Applied Mathematical Sciences, Vol. 2, 2008, no. 46, 2289 - 2296

A Takagi Factorization of a Real Symmetric Tridiagonal Matrix

Mohamed Elouafi

Preparatory Classes to High School Engineering
BP 3117, Souanni, Tangiers, Morocco
med_elouafi@hotmail.com

Ahmed Driss Aiat Hadj

Preparatory Classes to High School Engineering
BP 3117, Souannim Tangiers, Morocco
ait_hadj@yahoo.com

Abstract

This paper presents an explicit Takagi Factorization of a real symmetric tridiagonal matrix.

Keywords: Tridiagonal matrices; Takagi Factorization; Eigenvalues; Eigenvectors; Orthogonal Polynomials

1 Introduction

Complex symmetric matrices arise from many applications, such as chemical exchange in nuclear magnetic resonance and power systems. Singular value decomposition (SVD) reveals a great deal of properties of a matrix. A complex symmetric matrix has a symmetric SVD (SSVD), also called Takagi Factorization, which exploits the symmetry [3]. Let A be a complex symmetric matrix, its Takagi factorization has the form:

$$A = U\Sigma U^T \tag{1.1}$$

where U is unitary and the Σ diagonal singular value matrix. The columns of U are called the Takagi vectors.

As the case of a general matrix, computing the Takagi Factorization of a complex symmetric matrix consists of two stages: tridiagonalization and diagonalization. In the first stage, a complex symmetric matrix A is reduced to a complex symmetric tridiagonal T using two-side Householder transformations or Lanczos method [8, 9]. In the second stage, the Takagi Factorization

$$T = U\Sigma U^T \tag{1.2}$$

of the complex symmetric tridiagonal T resulted from the first stage is computed. The methods for the second stage include the implicit QR method [9] and the divide-andconquer method [12].

This paper presents a Takagi Factorization of a real symmetric tridiagonal matrix.

2 Preliminaries

For the symmetric tridiagonal matrix

$$T = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}),$$

where $\{a_i\}_{1\leq i\leq n}$ and $\{b_i\}_{1\leq i\leq n-1}$ are sequences of real numbers such that $b_i \neq 0$ for $i = 1, 2, ..., n-1$, we associate the polynomial sequence $\{P_i\}_{-1\leq i\leq n}$ characterized by a three-term recurrence relation

$$xP_i(x) = b_{i+1}P_{i+1}(x) + a_{i+1}P_i(x) + b_iP_{i-1}(x), \qquad i = 0, 1, \dots, n-1, \tag{2.1}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, where $b_n = b_0 = 1$. Let put for $i \geq 0$, $Q_i = \left(\prod_{k=0}^{i} b_k\right) P_i$. Then the polynomial sequence $\{Q_i\}_{-1\leq i\leq n}$ characterized by a three-term recurrence relation

$$xQ_i(x) = Q_{i+1}(x) + a_{i+1}Q_i(x) + b_i^2 Q_{i-1}(x). \tag{2.2}$$

with with initial conditions $Q_{-1}(x) = 0$ and $Q_0(x) = 1$. By a theorem of Favard [13] we know that the family of polynomials$\{Q_i(x)\}$ defined by (2.2) is orthogonal with respect to a probability distribution $F(x)$. A standard result in the theory of orthogonal polynomials (see [11,Theorem 3.2.1]) is that for each $i \geq 2$, the zeros of $Q_i(x)$ (an so for $P_i(x)$)are real, distinct, and separate those of $Q_{i-1}(x)$. We can write the relations (2.1) in a matrix form

$$x\mathbf{P}_{n-1}(x) = T\mathbf{P}_{n-1}(x) + P_n(x)E_n, \tag{2.3}$$

where $\mathbf{P}_{n-1}(x) = [P_0(x\ ),\ P_1(x\ ),\ \ldots,\ P_{n-1}(x\ )]^t$, and $E_n = [0,\ 0,\ \ldots 0\ ,\ 1]^t \in \mathbb{R}^n$. If $(\lambda_j)_{0\leq j\leq n-1}$ are the zeros of the polynomial P_n, then it follows from (2.2) that each λ_j is an eigenvalue of the corresponding tridiagonal matrix T and

$$\mathbf{P}_{n-1}(\lambda_j) = [P_0(\lambda_j\ ),\ P_1(\lambda_j\ ),\ \ldots,\ P_{n-1}(\lambda_j\ )]^t$$

is a corresponding eigenvector. Hence the (monic) characteristic polynomial of T is precisely Q_n, i.e.,

$$Q_n(x) = det(xI_n - T) = \left(\prod_{k=0}^{n} b_k\right) P_n, \quad n = 1, 2, ..., .$$

The polynomials $\{P_i\}_{-1\leq i\leq n}$ verify the well known *Christoffel-Darboux* formula:

Lemma 2.1 *We have*

$$\sum_{i=0}^{n-1} P_i\,(x)\,P_i\,(y) = \frac{P_n\,(y)\,P_{n-1}\,(x) - P_n\,(x)\,P_{n-1}\,(y)}{x-y} \text{ for } x \neq y. \tag{2.4}$$

$$\sum_{i=0}^{n-1} P_i\,(x)\,P_i\,(x) = P_n'\,(x)\,P_{n-1}\,(x) - P_n\,(x)\,P_{n-1}'\,(x)\,. \tag{2.5}$$

Proof. Using (2.1) we can write

$$(x - a_{i+1})\,P_i(x) = b_i P_{i-1}(x) + b_{i+1} P_{i+1}(x).$$

In particular,

$$\begin{aligned}
(x - a_{i+1})\,P_i(x)P_i\,(y) &= b_i P_{i-1}(x)P_i\,(y) + b_{i+1}P_{i+1}(x)P_i\,(y)\,,\\
(y - a_{i+1})\,P_i(x)P_i\,(y) &= b_i P_{i-1}(y)P_i\,(x) + b_{i+1}P_{i+1}(y)P_i\,(x)\,.
\end{aligned}$$

This implies that

$$\begin{aligned}
(y - x)\,P_i(x)P_i\,(y) &= b_i P_{i-1}(y)P_i\,(x) - b_{i+1}P_i\,(y)\,P_{i+1}(x)\\
&\quad + b_{i+1}P_{i+1}(y)P_i\,(x) - b_i P_i\,(y)\,P_{i-1}(x).
\end{aligned}$$

Hence

$$(y-x)\sum_{i=0}^{n-1} P_i(x)P_i(y) = b_0P_{-1}(y)P_0(x) - b_nP_{n-1}(y)P_n(x) + b_nP_n(y)P_{n-1}(x) - b_0P_{-1}(x)P_0(y).$$

Finally,

$$(y-x)\sum_{i=0}^{n-1} P_i(x)P_i(y) = P_n(y)P_{n-1}(x) - P_{n-1}(y)P_n(x).$$

Tending y to x we get formula (2.5). The proof is completed. □

Since all the eigenvalues $\lambda_0, \lambda_1, ..., \lambda_{n-1}$ of T are distincts, we can write

$$T = PDP^{-1},$$

where $D = diag(\lambda_0 , \lambda_1 ,..., \lambda_{n-1})$ and P is the transforming matrix. Namely $P = (p_{ij} = P_{i-1}(\lambda_{j-1}))_{1\leq i;j\leq n}$.

Lemma 2.2

$P^{-1} = (s_{ij})_{1\leq i;j\leq n}$ is defined as follows:.

$$s_{ij} = \frac{P_{j-1}(\lambda_{i-1})}{P_n'(\lambda_{i-1})\,P_{n-1}(\lambda_{i-1})}. \tag{2.6}$$

Proof. By using 2.4) and (2.5) we obtain

$$\sum_{k=1}^{n} s_{ik}p_{kj} = \sum_{k=1}^{n} \frac{P_{k-1}(\lambda_{i-1})\,P_{k-1}(\lambda_{j-1})}{P_n'(\lambda_{i-1})\,P_{n-1}(\lambda_{i-1})} = \delta_{ij},$$

where δ_{ij} is *the Kronecker symbol*. The result follows. □

Lemma 2.3

Let $(\lambda_k)_{0\leq k\leq n-1}$ are the eigenvalues of T, then

$$P_n'(\lambda_k)\,P_{n-1}(\lambda_k) > 0 \text{ for } k = 0, 1, ..., n-1. \tag{2.7}$$

Proof. Without loss of generality one can assume that

$$\lambda_0 < \lambda_0^{(n-1)} < \lambda_1 < .... < \lambda_{n-2} < \lambda_{n-2}^{(n-1)} < \lambda_{n-1},$$

where $\left(\lambda_k^{(n-1)}\right)_{0\leq k\leq n-2}$ are the roots of the polynomial P_{n-1}. Since $b_n = 1$, from the relation (2.1), P_n and P_{n-1} has the same leading coefficient $\beta \in \mathbb{R}$. We can write

$$P_n'(\lambda_k) = n\beta\prod_{\substack{i=0\\ i\neq k}}^{n-1}(\lambda_k - \lambda_i) \text{ and } P_{n-1}(\lambda_k) = \beta\prod_{j=0}^{n-2}\left(\lambda_k - \lambda_i^{(n-1)}\right)$$

Assuming $\beta > 0$, then the sign of $P_n'(\lambda_k)$ is that of $(-1)^k$ and the sign of $P_{n-1}'(\lambda_k)$ is that of $(-1)^k$. The result follows. □

3 Main Result

Let the matrix tridiagonal

$$T = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{pmatrix}$$

where $\{a_i\}_{1\leq i\leq n-1}$ and $\{b_i\}_{1\leq i\leq n-1}$ are sequences of real numbers such that $b_i \neq 0$ for all $n = 1, 2, ..., n-1$.

Theorem 3.1 *The Takagi Factorization of T is:*

$$T = U\Sigma U^T,$$

where $\Sigma = diag(\lambda_0, \lambda_1, ..., \lambda_{n-1})$, $\lambda_0, \lambda_1, ..., \lambda_{n-1}$ are the eigenvalues of T, and $U = (u_{ij})_{1\leq i,j\leq n}$ such that :

$$u_{ij} = \frac{P_{j-1}(\lambda_{i-1})}{\sqrt{P_n'(\lambda_{i-1})\,P_{n-1}(\lambda_{i-1})}}. \tag{2.8}$$

Proof. The proof follow from **Lemma 2.2.** and **Lemma 2.3.**. □

4 Example

A most important OPS is the Chebyshev polynomial of the second kind, $\{Un\}_{n\geq 0}$, which satisfies the three-term recurrence relations

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x),\ n = 1, 2, ...,$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. It is well known (cf. [4], e.g.) that each U_n also satisfies $U_n(x) = \frac{sin((n+1)\theta)}{sin(\theta)}$, $x = cos\,(\theta)$ $(0 \leq \theta < \pi)$ and by consequent the zeros of U_n are $\cos\left(\frac{(l+1)\pi}{n+1}\right)$, $\quad l = 0, 1, ..., n-1$.

Let T be an n-by-n real tridiagonal Toeplitz matrix defined by

$$T = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \ddots & \vdots \\ 0 & b & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & b & b & a \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}),$$

where $b \neq 0$. The polynomial sequence $\{P_i\}_{1\leq i\leq n-1}$ associated to T verifies

$$xP_i(x) = bP_{i+1}(x) + aP_i(x) + bP_{i-1}(x), \qquad i = 0, 1, 2 \ldots, n-1,$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$. By simple calculation we can show that $P_i(x) = U_n\left(\frac{x-a}{2b}\right)$, $i = 0, ..., n$. The eigenvalues of T are

$$\lambda_l = a + 2b\cos\left(\frac{(l+1)\,\pi}{n+1}\right), \quad l = 0, 1, ..., n-1.$$

Then:

$$T = U\Sigma U^T,$$

where $\Sigma = diag(\lambda_0\ , \lambda_1\ , ..., \lambda_{n-1})$ and $U = (u_{ij})$ such that :

$$u_{ij} = \frac{P_{j-1}\left(\lambda_{i-1}\right)}{\sqrt{P_n'\left(\lambda_{i-1}\right) P_{n-1}\left(\lambda_{i-1}\right)}} = \frac{\sin\left(\frac{(ij)\pi}{n+1}\right)}{\sqrt{(n+1)\sin\left(\frac{(i)\pi}{n+1}\right)}}.$$

We leave the details to the reader.

References

[1] A.D. Aiat Hadj, M. Elouafi, On the charactersitic polynomial, eigenvectors and determinant of a pentadiagonal matrix, Appl.Math. Compt, Vol 198/2 pp 634-642,(2008).

[2] A.D. Aiat Hadj, M. Elouafi, A fast numerical algorithm for the inverse of a tridiagonal and pentadiagonal matrix, Appl.Math. Compt. (2008), doi:10.1016/j.amc.2008.02.026

[3] A. Bunse-Gerstner andW.B. Gragg. Singular value decomposition of complex symmetric matrices. J. Computational and Applied Mathematics, 21:41–54, 1988.

[4] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[5] C. Davis and W.M. Kahan. The rotation of eigenvectors by a perturbation III. SIAM J. Numer. Anal., 7:1–46, 1970.

[6] J.S. Dhillon. A New O(n2) Algorithm for the Symmetric Tridiagonal Eigenvalue/Eigenvector Problem. Ph.D. Thesis, UCB, Berkeley , CA, 1997.

[7] F.T. Luk and S. Qiao. A fast singular value algorithm for Hankel matrices. In V. Olshevsky, editor, Fast Algorithms for Structured Matrices: Theory and Applications, Contemporary Mathematics 323. American Mathematical Society, 2003.

[8] S. Qiao G. Liu and W. Xu. Block Lanczos tridiagonalization of complex symmetric matrices. In Franklin T. Luk, editor, Advanced Signal Processing Algorithms, Architectures, and Implementations XV, Proc. of SPIE Vol. 5910, pp. 285-295 (2005).

[9] S. Qiao. Orthogonalization techniques for the Lanczos tridiagonalization of complex symmetric matrices. In Franklin T. Luk, editor, Advanced Signal Processing Algorithms, Architectures, and Implementations XIV, Proc. of SPIE Vol. 5559, pages 423–434, 2004.

[10] Roger A Horn and Charles R Johnson. Matrix Analysis. Cambridge University Press, Cambridge, 1985.

[11] G. Szegö, Orthogonal Polynomials, fourth ed., Amer. Math. Soc., Providence, RI, 1975.

[12] W. Xu and S. Qiao. A divide-and-conquer method for the Takagi factorization. Technical Report No. CAS 05-01-SQ, McMaster University, Hamilton, Ont. Canada, 2005.

[13] T. Whitehurst. An application of orthogonal polynomials to random walks. Pacific Journal of Mathematics. Volume 99, Number 1 (1982), 205-213.

Received: January 1, 2008