Applied Mathematical Sciences, Vol. 2, 2008, no. 39, 1919 - 1928

On Inequalities and Exponential Approximations for Residual Life Reliability Functions

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Abstract

Given that a unit is of age t, the remaining life after time t is random. The expected value of this random residual life is called the mean residual life at time t. Specifically, if T is the life of a component with distribution function F, then $\delta_F(t) = E(T-t|T>t)$ is called the mean residual life function (MRLF). It is well known that the class of distributions with decreasing mean residual life (DMR) contains the class of distributions with increasing hazard rate (IHR). In this note, exponential length-biased approximations, bounds and stability results on the distance between residual life reliability functions with monotone weight functions and the exponential counterpart in the class of distribution functions with increasing or decreasing hazard rate and mean residual life functions are established. Some examples are presented.

Mathematics Subject Classifications: 39C05

Keywords: Stochastic inequalities, Stochastic order, Weighted reliability functions, Integrable function

1 Introduction

The usefulness and applications of weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can be seen in Patil and Rao (1978), Gupta and Kirmani (1990), Gupta and Keating (1985), Oluyede (1999) and in references therein. When data is unknowingly sampled from a weighted distribution as opposed to the parent distribution, the survival function, hazard function, and mean residual life function (MRLF) may be under or overestimated depending on the weight function. It is well known that the length or size-biased distribution of an increasing failure rate (IFR) distribution is always IFR. The converse is not true. Also, if the weight function is monotone increasing and concave, then the weighted distribution of an IFR distribution is an IFR distribution. Similarly, the size-biased distribution of a decreasing mean residual (DMRL) distribution has decreasing mean residual life. The residual life at age t, is a weighted distribution, with survival function given by

$$\overline{F}_t(x) = \overline{F}(x+t)/\overline{F}(t), \tag{1}$$

for $x \geq 0$. The weight function is $W(x) = f(x+t)/f(x)$, where $f(u) = dF(u)/du$, the hazard function and mean residual life functions are $\lambda_{F_t}(x) = \lambda_F(x+t)$ and $\delta_{F_t}(x) = \delta_F(x+t)$. It is clear that if F is IFR (DMRL) distribution, then F_t is IFR (DMRL) distribution, where the hazard function $\lambda_F(x)$ and mean residual life function $\delta_F(x)$ of the distribution function F are given by $\lambda_F(x) = f(x)/\overline{F}(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$ respectively. The functions $\lambda_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent (Ross (1983)). Keilson (1979) suggested a measure of departure from exponentiality within the class of completely monotone distributions (mixture of exponential distributions). These measures of departure are given in terms of $\rho = |1 - \mu_2/2\mu^2|$, where $\mu_2 = E(X^2)$ and $\mu = E(X)$. This is due to the fact that the exponential distribution satisfies $\rho = 0$. The purpose of this article is to establish bounds and stability results on the distance between residual life reliability functions, as well as residual life distributions with monotone weight functions and the exponential counterpart including length-biased exponentials in the class of life distributions with increasing or decreasing hazard rate and mean residual life functions. In section 2 some basic results and utility notions are presented. Section 3 contain results on the residual exponential approximations for reliability measures under distributions with monotone weight functions. Section 4 contain some examples and applications.

2 Some Utility Notions

In this section, some basic definitions and utility notions are presented. In a weighted distribution problem, a realization x of X enters into the investigators record with probability proportional to a weight function $W(x)$. The recorded x is not an observation of X, but rather an observation on a weighted random variable X_W.

Let X be a nonnegative random variable with distribution function $F(x)$ and probability density function(pdf) $f(x)$. Let $W(x)$ be a positive weight function such that $0 < E(W(X)) < \infty$. The weighted survival or reliability function is given by

$$\overline{F}_W(x) = \frac{E_F[W(X)|X > x]}{E_F[W(X)]}\overline{F}(x). \tag{2}$$

Note that the survival or reliability function can also be expressed as

$$\overline{F}_W(x) = \overline{F}(x)\{W(x) + M_F(x)\}/E(W(X)), \tag{3}$$

where $M_F(x) = \int_x^\infty \{\overline{F}(t)W'(t)dt\}/\overline{F}(x)$, assuming $W(x)\overline{F}(x) \longrightarrow 0$ as $x \longrightarrow \infty$. The corresponding pdf of the weighted random variable X_W is

$$f_W(x) = W(x)f(x)/E(W(X)), \tag{4}$$

$x \geq 0$, where $0 < E(W(X) < \infty$. We now give some basic and important definitions.

Definition 2.1 . *Let X and Y be two random variables with distribution functions F and G respectively. We say $F <_{st} G$, stochastically ordered, if $\overline{F}(x) \leq \overline{G}(x)$, for $x \geq 0$ or equivalently, for any increasing function $\Phi(x)$,*

$$E(\Phi(X)) \leq E(\Phi(Y)). \tag{5}$$

Definition 2.2 . *A distribution function F is an increasing hazard rate (IHR) distribution if $\overline{F}(x + t)/\overline{F}(t)$ is decreasing in $0 < t < \infty$ for each $x \geq 0$. Similarly, a distribution function F is a decreasing hazard rate (DHR) distribution if $\overline{F}(x + t)/\overline{F}(t)$ is increasing in $0 < t < \infty$ for each $x \geq 0$.*

It is well known that IHR (DHR) implies DMRL (IMRL).

3 Residual Exponential Approximations

In this section we obtain useful inequalities for residual reliability functions. Let $\{X_i\}_{i=1}^\infty$ be a sequence of operating times from a repairable system that start functioning at time $t = 0$. The sequence of times $\{X_i\}_{i=1}^\infty$ form a renewal-type stochastic point process. Following Kijima (1989), if a system has virtual age $T_{m-1} = t$ immediately after the $(m-1)^{th}$ repair, then the length of the m^{th} cycle X_m has the distribution

$$F_t(x) = P(X_m \leq x|T_{m-1} = t) = \{F(x + t) - F(t)\}/\overline{F}(t), \tag{6}$$

$x \geq 0$, where $\overline{F}(x) = 1 - F(x)$ is the reliability function of a new system. When $t = \sum_{i=1}^{j} X_i$, $j = 1, 2, ..., m-1$, minimal repair is performed, keeping the virtual age intact and when $t = 0$ we have perfect repair. The virtual age of system is equal to its operating time for the case of minimal repair. The corresponding reliability function is given by

$$\overline{F}_t(x) = \overline{F}(x+t)/\overline{F}(t), \tag{7}$$

$x \geq 0$. Bounds and stability results on the distance between the residual life reliability functions and size-biased exponential distributions are established. These results are given in the context of life distributions with monotone hazard and mean residual life functions.

Theorem 3.1 *(Barlow et al.(1963)). If F has DMRL, then* $S_k(x) \leq S_k(0)e^{-x/\mu}, k = 1, 2, \ldots,$ *and* $S_k(x) \geq \mu S_{k-1}(0)e^{-x/\mu} - \mu S_{k-1}(0) + S_k(0), k = 2, 3, \ldots,$ *where*

$$S_k(x) = \begin{cases} \overline{F}(x) & \text{if } k = 0, \\ \int_0^\infty \overline{F}(x+t)t^{k-1}dt/(k-1)! & \text{if } k = 1, 2, \ldots, \end{cases}$$

is a sequence of decreasing functions for which F possess moments of order J, that is $\mu_k = E(X^k)$ *exists,* $k = 1, 2, ..., J$.

We let $S_{-1}(x) = f(x)$ be the pdf of F if it exists. Then $S_k(0) = \mu_k/k!$, and $S_k'(x) = -S_{k-1}(x)$, $k = 0, 1, 2, ...J$. The ratio $S_{k-1}(x)/S_k(x)$ is a hazard function of a distribution function with survival function $S_k(x)/S_k(0)$. The inequalities in Theorem 1 are reversed if F has increasing mean residual life (IMRL).

Theorem 3.2 *Let* $\overline{F}_t(x)$ *be an IHR residual reliability function. Then*

$$\int_0^\infty |\overline{F}_t(x) - (1 + x/\mu)e^{-x/\mu}|dx \leq 2\mu|1 + \mu - \mu_2/2\mu^2|. \tag{8}$$

Proof: Let $D = \{x|\overline{F}_t(x) \leq (1 + x/\mu)e^{-x/\mu}\}$. Then for $x > 0$,

$$\begin{aligned} \int_0^\infty |\overline{F}_t(x) - (1 + x/\mu)e^{-x/\mu}|dx &\leq 2\int_D ((1 + x/\mu)e^{-x/\mu} - \overline{F}_t(x))dx \\ &\leq 2\int_0^\infty \Big((1 + x/\mu)e^{-x/\mu} - \overline{F}(x+t)\Big)dx \\ &\leq 2\int_0^\infty \Big((1 + x/\mu)e^{-x/\mu} - S_1(x+t)/\mu\Big)dx \\ &= 2(\mu^2 + \mu - S_2(0)/\mu) \\ &= 2\mu(1 + \mu - \mu_2/2\mu^2). \end{aligned} \tag{9}$$

The first inequality is trivial and the second inequality is due to the fact that $\overline{F}_t(x) \geq \overline{F}(x+t)$ for $x \geq 0$ and $t > 0$. The result now follows from Theorem 3.1

Theorem 3.3 *Let $\overline{F}_t(x)$ be an IHR residual life reliability function, then*

$$\int_0^\infty |\overline{F}_t(x) - xe^{-x/\mu}|dx \leq 2\mu|1 - \mu_2/2\mu^2|. \tag{10}$$

Proof: Let $E = \{\overline{F}_t(x) \leq xe^{-x/\mu}\}$, then for fixed $t > 0$ and $x > 0$, we have

$$\begin{aligned}
\int_0^\infty |\overline{F}_t(x) - xe^{-x/\mu}|dx &= \int_E (xe^{-x/\mu} - \overline{F}_t(x))dx - \int_{E^c} (xe^{-x/\mu} - \overline{F}_t(x))dx \\
&\leq 2\int_E (xe^{-x/\mu} - \overline{F}_t(x))dx \\
&= 2\int_E (xe^{-x/\mu} - \frac{\overline{F}(x+t)}{\overline{F}(t)})dx \\
&\leq 2\int_0^\infty (xe^{-x/\mu} - \overline{F}(x+t))dx \\
&= \int_0^\infty (xe^{-x/\mu} - S_1(x+t)/\mu)dx \\
&= 2(\mu - \mu_2/2\mu) \\
&= 2\mu(1 - \mu_2/2\mu^2). \qquad (11)
\end{aligned}$$

The first inequality is trivial. The second inequality is due to the fact that $F_t(x)$ and $F(x+t)$ are stochastically ordered for all $x \geq 0$ and $t > 0$. The result follows from the application of Theorem 3.1

Theorem 3.4 *Let $\overline{F}_t(x)$ be an IHR residual life reliability function, then for $t > 0$,*

$$\int_0^\infty |\overline{F}_t(x) - (1 + x/(\mu+t))e^{-x/\mu}|dx \leq 2\mu|1 + \mu/(\mu+t) - \mu_2/2\mu^2|. \tag{12}$$

Proof: Let $E = \{\overline{F}_t(x) \leq (1 + x/(\mu+t))e^{-x/\mu}\}$, then for fixed $t > 0$ and $x > 0$, we have

$$\begin{aligned}
\int_0^\infty |\overline{F}_t(x) - (1 + x/(\mu+t))e^{-x/\mu}|dx &= \int_E \left((1 + x/(\mu+t))e^{-x/\mu} - \overline{F}_t(x)\right)dx \\
&- \int_{E^c} \left((1 + x/(\mu+t))e^{-x/\mu} - \overline{F}_t(x)\right)dx \\
&\leq 2\int_E \left((1 + \frac{x}{\mu+t})e^{-x/\mu} - \overline{F}_t(x)\right)dx \\
&= 2\int_E \left((1 + \frac{x}{\mu+t})e^{-x/\mu} - \frac{\overline{F}(x+t)}{\overline{F}(t)}\right)dx \\
&\leq 2\int_0^\infty \left((1 + \frac{x}{\mu+t})e^{-x/\mu} - \overline{F}(x+t)\right)dx \\
&= \int_0^\infty \left((1 + \frac{x}{\mu+t})e^{-x/\mu} - \frac{S_1(x+t)}{\mu}\right)dx \\
&= 2\mu(1 + \mu/(\mu+t) - \mu_2/2\mu^2). \qquad (13)
\end{aligned}$$

The first inequality is trivial. The second inequality is due to the fact that $F_t(x)$ and $F(x+t)$ are stochastically ordered for all $x \geq 0$, and $t > 0$. The result follows immediately from the application of Theorem 3.1.

Theorem 3.5 *Let $F_t(x)$ be a residual life distribution function. Suppose the weight function $f(x+t)/f(x)$ is log-convex and pdf $f_t(x) > 0$ for $x \geq x_0$. Furthermore, suppose the hazard function $\lambda_F(x)$ is such that $\lambda_F(x) \geq c/x$ for $x \geq x_0$, where c is a positive real number. If X is the original random variable then*

$$P(X - x \leq xt | X > t) \leq 1 - (1+t)^{-c},$$

for all $t > 0$ and $x \geq x_0$.

Proof: Let $f(x+t)/f(x)$ be log-convex, then $f(x+t)/f(x)$ increasing in x. Furthermore, the hazard function of the distribution function F satisfies the inequality

$$\lambda_{F_t}(x) \geq \lambda_F(x) \geq c/x,$$

for $x \geq x_0$. Then for $t > 0$,

$$\begin{aligned} \int_x^{(1+t)x} \lambda_{F_t}(y)dy &\geq c\int_x^{(1+t)x} (1/y)dy \\ &\geq 1 - (1+t)^{-1}, \end{aligned} \tag{14}$$

for $t > 0$. The last inequality follows from the fact that $ln(a) \geq 1 - a^{-1}$ for $a > 0$.

The result in Theorem 3.5 provides simple inequality for the lower bound of the residual life time distributions for large values of x from the use of the information about the hazard function of the residual life distribution function.

Theorem 3.6 *If $\overline{F}_t(x)$ is an DHR reliability function, then*

$$\int_0^\infty |\overline{F}_t(x) - (1 + x/\mu)e^{-x/\mu}|dx \geq 2max\{0, \mu e^{-(\epsilon+t)/\mu} - 1\}. \tag{15}$$

Proof: Let $\overline{F}_t(x)$ be a DHR survival function, then there exist $\epsilon \geq \mu$ such that $\overline{F}_t(x) \leq (1 + x/\mu)e^{-x/\mu}$ or $\overline{F}_t(x) \geq (1 + x/\mu)e^{-x/\mu}$ as $x \leq \epsilon$ or $x \geq \epsilon$. Now,

$$\begin{aligned} \int_0^\infty |\overline{F}_t(x) - (1 + x/\mu)e^{-x/\mu}|dx &= 2\int_\epsilon^\infty (\overline{F}_t(x) - (1 + x/\mu)e^{-x/\mu})dx \\ &= 2\int_\epsilon^\infty \left(\frac{\overline{F}(x+t)}{\overline{F}(t)} - (1 + x/\mu)e^{-x/\mu}\right)dx \\ &\geq 2\int_\epsilon^\infty (\overline{F}(x+t) - (1 + x/\mu)e^{-x/\mu})dx \end{aligned}$$

$$\geq 2\left(S_1(\epsilon+t) - \int_0^\infty (1+x/\mu)e^{-x/\mu}dx\right)$$
$$\geq 2\left(\mu e^{-(\epsilon+t)/\mu} - \int_0^\infty (1+x/\mu)e^{-x/\mu}dx\right)$$
$$= 2(\mu e^{-(\epsilon+t)/\mu} - 1). \qquad (16)$$

The first inequality is is due to the fact that $F_t(x) \geq F(x+t)$ for $x \geq 0$. The last inequality follows from Theorem 3.1.

Theorem 3.7 *Let $\overline{F}_t(x)$ be a DHR reliability function, then*

$$\int_0^\infty |\overline{F}_t(x) - e^{-x/\mu}|dx \geq max\{0, 2\mu e^{-\epsilon/\mu}|e^{-t/\mu} - 1|\}. \qquad (17)$$

Proof: Let $\overline{F}_t(x)$ be a DHR survival function, then there exist $\epsilon \geq \mu$ such that $\overline{F}_t(x) \leq e^{-x/\mu}$ or $\overline{F}_t(x) \geq e^{-x/\mu}$ as $x \leq \epsilon$ or $x \geq \epsilon$. Now,

$$\begin{aligned}
\int_0^\infty |\overline{F}_t(x) - e^{-x/\mu}|dx &= 2\int_\epsilon^\infty (\overline{F}_t(x) - e^{-x/\mu})dx \\
&= 2\int_\epsilon^\infty \left(\frac{\overline{F}(x+t)}{\overline{F}(t)} - e^{-x/\mu}\right)dx \\
&\geq 2\int_\epsilon^\infty (\overline{F}(x+t) - e^{-x/\mu})dx \\
&= 2(S_1(\epsilon+t) - \mu e^{-\epsilon/\mu}) \\
&\geq 2(\mu e^{-(\epsilon+t)/\mu} - \mu e^{-\epsilon/\mu}) \\
&= 2\mu e^{-\epsilon/\mu}(e^{-t/\mu} - 1). \qquad (18)
\end{aligned}$$

The first inequality follows from the fact that $\overline{F}_t(x) \geq \overline{F}(x+t)$ for all $x \geq 0$. The last inequality follow from Theorem 3.1.

Theorem 3.8 *If $\overline{F}_t(x)$ is a DHR reliability function, then*

$$\int_0^\infty |\overline{F}_t(x) - xe^{-x/\mu}|dx \geq max(0, 2\mu(e^{-(\epsilon+t)/\mu} - 1)). \qquad (19)$$

Proof: Using the fact that $\overline{F}_t(x)$ is a DHR survival function, we have, for $\epsilon \geq \mu$

$$\begin{aligned}
\int_0^\infty |\overline{F}_t(x) - xe^{-x/\mu}|dx &= 2\int_\epsilon^\infty (\overline{F}_t(x) - xe^{-x/\mu})dx \\
&= 2\int_\epsilon^\infty \left(\frac{\overline{F}(x+t)}{\overline{F}(t)} - xe^{-x/\mu}\right)dx \\
&\geq 2\int_\epsilon^\infty \left(\overline{F}(x+t) - xe^{-x/\mu}\right)dx \\
&= 2\left(S_1(\epsilon+t) - \int_\epsilon^\infty xe^{-x/\mu}dx\right) \\
&\geq 2\left(S_1(\epsilon+t) - \int_0^\infty xe^{-x/\mu}dx\right) \\
&= 2\mu(e^{-(\epsilon+t)/\mu} - 1), \qquad (20)
\end{aligned}$$

where $\mu = \int_0^\infty \overline{F}(x)dx$. The inequalities follows from the fact that $W(x)$ is increasing, so that $\overline{F}_t(x) \geq \overline{F}(x+t)$ for all $x \geq 0$, and Theorem 3.1.

Theorem 3.9 *If $\overline{F}_t(x)$ is a DHR reliability function, then*

$$\int_0^\infty |\overline{F}_t(x) - (1 + x/(\mu+t))e^{-x/\mu}|dx \geq 2max\Big\{0, \mu\Big(e^{-(\epsilon+t)/\mu} - \frac{\mu+t+1}{\mu+t}\Big)\Big\}. \tag{21}$$

Proof: Let $\overline{F}_t(x)$ be a DHR survival function, then there exist $\epsilon \geq \mu$ such that $\overline{F}_t(x) \leq (1 + x/(\mu+t))e^{-x/\mu}$ or $\overline{F}_t(x) \geq (1 + x/(\mu+t))e^{-x/\mu}$ as $x \leq \epsilon$ or $x \geq \epsilon$. Now,

$$\begin{aligned}
\int_0^\infty \Big|\overline{F}_t(x) - \Big(1 + \frac{x}{\mu+t}\Big)e^{-x/\mu}\Big|dx &= 2\int_\epsilon^\infty \big(\overline{F}_t(x) - (1 + x/(\mu+t))e^{-x/\mu}\big)\,dx \\
&\geq 2\int_\epsilon^\infty \Big(\overline{F}(x+t) - (1 + \frac{x}{\mu+t})e^{-x/\mu}\Big)\,dx \\
&= 2S_1(\epsilon+t) - 2\int_\epsilon^\infty \Big(1 + \frac{x}{\mu+t}\Big)\,e^{-x/\mu}dx \\
&\geq 2S_0(\epsilon+t) - 2\int_0^\infty \Big(1 + \frac{x}{\mu+t}\Big)\,e^{-x/\mu}dx \\
&\geq 2\Big(\mu e^{-(\epsilon+t)/\mu} - \mu - \frac{\mu}{\mu+t}\Big) \\
&= 2\mu\Big(e^{-(\epsilon+t)/\mu} - \frac{\mu+t+1}{\mu+t}\Big).
\end{aligned} \tag{22}$$

The first inequality follows from the fact that $\overline{F}_t(x) \geq \overline{F}(x+t)$ for all $x \geq 0$. The last inequality follow from Theorem 3.1.

4 Applications

In this section, we give some applications of the results presented in this paper.

Example 1. Gamma Distribution. Let

$$f(x;\alpha,\beta) = \begin{cases} \frac{x^{\alpha-1}\beta^\alpha}{\Gamma(\alpha)}e^{-x/\beta} & \text{if } x > 0,\ \alpha > 0,\ \beta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu = \alpha\beta$ and $\mu_2 = \alpha(\alpha+1)\beta^2$ and the hazard rate is increasing for $\alpha \geq 1$ and decreasing for $\alpha \leq 1$. If $W(x) = x$, then the weighted pdf is given by

$$f_W(x;\alpha,\beta) = \begin{cases} \frac{x^{\alpha}\beta^{\alpha+1}}{\Gamma(\alpha+1)}e^{-x/\beta} & \text{if } x > 0,\ \alpha > 0,\ \beta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 3.3, for $\alpha > 1$, we have

$$\begin{aligned}\int_0^\infty |\overline{F}_t(x) - xe^{-x/\alpha\beta}|dx &\leq 2\alpha\beta|1-(\alpha(\alpha+1)/2\alpha^2)| \\ &= \beta|\alpha-1|. \end{aligned} \tag{23}$$

With $\beta = 1/4$,

$$\begin{aligned} C(\alpha) &= \int_0^\infty |\overline{F}_t(x) - xe^{-2x/\alpha}|dx \\ &\leq |\alpha-1|/4. \end{aligned} \tag{24}$$

Example 2. Shifted Exponential Distribution. Consider the survival or reliability function given by

$$\overline{F}(x;\theta,\epsilon) = \begin{cases} e^{-(x-\theta)/(1-2\epsilon)^{1/2}} & \text{if } x > \theta,\ \theta = 1-(1-2\epsilon)^{1/2} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, the first and second moments of F are $\mu = 1$ and $\mu_2 = 2(1-\epsilon)$ respectively. Since the failure rate function $\lambda_F(x)$ is increasing, we obtain

$$\begin{aligned} L(\epsilon) &= \int_0^\infty |\overline{F}_t(x) - e^{-(x-\theta)/(1-2\epsilon)^{1/2}}|dx \\ &\leq 2\mu|1-\frac{\mu_2}{2\mu^2}| \\ &= 2|1-2(1-\epsilon)/2|. \end{aligned} \tag{25}$$

Consequently,

$$L(\epsilon) = \int_0^\infty |\overline{F}_t(x) - e^{-(x-\theta)/(1-2\epsilon)^{1/2}}|dx \leq 2\epsilon.$$

Acknowledgment: The authors expresses their gratitude to the editor and referee for many useful comments and suggestions.

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Received: December 10, 2007