamma, \eta), \tag{37}$$

Chern and Tenenblat [10] introduced several examples of (37) for pss equations. For use in the sequel, we list the IHNLS equation and their corresponding BT in the following.

The IHNLS equation

For any solution $u(x,t)$ of the IHNLS equation (23), the matrices S and T are

$$S = \begin{pmatrix} \frac{\eta}{2} & ue^{f} \\ -u^{\star}e^{-f} & -\frac{\eta}{2} \end{pmatrix}, \tag{38}$$

$$T = \begin{pmatrix} i|u|^2 + i\frac{\eta^2}{2} + 2b\eta t & (iu_x + 2btu + i\eta u)e^{f} \\ (iu_x^{\star} - 2btu^{\star} - i\eta u^{\star})e^{-f} & -i|u|^2 - i\frac{\eta^2}{2} - 2b\eta t \end{pmatrix}, \tag{39}$$

the above matrices S, T satisfy Eqs. (33). Then Eq. (35) becomes

$$\frac{\partial \Gamma}{\partial x} = \eta \Gamma + u e^{f} + u^{\star} e^{-f}\ \Gamma^2. \tag{40}$$

If we choose Γ' and u' as [22]

$$\Gamma' = \frac{1}{\Gamma^{\star}}, \qquad u' = u + 2\frac{\Gamma^2 \Gamma_x^{\star} - \Gamma_x}{1 - |\Gamma|^2}, \tag{41}$$

then Γ' and u' satisfies Eq. (40) for real η. The second equation in Eq. (41) reduces to simpler form by the aid of Mathematica

$$u' = -u - \frac{2\eta \Gamma e^{-f}}{1 + |\Gamma|^2}. \tag{42}$$

Now we shall choose some known solution of the IHNLS equation and substitute this solution into the corresponding matrices S and T. Next, we solve Eqs. (29) for ϕ_1 and ϕ_2. Then, by (34) and the corresponding BT we shall obtain the new solution for the IHNLS equation.

Substitute $u = 0$ into the matrices S and T in (38) and (39), then by (29) we have

$$d\phi = \phi_x dx + \phi_t dt = S\phi d\rho, \tag{43}$$

where

$$S = \begin{pmatrix} \frac{\eta}{2} & 0 \\ 0 & -\frac{\eta}{2} \end{pmatrix}, \tag{44}$$

$$\rho = x + kt, \quad k = \frac{\eta}{2} + 4ibt. \tag{45}$$

The solution of Eq. (43) is

$$\phi = e^{\rho S}\phi_0 = \left(I + \rho S + \frac{\rho^2 S^2}{2!} + \frac{\rho^3 S^3}{3!} + \cdots \right)\phi_0, \tag{46}$$

where ϕ_0 is a constant column vector. The solution of Eq. (46) is

$$\phi = \begin{pmatrix} \cosh\frac{\eta}{2}\rho + \sinh\frac{\eta}{2}\rho & 0 \\ 0 & \cosh\frac{\eta}{2}\rho - \sinh\frac{\eta}{2}\rho \end{pmatrix}\phi_0. \tag{47}$$

Now, we choose $\phi_0 = (1,1)^T$ in (47), then we have

$$\phi = \begin{pmatrix} e^{\frac{\eta\rho}{2}} \\ e^{-\frac{\eta\rho}{2}} \end{pmatrix}. \tag{48}$$

Substitute (48) into (34), then by (42), we obtain the new solutions of the IHNLS equation (23)

$$u' = -\eta\, e^{i\eta^2 t}\, e^{-f}\operatorname{sech}(\eta\rho). \tag{49}$$

We can calculate the gauge potential A_μ and the gauge field strengths $F_{\mu\nu}$, from equations (6)-(10) and (20)-(22), then

$$A_y = \begin{bmatrix} 0 & -\eta e^{a_1}\operatorname{sech}(\eta\rho) \\ \eta e^{b_1}\operatorname{sech}(\eta\overline{\rho}) & 0 \end{bmatrix}, \quad A_{\overline{y}} = 0, \quad A_z = \begin{bmatrix} a_2 & b_2 \\ b_3 & -a_2 \end{bmatrix}, \quad A_{\overline{z}} = \begin{bmatrix} \frac{-i}{2} & 0 \\ 0 & \frac{i}{2} \end{bmatrix}, \tag{50}$$

where $a_1 = i\eta^2 t, \quad b_1 = -\frac{8}{3}b^2t^3 - a_1, \quad a_2 = 4i\eta e^{-\frac{8}{3}b^2t^3}\operatorname{sech}(\eta\overline{\rho})\operatorname{sech}(\eta\rho),$
$b_2 = [-8bt\eta - 2i\eta^2\tanh(\eta\rho)]e^{i\eta^2 t}\operatorname{sech}(\eta\rho), \quad b_3 = [8bt\eta - 2i\eta^2\tanh(\eta\overline{\rho})]e^{b_1}\operatorname{sech}(\eta\overline{\rho}).$

Consequently, we obtain the gauge field strengths $F_{\mu\nu}$ as follows:

$$F_{y\overline{y}} = -\partial_x A_y, \quad F_{z\overline{z}} = -\partial_t A_z - [A_z, A_{\overline{z}}],\ F_{yz} = \partial_x A_z - \partial_t A_y - [A_y, A_z],\ F_{\overline{y}\overline{z}} = \partial_x A_{\overline{z}} - \partial_t A_{\overline{y}} - [A_{\overline{y}}, A_{\overline{z}}]. \tag{51}$$

We note that the self-dual $SU(2)$ Yang Mills equations holds.

4. Conclusions

A soliton is a localized pulse-like nonlinear wave that possesses remarkable stability properties. Typically, problems that admit soliton solutions are in the form of evolution equations that describe how some variable or set of variables evolve in time from a given state. The equations may take a variety of forms, for example, PDEs, differential difference equations, partial difference equations, and integro-differential equations, as well as coupled ODEs of finite order.

In this paper, we considered the construction of exact solutions to IHNLS equation. We obtain travelling wave solutions for the above equations by using BTs method with the aid of Mathematica.

The soliton phenomena and integrable NLEEs represent an important and well established field of modern physics, mathematical physics and applied mathematics. Solitons are found in various areas of physics from hydrodynamics and plasma physics, nonlinear optics and solid state physics, to field theory and gravitation. NLEEs which describe soliton phenomena have an universal character.

A travelling wave of permanent form has already been met; this is the solitary wave solution of the NLEE itself. Such a wave is a special solution of the governing equation which does not change its shape and which propagates at constant speed.

The SDYM equations play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics.

In addition the SDYM equations are a rich source of integrable systems suggested by the fact that they are the compatibility condition of an associated linear problem which admits enormous freedom if one allows the associated gauge algebra to be arbitrary. The classical soliton equations in 1+1, 2+1 and 3+1 dimensions are reductions of the SDYM equations with finite-dimensional gauge algebra. In this paper we have demonstrated the reductions of the SDYM equations to IHNLS equaton and also obtained travelling wave solution.

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