Applied Mathematical Sciences, Vol. 2, 2008, no. 18, 883 - 899

Sufficient and Necessary Conditions for Existence of Positive Solutions to p-Laplacian Systems

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Abstract

In this work, we are concerned with the study of existence of positive solutions to some p-Laplacian systems on an unbounded domain, with non-linearities without a necessary variational character. Under some sufficient constraints, we use the Leray-Schauder Theorem to establish the existence and we present a necessary condition for existence at the end of this work.

Mathematics Subject Classification: 35B40, 35L70

Keywords: p-Laplacian operator, variational method, sub and super-solution for the system, Mountain Pass Geometry, Leray-Schauder Theorem, Picone Identity

1 Introduction

Consider the following non-linear elliptic system

$$(\mathcal{S})\begin{cases} -\Delta_p u = \mu f(x,u,v) & x \in \mathbb{R}^N \\ -\Delta_q v = \mu g(x,u,v) & x \in \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) = 0 = \lim\limits_{|x|\to\infty} v(x) & \end{cases}$$

where $max(p,q) < N,\ p,\ q > 1$, and $f,\ g$ are non-negative continues functions which are monotone in relation to v and u respectively. Our purpose is to establish sufficient condition on the parameter μ for the existence of positive solutions to $(\mathcal{S})$. Via a variational method, we may apply the pass mountain theorem to check on the existence of super-solution. Whereas, we look for

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a positive sub-solution by imposing a supplementary condition and we can set up the frame work to apply the Leray-Schauder's theorem. In the end of this work, we prove a necessary condition on the parameter μ for existence of positive solutions to the problem $(\mathcal{S})$.

A. Anane ([3]), G. Barles ([4]) and S. Sakaguchi ([9]) have proved the existence of a unique solution to the eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } B$$

where B is a bounded domain of $\mathbb{R}^N$, and the same result was established by J.I. Diaz and J.E. Saa ([6]) for the equation

$$-\Delta_p u = f(x,u) \quad \text{in } B$$

where the function $r \to \frac{f(x,r)}{r^{p-1}}$ is non-increasing. As for the case of systems, A. Ahammou ([1]) has studied the non-necessarily potential elliptic system

$$\begin{cases} -\Delta_p u = f(x,u,v) & \text{in } B \\ -\Delta_q v = g(x,u,v) & \text{in } B \\ u = v = 0 & \text{on } \partial B \end{cases} \tag{1}$$

under some lower limit conditions associated to F and G, where

$$F(x,u,v) = \int_0^u f(x,t,v)dt, \qquad G(x,u,v) = \int_0^v g(x,u,s)ds,$$

he proved existence of bounded solutions to (1) using a shooting technique. When we are concerned with potential systems of the form

$$\begin{cases} -\Delta_p u = \dfrac{\partial H}{\partial u}(u,v) + h_1 & \text{in } B \\ \\ -\Delta_q v = \dfrac{\partial H}{\partial v}(u,v) + h_2 & \text{in } B \\ u = v = 0 & \text{on } \partial B \end{cases} \tag{2}$$

A. Ahammou ([2]) placed only some lower limit conditions on the potential function H, which is associated to (2) and assumed to have an oscillatory behavior at infinity, that lead to the existence of infinitely many solutions for the system (2).

We point out that our system is not necessarily hamiltonian as in the work of A. El Khalil, M. Ouanan and A. Touzani ([7]) who have studied the question of the existence of positive solutions in a bounded domain, therefore, we may not apply directly the Mountain Pass Theorem to prove existence of such solutions. Concerning the unbounded case, see the work of K. Chaib ([5]) where he proved the existence of non-negative non-trivial solutions by imposing a variational character on the polynomial functions controlling f and g, but when the polynomial functions are not necessary variational, he used a special symmetric p-Laplacian system $(p = q)$ to prove the existence of a super-solution. In this paper, we consider general non-linearities which verify some monotony without imposing the condition: $f(x,0,0) + g(x,0,0) \not\equiv 0$.

2 Preliminaries

In the sequel, $D^{1,p}\left(\mathbb{R}^N\right)$ denotes the adherence of $C_0^\infty\left(\mathbb{R}^N\right)$ with respect to the norm

$$\|u\|_{D^{1,p}(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

and henceforth, p^* and q^* denote the critical Sobolev exponents

$$p^* = \tfrac{Np}{N-p} \quad , \quad q^* = \tfrac{Nq}{N-q}.$$

Next, we define two fundamental notions related to the application of the Mountain Pass Theorem.

Definition 1 *A functional J in $C^1\left(D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)\right)$ is said to have a Mountain Pass Geometry if the following conditions hold:*
1. There exist $r > 0$ and $c > 0$ such that

$$\|u\|_{D^{1,p}(\mathbb{R}^N)} + \|v\|_{D^{1,q}(\mathbb{R}^N)} = r \quad \textit{implies} \quad J(u,v) > c$$

2. There exists $(\tilde{u},\tilde{v})$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ such that

$$\|\tilde{u}\|_{D^{1,p}(\mathbb{R}^N)} + \|\tilde{v}\|_{D^{1,q}(\mathbb{R}^N)} > r \quad \textit{and} \quad J(\tilde{u},\tilde{v}) < c.$$

Definition 2 *A functional $J \in C^1\left(D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right), \mathbb{R}\right)$ verifies the Palais-Smale condition if any sequence (u_n, v_n) of $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$ verifying*
1. $|J(u_n,v_n)| < C$;
2. $J'(u_n,v_n)$ converges to 0 in $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$;
possesses a convergent sub-sequence in $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$.

Let α and β be some exponents verifying

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} < 1 < \frac{\alpha+1}{p} + \frac{\beta+1}{q},$$

and consider the functions $F,\ G$ and H defined as:

$$F(x,u,v)=\int_0^u f(x,s,v)ds, \qquad G(x,u,v)=\int_0^v g(x,u,s)ds,$$

and

$$H(x,u,v)=F(x,u,v)+G(x,u,v).$$

The main hypotheses of this paper are formulated as follows:

$(\mathcal{H}1)$ The functions $(r,s)\to f(x,r,s)$ and $(r,s)\to g(x,r,s)$ are continues on $[0,+\infty)\times[0,+\infty)$ for almost every $x\in\mathbb{R}^N$, $s\to f(x,r,s)$ and $r\to g(x,r,s)$ are non-decreasing, $x\to f(x,r,s)\,,\ x\to g(x,r,s)$, $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial g}{\partial r}$ are measurable for almost every $x\in\mathbb{R}^N$.

$(\mathcal{H}2)$ $i/$ For any compact $K\subseteq\mathbb{R}^2$ there exists function $C_K\in L^1(\mathbb{R}^N)$ such that $\forall\ (r,s)\in K$:

$$H(x,r,s)<C_K(x) \quad \text{a.e } x\in\mathbb{R}^N$$

$ii/$ $max(\lim\limits_{|r||s|\to+\infty}\dfrac{\frac{\partial H}{\partial r}(x,r,s)}{r^{\alpha}s^{\beta+1}},\lim\limits_{|r||s|\to+\infty}\dfrac{\frac{\partial H}{\partial s}(x,r,s)}{r^{\alpha+1}s^{\beta}})<b(x)$ a.e $x\in\mathbb{R}^N$ with $b\in L^{\eta_b}\left(\mathbb{R}^N\right)\cap L^{\infty}_{loc}\left(\mathbb{R}^N\right)$ and $\eta_b=\frac{p^*q^*}{p^*q^*-(\alpha+1)q^*-(\beta+1)p^*}$

$(\mathcal{H}3)$ $i/$ There exist positive numbers $r_0,\ \theta$ and λ with $min(\theta,\lambda)>max(p,q)$ such that for all $|r|+|s|>r_0$ we have

$$r^{\alpha+1}s^{\beta+1}\le H(x,r,s)\le\frac{1}{\theta}\frac{\partial H}{\partial r}(x,r,s)r+\frac{1}{\lambda}\frac{\partial H}{\partial s}(x,r,s)s \quad \text{a.e on } \mathbb{R}^N$$

$ii/$ For any compacts $K'\subseteq\mathbb{R}^2$) there exists function $C_{K'}$ such that $\forall\ (r,s)\in K'$:

$$max\left(\frac{\partial H}{\partial r}(x,r,s),\frac{\partial H}{\partial s}(x,r,s)\right)<C_{K'}(x) \quad \text{a.e } x\in\mathbb{R}^N$$

with $C_{K'}\in L^{\eta_p}\left(\mathbb{R}^N\right)\cap L^{\eta_q}\left(\mathbb{R}^N\right)\cap L^{\infty}_{loc}\left(\mathbb{R}^N\right),\ \eta_p=\frac{p^*}{p^*-1}$ and $\eta_q=\frac{q^*}{q^*-1}$.

$(\mathcal{H}4)$ There exists $A\in L^{\frac{N}{\varrho+\sigma+2}}\left(\mathbb{R}^N\right)\cap L^{\infty}_{loc}\left(\mathbb{R}^N\right)$ with $A>0$ such that for any $u,\ v\ge 0$

$$f(x,u,v)\ge\lambda_1 A(x)u^{\varrho}v^{\sigma+1} \quad \text{a.e on } \mathbb{R}^N$$
$$g(x,u,v)\ge\lambda_1 A(x)u^{\varrho+1}v^{\sigma} \quad \text{a.e on } \mathbb{R}^N,$$

where is related to the eigenvalue problem

$$(\mathcal{V}_p)\begin{cases}-\Delta_p\varphi=\lambda_1 B(x)\varphi^{\varrho}\psi^{\sigma+1} & \text{on } \mathbb{R}^N\\ \\ -\Delta_q\psi=\lambda_1 B(x)\varphi^{\varrho+1}\psi^{\sigma} & \text{on } \mathbb{R}^N\end{cases}$$

with $B \in L^{\frac{N}{\varrho+\sigma+2}}\left(\mathbb{R}^N\right) \cap L^{\infty}_{loc}\left(\mathbb{R}^N\right),\ B \not\equiv 0,\ B \leq A$ and $\frac{\varrho+1}{p} + \frac{\sigma+1}{q} = 1.$

3 Sufficient condition for existence

This part is devoted to the sufficient condition on the parameter μ for existence of non-negative solutions to the problem $(\mathcal{S})$.

Theorem 3.1 *Under the hypotheses* $(\mathcal{H}1), (\mathcal{H}2), (\mathcal{H}3)$ *and* $(\mathcal{H}4)$, *there exists a constant* $\mu^*(N,\ p,\ q,\ \alpha,\ \beta,\ ||b||_{L^{\eta_b}})$ *such that for any* $\mu \in (0, \mu^*)$, *the problem* $(\mathcal{P})$ *has a non-negative non-trivial solution in* $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$.

First of all, we show existence of sub,super-solution for the system $(\mathcal{S})$.

Existence of super-solution

The existence of super-solution remains to show existence of non-negative non-trivial solutions to the problem

$$(\mathcal{S}_h)\begin{cases} -\Delta_p u = \mu \dfrac{\partial H}{\partial u} = \mu\left[f(x,u,v) + \displaystyle\int_0^v \frac{\partial g}{\partial u}(x,u,s)ds\right] & \text{on } \mathbb{R}^N \\ -\Delta_q v = \mu \dfrac{\partial H}{\partial v} = \mu\left[g(x,u,v) + \displaystyle\int_0^u \frac{\partial f}{\partial v}(x,s,v)ds\right] & \text{on } \mathbb{R}^N \end{cases}$$

where

$$\begin{aligned} H(x,u,v) &= \int_0^u f(x,s,v)ds + \int_0^v g(x,u,s)ds \\ &= F(x,u,v) + G(x,u,v). \end{aligned}$$

Its well known that weak solutions of $(\mathcal{S}_h)$ correspond to critical points of the C^1 functional $J : D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right) \to \mathbb{R}$, given by:

$$J(u,v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla v|^q dx - \mu \int_{\mathbb{R}^N} H(x,u,v)dx.$$

Since $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$ is a reflexive Banach space, the Mountain Pass Theorem allows to prove only tow fundamental points:
- J satisfies the Palais-Smale condition;
- J has a Mountain Pass Geometry.

Proposition 1 J *satisfies the Palais-Smale condition.*

Proof:

Consider a sequence $(u_n, v_n)_n$ of $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$ verifying the properties of Definition(1). We show first that (u_n, v_n) is bounded in $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$. Here we proceed by contradiction and we suppose that $(u_n, v_n)_n \underset{n\to\infty}{\longrightarrow} +\infty$.

Since $J'(u_n, v_n)$ converges to 0 in $D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$, there exists a non-increasing sequence $(\epsilon_n)_n$ converging to 0 such that

$$|J'(u_n, v_n)(\varphi, \psi)| \le \epsilon_n \|(\varphi, \psi)\|_{D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)}$$

for any $(\varphi, \psi) \in D^{1,p}\left(\mathbb{R}^N\right) \times D^{1,q}\left(\mathbb{R}^N\right)$.

If we choose $\varphi = u_n$ and $\psi = v_n$, we get

$$|J'(u_n, v_n)(u_n, v_n)| \le \epsilon_n \|u_n, v_n\|_{D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)}$$

thus

$$\begin{aligned}\frac{-\epsilon_n \|u_n, v_n\|_{D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)}}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}} &\le 1 - \frac{\mu \displaystyle\int \left(\frac{\partial H}{\partial u}(x, u_n, v_n)u_n + \frac{\partial H}{\partial v}(x, u_n, v_n)v_n\right)}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}} \\ &\le \frac{\epsilon_n \|u_n, v_n\|_{D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)}}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}}\end{aligned}$$

consequently

$$\lim_{n\to\infty} \frac{\mu \displaystyle\int \left(\frac{\partial H}{\partial u}(x, u_n, v_n)u_n + \frac{\partial H}{\partial v}(x, u_n, v_n)v_n\right)}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}} = 1. \tag{3}$$

In addition, $|J(u_n, v_n)| < C$, so

$$\begin{aligned}\frac{-C}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}} &\le \frac{\frac{1}{p}\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{1}{q}\|v_n\|^q_{D^{1,q}(\mathbb{R}^N)} - \mu \displaystyle\int H(x, u_n, v_n)}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}} \\ &\le \frac{C}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}},\end{aligned}$$

and then

$$\lim_{n\to\infty} \frac{\mu \displaystyle\int H(x, u_n, v_n)}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)} + \|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}} \ge min\left(\frac{1}{p}, \frac{1}{q}\right). \tag{4}$$

We deduce from $(\mathcal{H}3)_i$, (3) and (4) that

$$\begin{aligned} min\left(\frac{1}{p},\frac{1}{q}\right) &\leq \frac{\mu\int\left(\frac{1}{\theta}\frac{\partial H}{\partial u}(x,u_n,v_n)u_n+\frac{1}{\lambda}\frac{\partial H}{\partial v}(x,u_n,v_n)v_n\right)}{\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)}+\|v_n\|^q_{D^{1,q}(\mathbb{R}^N)}} \\ &\leq max\left(\frac{1}{\theta},\frac{1}{\lambda}\right) \\ &< min\left(\frac{1}{p},\frac{1}{q}\right) \end{aligned}$$

hence a contradiction. Consequently $(u_n,v_n)_n$ is bounded in $D^{1,p}\left(\mathbb{R}^N\right)\times D^{1,q}\left(\mathbb{R}^N\right)$.

Next, we prove that $(u_n)_n$ is relatively compact in $D^{1,p}(\mathbb{R}^N)$ (it will be possible to argue similarly about $(v_n)_n$). In this respect, we should prove that $(u_n)_n$ is a Cauchy sequence. We have

$$\begin{aligned} 0 &\leq \int\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_n-\nabla u_m) \\ &\leq \left(J'(u_n,v_n)-J'(u_m,v_m)\right)(u_n-u_m,0) \\ &\quad +\mu\int\left(\frac{\partial H}{\partial u}(x,u_n,v_n)-\frac{\partial H}{\partial u}(x,u_m,v_m)\right)(u_n-u_m)\,dx \end{aligned}$$

where $\left(J'(u_n,v_n)-J'(u_m,v_m)\right)(u_n-u_m,0)$ converges to zero when n and m tend to ∞.

From $(\mathcal{H}2)_{ii}$, there exists $k_0>0$ such that for all $|r|+|s|>k_0$

$$\frac{\partial H}{\partial r}(x,r,s)\leq b(x)r^{\alpha}s^{\beta+1}. \tag{5}$$

Set

$$A_n=\{x\in\mathbb{R}^N/|u_n(x)|+|v_n(x)|\leq k_0\}.$$

From (5) we get

$$\forall x\in A_n^c:\quad \frac{\partial H}{\partial u}(x,u_n,v_n)\leq b(x)u_n^{\alpha}v_n^{\beta+1}\quad \text{a.e}\quad x\in\mathbb{R}^N, \tag{6}$$

and by $(\mathcal{H}3)_{ii}$, there exists function c_{k_0} such that

$$\forall x\in A_n:\quad \frac{\partial H}{\partial u}(x,u_n,v_n)\leq c_{k_0}(x)\quad \text{a.e}\quad x\in\mathbb{R}^N. \tag{7}$$

Hence, from (6) we can write

$$\begin{aligned}
&\int_{A_n^c}\left(\frac{\partial H}{\partial u}(x,u_n,v_n)-\frac{\partial H}{\partial u}(x,u_m,v_m)\right)(u_n-u_m)\,dx\\
&\quad\le \int_{A_n^c} b(x)\left(u_n^\alpha v_n^{\beta+1}+u_m^\alpha v_m^{\beta+1}\right)(u_n-u_m)\,dx\\
&\quad\le \int_{\mathbb{R}^N} b(x)\left(u_n^\alpha v_n^{\beta+1}+u_m^\alpha v_m^{\beta+1}\right)(u_n-u_m)\,dx\\
&\quad\le \|b\|_{L^\infty(B_R)}\left(\|u_n^\alpha v_n^{\beta+1}\|_{L^r(B_R)}+\|u_m^\alpha v_m^{\beta+1}\|_{L^r(B_R)}\right)\|u_n-u_m\|_{L^{r'}(B_R)}\\
&\qquad+\|b\|_{L^{\eta_b}(B_{R^c})}\Big[\|u_n\|^\alpha_{L^{p^*}(B_{R^c})}\|v_n\|^{\beta+1}_{L^{q^*}(B_{R^c})}\\
&\qquad\qquad+\|u_m\|^\alpha_{L^{p^*}(B_{R^c})}\|v_m\|^{\beta+1}_{L^{q^*}(B_{R^c})}\Big]\left(\|u_n\|_{L^{p^*}(B_{R^c})}+\|u_m\|_{L^{q^*}(B_{R^c})}\right)\\
&\quad\le \underbrace{\|b\|_{L^\infty(B_R)}\left(\|u_n\|^\alpha_{L^{\alpha rs}(B_R)}\|v_n\|^{\beta+1}_{L^{(\beta+1)rs'}(B_R)}+\|u_m\|^\alpha_{L^{\alpha rs}(B_R)}\|v_m\|^{\beta+1}_{L^{(\beta+1)rs'}(B_R)}\right)\|u_n-u_m\|_{L^{r'}(B_R)}}_{(I_1)}\\
&\qquad+\underbrace{\|b\|_{L^{\eta_b}(B_{R^c})}\left[\|u_n\|^\alpha_{L^{p^*}(\mathbb{R}^N)}\|v_n\|^{\beta+1}_{L^{q^*}(\mathbb{R}^N)}+\|u_m\|^\alpha_{L^{p^*}(\mathbb{R}^N)}\|v_m\|^{\beta+1}_{L^{q^*}(\mathbb{R}^N)}\right]\left(\|u_n\|_{L^{p^*}}+\|u_m\|_{L^{q^*}}\right)}_{(I_1')}
\end{aligned}$$

where r, s, r' and s' are chosen as follows:

$$\frac{1}{r}=\frac{\alpha}{p^*}+\frac{\beta+1}{q^*}\quad,\quad r'=\frac{r}{r-1}$$

$$s=\frac{p^*}{\alpha r}\quad,\quad s'=\frac{s}{s-1}=\frac{q^*}{(\beta+1)r}$$

with $\alpha rs=p^*$, $(\beta+1)rs'=q^*$, and as $\frac{\alpha+1}{p^*}+\frac{\beta+1}{q^*}<1$ we can verify that $r'<p^*$.

Next, by (7) we get

$$\begin{aligned}
&\int_{A_n}\left(\frac{\partial H}{\partial u}(x,u_n,v_n)-\frac{\partial H}{\partial u}(x,u_m,v_m)\right)(u_n-u_m)\,dx\\
&\quad\le \int_{A_n} c_{k_0}(x)\,(u_n-u_m)\,dx\\
&\quad\le \int_{\mathbb{R}^N} c_{k_0}(x)\,(u_n-u_m)\,dx\\
&\quad\le \underbrace{\|c_{k_0}\|_{L^\infty(B_R)}\|u_n-u_m\|_{L^1(B_R)}}_{(I_2)}+\underbrace{\|c_{k_0}\|_{L^{\eta_p}(B_R^c)}\left(\|u_n\|_{L^{p^*}(\mathbb{R}^N)}+\|u_m\|_{L^{p^*}(\mathbb{R}^N)}\right)}_{(I_2')}
\end{aligned}$$

Now, observe that (I_1) and (I_2) converge to zero when n and m tend to ∞ since $(u_n)_n, (v_n)_n$ are bounded respectively in $L^{p^*}(B_R)$ and $L^{q^*}(B_R)$, and $(u_n)_n$ converges in $L^{p^*}(B_R)$. Indeed, since we have proved that $(u_n)_n$ is bounded in $D^{1,p}\left(\mathbb{R}^{\mathbb{N}}\right)$, it is also bounded in $D^{1,p}(B_R)$ and so in $L^\zeta(B_R)$ $(1\le\zeta\le p^*)$, consequently, we can extract a subsequence $(u_n)_n$ converging in $L^{\zeta}(B_R)$. The remaining term (I_1') and (I_2') can be token as small as we need by choosing R sufficiently large. Hence, (∇u_n) and the same for (∇v_n), are Cauchy sequences in $\left(L^p(\mathbb{R}^N)\right)^N$ and $\left(L^q(\mathbb{R}^N)\right)^N$ respectively, i.e (u_n, v_n) converges strongly in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.

Proposition 2 *J has a Mountain Pass Geometry.*

Proof:
1. Consider a pair (u, v) and a positive number r such that

$$\|u\|_{D^{1,p}(\mathbb{R}^N)} + \|v\|_{D^{1,q}(\mathbb{R}^N)} = r$$

then we obtain

$$\frac{1}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} + \frac{1}{q}\|v\|^q_{D^{1,q}(\mathbb{R}^N)} \geq \min\left(\frac{r^p}{p2^p}, \frac{r^q}{q2^q}\right).$$

Now, from $(\mathcal{H}2)_{ii}$ and $(\mathcal{H}3)_i$, there exists $k_0' > 0$ such that for all $|r|+|s| > k_0'$

$$H(x, r, s) \leq b(x) r^{\alpha+1} s^{\beta+1} \quad \text{a.e on } \mathbb{R}^N. \tag{8}$$

Set

$$A_n = \{x \in \mathbb{R}^N / |u_n(x)| + |v_n(x)| \leq k_0'\}$$

From (8) we get

$$\begin{aligned}\int_{A_n^c} H(x, u_n, v_n)dx &\leq \int_{A_n^c} b(x) u_n^{\alpha+1}(x) v_n^{\beta+1}(x)dx \\ &\leq \|b\|_{L^{\eta_b}(\mathbb{R}^N)} C_{N,p}^{\alpha+1} \|u_n\|^{\alpha+1}_{D^{1,p}(\mathbb{R}^N)} C_{N,q}^{\beta+1} \|v_n\|^{\beta+1}_{D^{1,q}(\mathbb{R}^N)} \\ &\leq C_{N,p,q,b} r^{\alpha+\beta+2}\end{aligned} \tag{9}$$

Next, from $(\mathcal{H}2)_i$, there exists function $c_{k_0'}$ such that

$$\begin{aligned}\int_{A_n} H(x, u_n, v_n)dx &< \int_{A_n} c_{k_0'}(x)dx \\ &< \infty\end{aligned} \tag{10}$$

Then, from (9) and (10) we conclude

$$J(u, v) \geq min\left(\frac{r^p}{p2^p}, \frac{r^q}{q2^q}\right) - \mu C_{N,p,q,||b||_{L^{\eta_b}}} r^{\alpha+\beta+2} - \mu C$$

hence, there exist $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, we can find $r > 0$ verifying $\|u, v\|_{D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)} = r$ and $J(u, v) > 0$.

2. Let (w,z) be a fixed positive pair such that $wz>0$ on a bole of $\mathbb{R}^N$, and k a positive constant, then we have

$$J(k^{\frac{1}{p}}w,k^{\frac{1}{q}}z)=\frac{k}{p}\int_{\mathbb{R}^N}|\nabla w|^p dx+\frac{k}{q}\int_{\mathbb{R}^N}|\nabla z|^q dx-\mu\int_{\mathbb{R}^N}H(x,k^{\frac{1}{p}}w,k^{\frac{1}{q}}z)dx.$$

From $(\mathcal{H}3)_i$, we obtain for all k sufficiently large

$$H(x,k^{\frac{1}{p}}w,k^{\frac{1}{q}}z)\geq k^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}w^{\alpha+1}z^{\beta+1}$$

with $\frac{\alpha+1}{p}+\frac{\beta+1}{q}>1$, hence

$$J(k^{\frac{1}{p}}w,k^{\frac{1}{q}}z)\leq\ \frac{k}{p}\int_{\mathbb{R}^N}|\nabla w|^p dx+\frac{k}{q}\int|\nabla z|^q dx-\mu k^{\frac{\alpha+1}{p}+\frac{\beta+1}{q}}\int_{\mathbb{R}^N}w^{\alpha+1}z^{\beta+1}dx$$

which converges to $-\infty$ as k tends to $+\infty$.

Finally, from Proposition(1), Proposition(2) and the Mountain Pass Theorem, there exists a positive solution (u^0,v^0) to the system $(\mathcal{S}_h)$, which is a super-solution for the system $(\mathcal{S})$.

Construction of sub-solution

Consider the following eigenvalue system

$$(\mathcal{V}_p)\begin{cases}-\Delta_p\varphi=\lambda_1 B(x)\varphi^{\varrho}\psi^{\sigma+1} & \text{on } \mathbb{R}^N\\ \\ -\Delta_q\psi=\lambda_1 B(x)\varphi^{\varrho+1}\psi^{\sigma} & \text{on } \mathbb{R}^N\end{cases}$$

with $B\in L^{\frac{N}{\varrho+\sigma+2}}\left(\mathbb{R}^N\right)\cap L^{\infty}_{loc}\left(\mathbb{R}^N\right)$, $B\not\equiv 0$ and $B\leq A$.

We can show that there exist positive numbers r and s such that for any solution (φ,ψ) of the problem $(\mathcal{V}_p)$ we get

$$\begin{cases}-\Delta_p(\varepsilon^r\varphi)=\lambda_1 B(x)(\varepsilon^r\varphi)^{\varrho}(\varepsilon^s\psi)^{\sigma+1} & \text{on } \mathbb{R}^N\\ \\ -\Delta_q(\varepsilon^s\psi)=\lambda_1 B(x)(\varepsilon^r\varphi)^{\varrho+1}(\varepsilon^s\psi)^{\sigma} & \text{on } \mathbb{R}^N\end{cases}$$

for every $\varepsilon>0$. Hence, for ε sufficiently small, we obtain $0<\varepsilon^r\varphi<u^0$ and $0<\varepsilon^s\psi<v^0$ where (u^0,v^0) is the pair of super-solution of the problem $(\mathcal{S})$.

From the hypothesis $(\mathcal{H}4)$ we can write

$$f\left(x,\varepsilon^r\varphi,v\right)\geq\lambda_1 B(x)(\varepsilon^r\varphi)^{\varrho}(\varepsilon^s\psi)^{\sigma+1}\qquad\forall v\geq\varepsilon^s\psi$$

and

$$g\left(x,u,\varepsilon^s\psi\right)\geq\lambda_1 B(x)(\varepsilon^r\varphi)^{\varrho+1}(\varepsilon^s\psi)^{\sigma}\qquad\forall u\geq\varepsilon^r\varphi$$

hence $(u_0, v_0) = (\varepsilon^r\varphi, \varepsilon^s\psi)$ verifies

$$\begin{cases} -\Delta_p u_0 - \mu f(x, u_0, v) \le 0 & \forall v \ge v_0 \\ \\ -\Delta_q v_0 - \mu g(x, u, v_0) \le 0 & \forall u \ge u_0. \end{cases}$$

Consequently, the pair (u_0, v_0) is a positive sub-solution to the problem $(\mathcal{S})$.

Next, to use the fixed point theorem of Leray-Schauder, we define an operator $T : K \to E = L^{p^*}(\mathbb{R}^N) \times L^{q^*}(\mathbb{R}^N),\ K = [u_0, u^0] \times [v_0, v^0] = K_u \times K_v \subset E$. For any $(\bar{u}, \bar{v}) \in K,\ (w, z) = T(\bar{u}, \bar{v})$ is the unique solution to the system $(\mathcal{P}_d)$

$$(\mathcal{P}_d)\begin{cases} -\Delta_p w + Mk_p(x)|w|^{p-2}w = \mu f(x, \bar{u}, \bar{v}) + Mk_p(x)\bar{u}^{p-1} & \text{on } \mathbb{R}^N \\ \\ -\Delta_q z + Mk_q(x)|z|^{q-2}z = \mu g(x, \bar{u}, \bar{v}) + Mk_q(x)\bar{v}^{q-1} & \text{on } \mathbb{R}^N \end{cases}$$

where we choose the functions k_p and k_q in $L^{\frac{N}{p}}(\mathbb{R}^N)$ and $L^{\frac{N}{q}}(\mathbb{R}^N)$, such that: $\inf\limits_{x\in\mathbb{R}^N} \{k_p(x), k_q(x)\} > 0$, and M is a positive constant.

We have the following result.

Proposition 3 *T has a fixed point in K if the following conditions are satisfied:*
$(\mathcal{C}1)$ *K is a bounded closed and convex subset of E;*
$(\mathcal{C}2)$ *T is well defined on K;*
$(\mathcal{C}3)$ *K is invariant under T;*
$(\mathcal{C}4)$ *T is a compact operator on K.*

Proof:
$(\mathcal{C}1)$ It's easy to verify this condition since $(u_0, u^0) \in \left(D^{1,p}(\mathbb{R}^N)\right)^2$ and $(v_0, v^0) \in \left(D^{1,q}(\mathbb{R}^N)\right)^2$ which are a reflexive Banach spaces.

$(\mathcal{C}2)$ Here, it remains to prove that $(\mathcal{P}_d)$ has an unique solution for any $(\bar{u}, \bar{v})$ in K. We should study the existence and the uniqueness in each equation separately, which can be obtained by minimization of a s.c.i convex function (O. Kavian ([8])).

$(\mathcal{C}3)$ To prove the invariance of K under T, we will show that $w \ge u_0$ (we can use the same method to prove the other inequality: $z \ge v_0$).

The pair $(u_0, v_0)\,(= (\varepsilon^r\varphi, \varepsilon^s\psi))$ is a sub-solution to the problem $(\mathcal{S})$, and satisfies

$$\begin{cases} -\Delta_p u_0 - \mu f(x, u_0, v) \le 0, & \forall v \in (v_0, v^0) \\ \\ -\Delta_q v_0 - \mu g(x, u, v_0) \le 0, & \forall u \in (u_0, u^0) \end{cases} \tag{11}$$

The pair (w,z) is defined by $(w,z)=T(\bar u,\bar v)$ where $(\bar u,\bar v)\in K$ and satisfies

$$\begin{cases} -\Delta_p w + Mk_p(x)|w|^{p-1} = \mu f(x,\bar u,\bar v) + Mk_p(x)\bar u^{p-1} \\ -\Delta_q z + Mk_q(x)|z|^{q-1} = \mu g(x,\bar u,\bar v) + Mk_q(x)\bar v^{q-1} \end{cases}$$

Since $(\bar u,\bar v)\in K$, we have

$$\begin{cases} -\Delta_p w + Mk_p(x)|w|^{p-1} \geq \mu f(x,\bar u,\bar v) + Mk_p(x)\bar u_0^{p-1} \\ \\ -\Delta_q z + Mk_q(x)|z|^{q-1} \geq \mu g(x,\bar u,\bar v) + Mk_q(x)\bar v_0^{q-1}. \end{cases} \tag{12}$$

Suppose that $(u_0-w)_+ = \max(0,u_0-w)\neq 0$. Multiplying the first inequalities of (11) and (12) by $(u_0-w)_+$, integrating on $\mathbb{R}^N$ and subtracting the inequalities as resulted

$$\int(-\Delta_p u_0 + \Delta_p w)(u_0-w)_+\,dx$$
$$+\int\left[\mu\left(f(x,\bar u,\bar v)-f(x,\bar u_0,\bar v)\right)+Mk_p(x)\left(u_0^{p-1}-w^{p-1}\right)\right](u_0-w)_+\,dx\leq 0.$$

Since $k_p(x)\left(u_0^{p-1}-w^{p-1}\right)(u_0-w)_+\neq 0$, we can choose M large enough to make the second integral positive, that is

$$\int\left[\mu\left(f(x,\bar u,\bar v)-f(x,\bar u_0,\bar v)\right)+Mk_p(x)\left(u_0^{p-1}-w^{p-1}\right)\right](u_0-w)_+\,dx>0,$$

thus

$$\int(-\Delta_p u_0+\Delta_p w)(u_0-w)_+\,dx<0$$

which is a contradiction with the monotony of the operator $-\Delta_p$, then $(u_0-w)_+ = 0$. Consequently, K is invariant under T.

$(\mathcal{C}4)$ Now, we should prove the continuity of T from K in K and the image of any bounded subset of K is relatively compact in K. Here, we prove the second point, as for the continuity, it is due to the regularity of f and g, indicated in the hypothesis $(\mathcal{H}1)$.

Let $(\bar u_n,\bar v_n)_n$ be a sequence in K converging weakly to $(\bar u,\bar v)$ in $L^{p^*}(\mathbb{R}^N)\times L^{q^*}(\mathbb{R}^N)$. From the previous property of T, we may assert that $(T(\bar u_n,\bar v_n))_n = (w_n,z_n)_n\in K$. We show that there exists a subsequence of $(w_n)_n$ converging strongly in K_u with respect to the norm of L^{p^*}, as for the convergence of $(z_n)_n$ the method is similar.

First, we prove that $(w_n)_n$ is bounded in K_u according to the norm of $D^{1,p}(\mathbb{R}^N)$

$$\int|\nabla w_n|^p dx + M\int k_p(x)w_n^p dx = \mu\int f(x,\bar u_n,\bar v_n)w_n dx + M\int k_p(x)\bar u_n^{p-1}w_n dx$$
$$\leq \mu\int\frac{\partial H}{\partial u}(x,\overline{u}_n,\overline{v}_n)u^0 + M\int k_p u^{0^p}dx \tag{13}$$

From $(\mathcal{H}2)_{ii}$, there exists $k_0 > 0$ such that for all $|r| + |s| > k_0$

$$\frac{\partial H}{\partial r}(x, r, s) \leq b(x) r^{\alpha} s^{\beta+1}. \tag{14}$$

Set

$$A_n = \{x \in \mathbb{R}^N / |\overline{u}_n(x)| + |\overline{v}_n(x)| \leq k_0\}$$

From (14) we get

$$\forall x \in A_n^c : \quad \frac{\partial H}{\partial u}(x, \overline{u}_n, \overline{v}_n) \leq b(x) \overline{u}_n^{\alpha} \overline{v}_n^{\beta+1} \quad \text{a.e} \quad x \in \mathbb{R}^N, \tag{15}$$

and by $(\mathcal{H}3)_{ii}$, there exists function c_{k_0} such that

$$\forall x \in A_n : \quad \frac{\partial H}{\partial u}(x, \overline{u}_n, \overline{v}_n) \leq c_{k_0}(x) \quad \text{a.e} \quad x \in \mathbb{R}^N. \tag{16}$$

Hence, from (15) we can write

$$\begin{aligned}
&\int_{A_n^c} \frac{\partial H}{\partial u}(x, \overline{u}_n, \overline{v}_n) u^0 dx \\
&\quad \leq \int_{A_n^c} b(x) \overline{u}_n^{\alpha} \overline{v}_n^{\beta+1} u^0 dx \\
&\quad \leq \int_{\mathbb{R}^N} b(x) {u^0}^{\alpha+1} {v^0}^{\beta+1} dx \\
&\quad \leq \|b\|_{L^{\eta_b}(\mathbb{R}^N)} \|u^0\|_{L^{p^*}(\mathbb{R}^N)}^{\alpha+1} \|v^0\|_{L^{q^*}(\mathbb{R}^N)}^{\beta+1}.
\end{aligned} \tag{17}$$

Next, by (16) we get

$$\begin{aligned}
&\int_{A_n} \frac{\partial H}{\partial u}(x, \overline{u}_n, \overline{v}_n) u^0 dx \\
&\quad \leq \int_{A_n} c_{k_0}(x) u^0 dx \\
&\quad \leq \int_{\mathbb{R}^N} c_{k_0}(x) u^0 dx \\
&\quad \leq \|c_{k_0}\|_{L^{\eta_p}(\mathbb{R}^N)} \|u^0\|_{L^{p^*}(\mathbb{R}^N)}.
\end{aligned} \tag{18}$$

In this way, we use (17) (18) to conclude

$$\int_{\mathbb{R}^N} \frac{\partial H}{\partial u}(x, \overline{u}_n, \overline{v}_n) u^0 < \infty$$

that is to say

$$\int_{\mathbb{R}^N} |\nabla w_n|^p dx < C.$$

Consequently, we can extract a subsequence which we note $(w_n)_n$ converging weakly in $D^{1,p}(\mathbb{R}^N)$. Then $(w_n)_n$ converges strongly in $L^{\zeta}(B_R)$ for any $R > 0$ and any $1 \leq \zeta < p^*$, it converges, then, almost every where and it's dominated by the super-solution $u^0 \in L^{p^*}(\mathbb{R}^N)$. Consequently, it converges in K_u with respect to the norm of $L^{p^*}(\mathbb{R}^N)$. We do the same work for $(z_n)_n$ to end the proof of the compactness of T.

Proof of Theorem(3.1)

The proof follows immediately from Proposition(3). We apply the Leray-Schauder fixed point theorem to the operator T, then we get existence of a positive solution (u, v) in K for the problem $(\mathcal{S})$. ■

4 Necessary condition for existence

This part deals with necessary condition for existence of positive solutions to $\mathcal{S}$.

Consider the eigenvalue problem

$$(\mathcal{V}_p)\begin{cases} -\Delta_p \varphi = \lambda_1 B(x)\varphi^{\varrho}\psi^{\sigma+1} & \text{on } \mathbb{R}^N \\ -\Delta_q \psi = \lambda_1 B(x)\varphi^{\varrho+1}\psi^{\sigma} & \text{on } \mathbb{R}^N \end{cases}$$

with $B \in L^{\frac{N}{\varrho+\sigma+2}}\left(\mathbb{R}^N\right) \cap L^{\infty}_{loc}\left(\mathbb{R}^N\right),\ B \not\equiv 0,\ B \leq A$ and $\frac{\varrho+1}{p} + \frac{\sigma+1}{q} = 1$.

We will use the following generalized Picone Identity to prove our result:

Proposition 4 ([5]) *Let u, v be a differentiable functions and $v > 0$ a.e on $\Omega \subseteq \mathbb{R}^N$. We note*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{|u|^p}{v^p}|\nabla v|^p - p\frac{|u|^{p-2}u}{v^{p-1}}\nabla u \cdot \nabla v\, |\nabla v|^{p-2} \quad a.e \text{ on } \Omega$$
$$R(u,v) = |\nabla u|^p - \nabla\left(\frac{|u|^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v \quad a.e \text{ on } \Omega.$$

Then L(u,v)=R(u,v)≥ 0.
Moreover, L(u,v)=0 a.e on Ω if and only if $\nabla\left(\frac{u}{v}\right) = 0$ a.e on Ω.

This is the main result of this section

Theorem 4.1 *We suppose that the hypotheses $(\mathcal{H}1)$ and $(\mathcal{H}4)$ are satisfied. If (u, v) is a positive solution to the problem $(\mathcal{S})$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N) \cap C^{1,\rho}(\mathbb{R}^N) \times C^{1,\tau}(\mathbb{R}^N)$ with $(\rho, \tau) \in (0, 1)^2$, then $\mu \leq 1$.*

Proof:

Let (u,v) be a positive solution to $(\mathcal{S})$, and consider two positive sequences $(\varphi_n)_{n\in N}$ and $(\psi_n)_{n\in N}$ in $C_0^\infty\left(\mathbb{R}^N\right)$ converging, respectively, to φ and ψ solutions of $(\mathcal{V}_p)$. We apply the Picone identity to the functions φ_n, u and ψ_n, v to get

$$\begin{aligned}
0 &\le \frac{\varrho+1}{p}\int_{\mathbb{R}^N} L\left(\varphi_n,u\right)dx + \frac{\sigma+1}{q}\int_{\mathbb{R}^N} L\left(\psi_n,v\right)dx \\
&\le \frac{\varrho+1}{p}\int_{\mathbb{R}^N} \left|\nabla\varphi_n\right|^p dx - \frac{\varrho+1}{p}\int_{\mathbb{R}^N} \nabla\left(\frac{\varphi_n^p}{u^{p-1}}\right)\left|\nabla u\right|^{p-2}\nabla u dx \\
&+ \frac{\sigma+1}{q}\int_{\mathbb{R}^N} \left|\nabla\psi_n\right|^q dx - \frac{\sigma+1}{q}\int_{\mathbb{R}^N} \nabla\left(\frac{\psi_n^q}{v^{q-1}}\right)\left|\nabla v\right|^{q-2}\nabla v dx
\end{aligned}$$

where the functions $\frac{\varphi_n^p}{u^{p-1}}$ and $\frac{\psi_n^q}{v^{q-1}}$ are admissible since $u,v>0$ and $\varphi_n,\psi_n \in C_0^\infty\left(\mathbb{R}^N\right)$, next we integrate by parts to obtain

$$\begin{aligned}
0 &\le \frac{\varrho+1}{p}\int_{\mathbb{R}^N} \left|\nabla\varphi_n\right|^p dx + \frac{\sigma+1}{q}\int_{\mathbb{R}^N} \left|\nabla\psi_n\right|^q dx \\
&+ \frac{\varrho+1}{p}\int_{\mathbb{R}^N} \frac{\varphi_n^p}{u^{p-1}}\Delta_p u dx + \frac{\sigma+1}{q}\int_{\mathbb{R}^N} \frac{\psi_n^q}{v^{q-1}}\Delta_q v dx \\
&\le \frac{\varrho+1}{p}\int_{\mathbb{R}^N} \left|\nabla\varphi_n\right|^p dx + + \frac{\sigma+1}{q}\int_{\mathbb{R}^N} \left|\nabla\psi_n\right|^q dx \\
&- \mu\left(\frac{\varrho+1}{p}\int_{\mathbb{R}^N} \frac{\varphi_n^p}{u^{p-1}} f(x,u,v)dx + \frac{\sigma+1}{q}\int_{\mathbb{R}^N} \frac{\psi_n^q}{v^{q-1}} g(x,u,v)dx\right).
\end{aligned}$$

From the hypothesis $(\mathcal{H}4)$ we have

$$f(x,u,v) \ge \lambda_1 A(x) u^\varrho v^{\sigma+1}$$

$$g(x,u,v) \ge \lambda_1 A(x) u^{\varrho+1} v^\sigma,$$

then

$$\begin{aligned}
0 \le & \frac{\varrho+1}{p}\int_{\mathbb{R}^N} \left|\nabla\varphi_n\right|^p dx + \frac{\sigma+1}{q}\int_{\mathbb{R}^N} \left|\nabla\psi_n\right|^q dx \\
& - \mu\lambda_1\Big[\frac{\varrho+1}{p}\int_{\mathbb{R}^N} A(x)\varphi_n^p u^{\varrho+1-p} v^{\sigma+1} dx + \frac{\sigma+1}{q}\int_{\mathbb{R}^N} A(x)\psi_n^q u^{\varrho+1} v^{\sigma+1-q} dx\Big].
\end{aligned} \tag{19}$$

Now we have

$$\left|\int_{\mathbb{R}^N} A(x) u^{\varrho+1-p} v^{\sigma+1}\left(\varphi_n^p - \varphi^p\right)dx\right| \le \underbrace{\|A\|_{L^{\frac{N}{\varrho+\sigma+2}}(\mathbb{R}^N)} \|u\|_{L^{p^*}(\mathbb{R}^N)}^{\varrho+1-p} \|v\|_{L^{q^*}(\mathbb{R}^N)}^{\sigma+1} \left\|\varphi_n^p - \varphi^p\right\|_{L^{\frac{p^*}{p}}(\mathbb{R}^N)}}_{(I_3)}$$

and

$$\left|\int_{\mathbb{R}^N} A(x)u^{\varrho+1}v^{\sigma+1-q}\left(\psi_n^p-\psi^p\right)dx\right| \leq \underbrace{\|A\|_{L^{\frac{N}{\varrho+\sigma+2}}(\mathbb{R}^N)}\|u\|^{\varrho+1}_{L^{p^*}(\mathbb{R}^N)}\|v\|^{\sigma+1-q}_{L^{q^*}(\mathbb{R}^N)}\|\psi_n^p-\psi^p\|_{L^{\frac{p^*}{p}}(\mathbb{R}^N)}}_{(I_4)}$$

where (I_3) and (I_4) tend to zero, as n increases to ∞, since $(\varphi_n,\psi_n)_{n\in N}$ converges to (φ,ψ) in $L^{p^*}\left(\mathbb{R}^N\right)\times L^{q^*}\left(\mathbb{R}^N\right)$. Hence, from (19), we obtain by the Dominated Convergence Theorem

$$\begin{aligned}\mu\lambda_1\Big(\frac{\varrho+1}{p}\int_{\mathbb{R}^N} A(x)\varphi^p u^{\varrho+1-p}v^{\sigma+1}dx\\ +\frac{\sigma+1}{q}\int_{\mathbb{R}^N} A(x)\psi^q u^{\varrho+1}v^{\sigma+1-q}dx\Big) &\leq \frac{\varrho+1}{p}\int_{\mathbb{R}^N}|\nabla\varphi|^p\,dx+\frac{\sigma+1}{q}\int_{\mathbb{R}^N}|\nabla\psi|^q\,dx\\ &=\lambda_1\int_{\mathbb{R}^N} B(x)\varphi^{\varrho+1}\psi^{\sigma+1}dx. \qquad (20)\end{aligned}$$

From other side, we have by Hölder-Young inequality

$$\begin{aligned}\left(\varphi u^{\frac{\varrho+1}{p}-1}v^{\frac{\sigma+1}{q}}\right)^{\varrho+1}\left(\psi u^{\frac{\varrho+1}{q}}v^{\frac{\sigma+1}{q}-1}\right)^{\sigma+1} \leq& \frac{\varrho+1}{p}\varphi^p u^{\varrho+1-p}v^{\sigma+1}\\ &+\frac{\sigma+1}{q}\psi^q u^{\varrho+1}v^{\sigma+1-q}. \qquad (21)\end{aligned}$$

Since $\frac{\varrho+1}{p}-1=-\frac{\sigma+1}{q}$ and $\frac{\sigma+1}{q}-1=-\frac{\varrho+1}{p}$, we derive

$$\left(u^{\frac{\varrho+1}{p}-1}v^{\frac{\sigma+1}{p}}\right)^{\varrho+1}=\left(u^{-\frac{1}{q}}v^{\frac{1}{p}}\right)^{(\varrho+1)(\sigma+1)}$$

and

$$\left(u^{\frac{\varrho+1}{q}}v^{\frac{\sigma+1}{q}-1}\right)^{\sigma+1}=\left(u^{\frac{1}{q}}v^{-\frac{1}{p}}\right)^{(\varrho+1)(\sigma+1)},$$

then

$$\left(u^{\frac{\varrho+1}{p}-1}v^{\frac{\sigma+1}{p}}\right)^{\varrho+1}\left(u^{\frac{\varrho+1}{q}}v^{\frac{\sigma+1}{q}-1}\right)^{\sigma+1}=1,$$

consequently, the inequality (21) becomes

$$\varphi^{\varrho+1}\psi^{\sigma+1}\leq\frac{\varrho+1}{p}\varphi^p u^{\varrho+1-p}v^{\sigma+1}+\frac{\sigma+1}{q}\psi^q u^{\varrho+1}v^{\sigma+1-q}. \qquad (22)$$

Now, using (20) and (22) we deduce

$$\mu\lambda_1\int_{\mathbb{R}^N} B(x)\varphi^{\varrho+1}\psi^{\sigma+1}dx\leq\lambda_1\int_{\mathbb{R}^N} B(x)\varphi^{\varrho+1}\psi^{\sigma+1}dx$$

that is to say

$$\mu\leq 1.$$

References

[1] A. Ahammou, *On the existence of bounded solutions of non-linear elliptic systems*, IJMMS 30:8 (2002) 479-490.

[2] A. Ahammou, *A multiplicity result for a quasilinear gradient elliptic system*, Journal of Applied Mathematics, vol. 1, no. 3, pp. 91-106, 2001.

[3] A. Anane. *Simplicite et isolation de la premiere valeur propre du p-Laplacien avec poids*. C. R. Acad. Sci. Paris Ser. I Math., 305(16) : 725-728, 1987.

[4] G. Barles. *Remarks on uniqueness results of the first eigenvalue of the p-Laplacian*. Ann. Fac. Sci. Toulouse Math., 9(1) : 65-75, 1988.

[5] K. Chaib, *Quelques resultats sur les systemes d'equations aux deriivatives partielles faisant intervenir l'operateur p-Laplacien,* these Universite Paul Sabatier de Toulouse-France, 2002, p:47-82.

[6] J.I. Diaz J.E. Saa. *Existence et unicite de solutions positives pour certaines equtaions elliptiques quasilineaires*. C. R. Acad. Sci. Paris, 305 Ser. I : 521-524, 1987.

[7] A. El Khalil, M. Ouanan and A. Touzani, *Existence and regularity of positive solutions for an elliptic system.* Electronic Journal of DifferentialEquations, Conference 09, 2003, pp 171-182.

[8] O. Kavian, *Introduction la theorie des points critiques.* Springer Verlag, 1993.

[9] S. Sakaguchi. *Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems*. Ann. Scuola Norm. Sup. Pisa Cl. Sci., 14(4) : 403-421, 1987.

[10] P. Tolksdorf *Regularity for more general class of quasilinear elliptic equations,* J. Differential Equations, 51(1):126-150, 1984.

Received: October 6, 2007