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k-Level Menage-type Hamiltonian Cycles Derived from a Given $(k-1)$-Level Cycle

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Abstract

The Menage problem can be viewed as finding particular Hamiltonian cycles representing a $2\times n$ array. As a generalization of this problem we introduce the k-level Menage problem which considers $k \times n$ arrays. Using the inclusion-exclusion principle, we give an efficient algorithm to derive solutions for the k-level Menage problem of order n from a given solution for the $(k-1)$-level Menage problem of the same order.

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Keywords: Menage problem, Hamiltonian cycle, Inclusion-exclusion principle

1 Introduction

The well-known traditional Menage problem, introduced by the French mathematician E. Lucas in 1891 [1], states that in how many ways can n married couples sit around a table with $2n$ seats under the constraint that no two men, no two women and no wife and husband can sit next to each other? [2, 3]. The solution of this problem, denoted $M(n)$, is $M(n) = 2(n!)g(n)$ [4], where

$$g(n) = n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \ldots + (-1)^n \frac{2n}{n}\binom{n}{n}0!$$

In this paper we introduce the Menage problem as finding a specific form of a length $2n$ Hamiltonian cycle that represents a $2 \times n$ array. Then this problem is extended to a $k \times n$ array with $k > 2$. For instance, a $3 \times n$ array can be associated with a simple generalization of the Menage problem: n married couples, each couple with a child, are to sit around a table with $3n$ seats under the constraint that no two members of a family, no two men, no two women, and no two children can sit next to each other. We refer to this as a 3-level Menage problem. Given a length $(k-1)n$ cycle C as a solution for the Menage problem on a $(k-1) \times n$ array, we use C to construct length kn cycles for a k-level Menage problem.

2 Menage-type problem and Hamiltonian cycles representing a $k \times n$ array

2.1 Menage problem and $2 \times n$ arrays

We consider the $2 \times n$ array given by

	F_1	F_2	$\ldots$	F_n
M	1	3	$\ldots$	$2n-1$
W	2	4	$\ldots$	$2n$

, (1)

as a representation of n married couples wherein the ith column, denoted F_i, represents the ith couple, the first row represents men and the second row represents women. A simple observation shows that we have the following trivial but essential theorem.

Theorem 2.1. *A proper arrangement of the n couples in the Menage problem is equivalent to starting from the top-left entry of the given $2 \times n$ array and moving diagonally with arbitrary length at each step (no vertical or horizontal movement), visiting all entries exactly once and returning back to the starting point.*

We refer to such a closed path as a *proper Hamiltonian cycle* for the Menage problem, or a 2-*level Menage-type length* $2n$ *Hamiltonian* cycle [5].

For example, Fig. 1 is a 2-level Menage-type length 8 Hamiltonian cycle associated with the 2×4 array introduced by the first two rows of the following

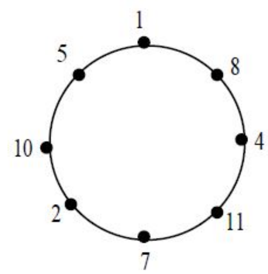

Figure 1: A 2-level Menage-type length 8 Hamiltonian cycle associated with the first two rows of the array (2).

array

	F_1	F_2	F_3	F_4
W	1	4	7	10
H	2	5	8	11
C	3	6	9	12

. (2)

2.2 Menage-type problem and $k \times n$ arrays

Definition 2.2. *Consider a $k \times n$ array, and suppose a particle standing on the top-left entry of the array starts moving diagonally (that is vertical and horizontal movements are not allowed) with arbitrary length at each movement to visit all entries exactly once before the final diagonal movement by which it returns back to the initial position. We refer to the so obtained length kn cycle as a k-level Menage-type length kn Hamiltonian cycle.*

According to the idea behind Theorem 1, any $k \times n$ array can be used to pose a Menage-type problem. For a $4 \times n$ array with distinct elements, we may consider each column as a 4-member family (say, a married couple with two children one boy and one girl), the first row as men, the second row as women, the third row as girls and the fourth row as boys. These $4n$ people are to sit around a table under the constraint that no two men, no two women, no two boys, no two girls, and no two members of a family can sit next to each other. Obviously, any 4-level Menage-type length $4n$ Hamiltonian cycle is a solution for this Menage-type problem.

3 Cycle construction

This section is devoted to giving a recursive method to construct k-level Menage-type length kn Hamiltonian cycles. We present a method to derive k-level Menage-type length kn Hamiltonian cycles from a given $(k-1)$-level

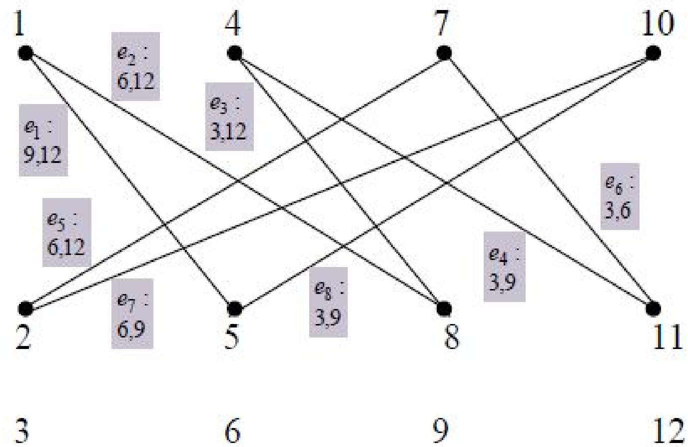

Figure 2: A graph with 12 vertices and 8 edges that form a 2-level Menage-type length 8 Hamiltonian cycle associated with the first two rows of the array (2).

Menage-type length $(k-1)n$ Hamiltonian cycle. The nature of the given method allows one to determine the number of cycles using the principle of inclusion-exclusion [4].

Example 3.1 (3-level cycle)**.** *The length* 8 *circle in Fig. 2 is a proper Hamiltonian cycle which gives a solution for the* 2*-level Menage problem associated with the first two rows of the* 3×4 *array (2). The edges of this circle are labeled by* e_i, $1 \leq i \leq 8$. *To extend this circle to a* 3*-level Menage-type length* 12 *Hamiltonian cycle as a solution for the* 3*-level Menage problem corresponding to the given* 3×4 *array, the edges of the length* 8 *cycle are labeled properly. As the array and cycle show, on edge* e_1 *we can insert a new vertex that represents either entry* 9 *or* 12 *of the array, but none of the two entries* 3 *and* 6*; and this is the reason for using label* $e_1 : 9, 12$. *All edges are labeled using the same idea. Let* $i \to e_j$ *denote the insertion of a new vertex on* e_j *labeled by* i. *With this notation, as examples, the two sets of insertions* $\{3 \to e_3, 6 \to e_7, 9 \to e_1, 12 \to e_2\}$ *and* $\{3 \to e_6, 6 \to e_2, 9 \to e_4, 12 \to e_5\}$ *produce two* 3*-level Menage-type length* 12 *Hamiltonian cycles for the given* 3×4 *Menage problem. These two cycles can be expressed as*

$$\begin{cases} C: \ 1-9-5-10-6-2-7-11-4-3-8-12-1, \\ C': \ 1-5-10-2-12-7-3-11-9-4-8-6-1. \end{cases}$$

Cycle enumeration
As mentioned above, it follows from the structure of the length 8 cycle that vertex 3 can be inserted on each of the four elements of $R_3 := \{e_3, e_4, e_6, e_8\}$.

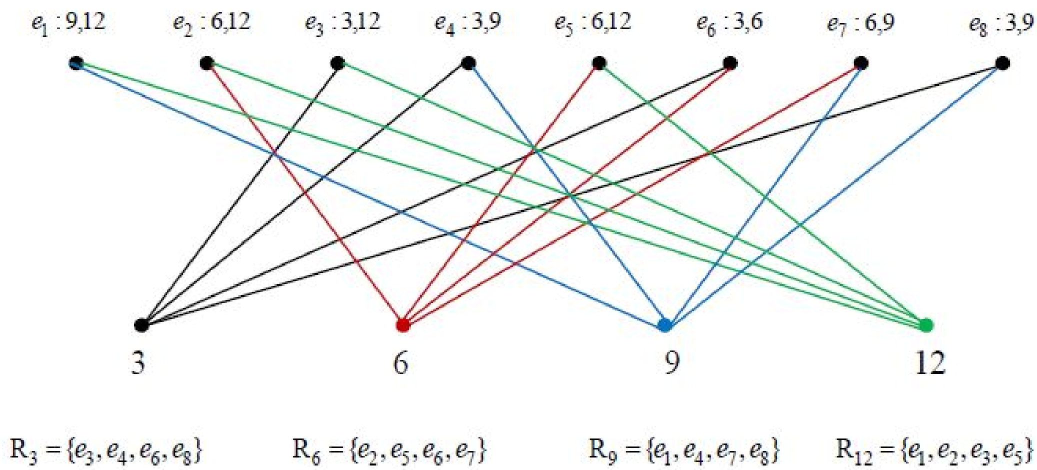

Figure 3: Allocation of elements of $\{3, 6, 9, 12\}$ to the elements of $\{1, 2, \ldots, 8\}$ under the specified constraints.

Similarly, for vertex 6 the suitable set of edges is $R_6 := \{e_2, e_5, e_6, e_7\}$, and also $R_9 := \{e_1, e_4, e_7, e_8\}$ and $R_{12} := \{e_1, e_2, e_3, e_5\}$. These constraints are represented by Fig. 3 in which, for instance, vertex 3 is adjacent to precisely the vertices labeled by e_3, e_4, e_6 and e_8. To form a 3-level cycle we also have the constraint that no two elements of $\{3, 6, 9, 12\}$ can be inserted simultaneously on one edge of the length 8 cycle.

Therefore, the number of 3-level Menage-type length 12 Hamiltonian cycles that can be constructed is precisely the number of maximum matchings of the bipartite graph in Fig. 3. A set of edges in a graph G is called a matching if no two of them are adjacent, and the maximum number of such edges in G is called a maximum matching for G [5]. For simplicity e_i is denoted by i, and hence, for instance, we have $R_3 = \{3, 4, 6, 8\}$. With this notation, the number of maximum matchings of the bipartite graph is the number of one to one functions from the set $\{3, 6, 9, 12\}$ to $\bigcup_{i \in \{3,6,9,12\}} R_i = \{1, 2, \ldots, 8\}$ under the constraints that 3 can be only mapped to the elements of $R_3 = \{3, 4, 6, 8\}$, and so on.

To solve the problem we use the inclusion-exclusion principle [4]. We say a function f from $\{3, 6, 9, 12\}$ to $\{1, 2, \ldots, 8\}$, under the constraints given by the graph, satisfies property p_i for $1 \leq i \leq 8$ if i appears at least two times in the range of f, that is $|f^{-1}(i)| > 1$. As the graph shows, the set of all functions, denoted S, is of size $|S| = 4^4$, since each element of $\{3, 6, 9, 12\}$ can be mapped to four elements of $\{1, 2, \ldots, 8\}$.

Let $N(i)$ denote the number of functions that satisfy p_i and $N(i, j)$ denote the number of functions that satisfy both p_i and p_j. With this notation, it is obvious that $N(i, j, k) = 0$ for $1 \leq i < j < k \leq 8$, since the domain of

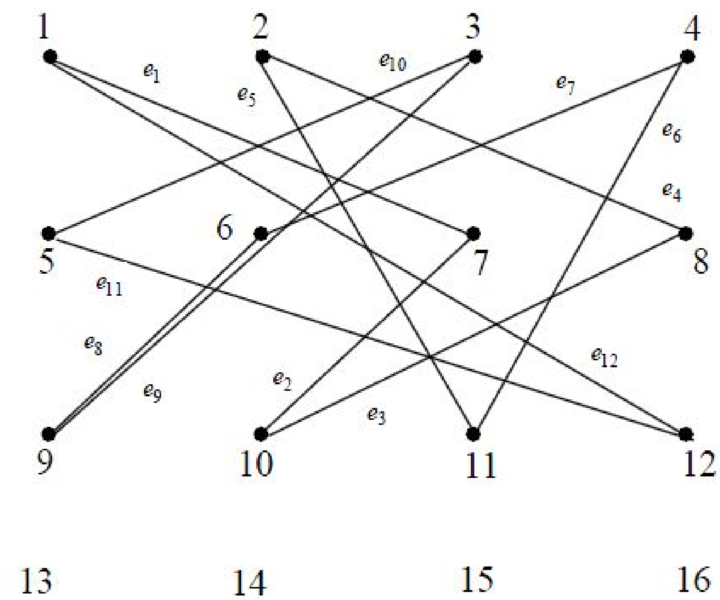

Figure 4: A graph with 16 vertices and 12 edges that form a length 12 Menage-type Hamiltonian cycle associated with the first three rows of the vertices.

the functions is of size 4. Setting $f(12) = f(9) = 1$, $f(3) \in \{3, 4, 6, 8\}$ and $f(6) \in \{2, 5, 6, 7\}$ we get $N(1) = 4^2$. The same reasoning shows that $N(i) = 4^2$ for $1 \leq i \leq 8$.

We may set $f(9) = f(12) = 1$ and $f(3) = f(6) = 6$ showing that $N(1, 6) = 1$. The sets R_i show that

$$\begin{cases} N(1,2) = N(1,3) = N(1,4) = N(1,5) = N(1,7) = N(1,8) = 0,\ N(1,6) = 1; \\ N(2,3) = N(2,5) = N(2,6) = N(2,7) = 0,\ N(2,4) = N(2,8) = 1; \\ N(3,4) = N(3,5) = N(3,6) = N(3,8) = 0,\ N(3,7) = 1; \\ N(4,6) = N(4,7) = N(4,8) = 0,\ N(4,5) = 1; \\ N(5,6) = N(5,7) = 0,\ N(5,8) = 1; \\ N(6,7) = N(6,8) = N(7,8) = 0. \end{cases}$$

Hence $\sum_i N(i) = 8 \times 4^2$ and $\sum_{i<j} N(i,j) = 6$. Therefore, the number of maximum matchings of the graph is

$$\alpha = |S| - \sum_i N(i) + \sum_{i<j} N(i,j) = 4^4 - 8 \times 4^2 + 6 = 134.$$

Example 3.2 (4-level cycle). *We now consider the construction of* 4*-level Menage-type length* 16 *Hamiltonian cycles by using a given length* 12 *Menage-type cycle associated with a* 3×4 *array.*

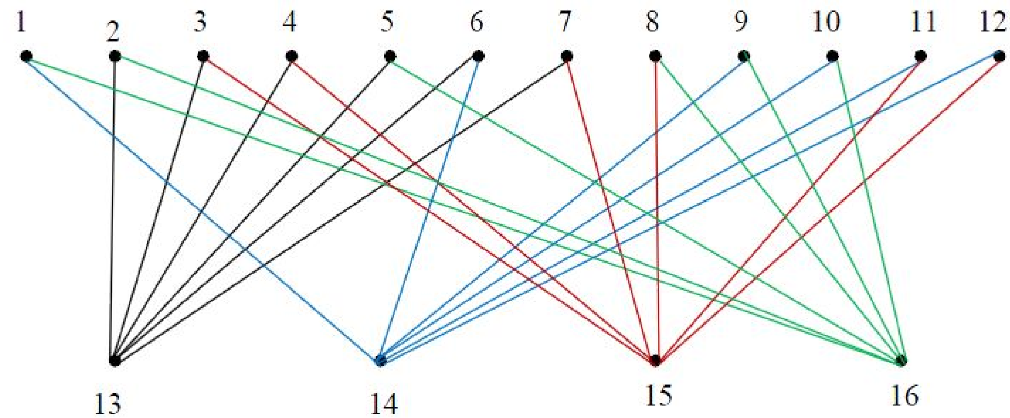

Figure 5: Allocation of elements of $\{13, 14, 15, 16\}$ to the elements of $\{1, 2, \ldots, 12\}$ under the specified constraints.

The cycle formed by the edges of the graph in Fig. 4 is a 3*-level Menage-type length* 12 *Hamiltonian cycle associated with the* 3×4 *array formed by the first* 3 *rows of the* 4×4 *array*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

. (3)

It is clear from Fig. 4 that vertex 13 *cannot be inserted on precisely the edges incident with vertices* 1, 5 *and* 9*. Thus, the edges suitable for accepting vertex* 13 *are the elements of* $R_{13} = \{e_2, e_3, e_4, e_5, e_6, e_7\}$*, denoted for simplicity by* $R_{13} = \{2, 3, 4, 5, 6, 7\}$*. A similar argument shows that* $R_{14} = \{1, 6, 9, 10, 11, 12\}$*,* $R_{15} = \{3, 4, 7, 8, 11, 12\}$*, and* $R_{16} = \{1, 2, 5, 8, 9, 10\}$*.*

Cycle enumeration

According to the constraints, we have the bipartite graph G_2 given by Fig. 5 wherein vertex i in the top vertices represents edge e_i in Fig. 4.

The maximum matchings of graph G_2 are of size 4, and any such matching gives a 4-level Menage-type length 16 Hamiltonian cycle for the 4×4 array. For instance, the matching $\{e_{13,3}, e_{14,6}, e_{15,8}, e_{16,10}\}$ says that vertices 13, 14, 15 and 16 are inserted on the edges e_3, e_6, e_8 and e_{10} of the length 12 cycle, respectively. This insertion gives the length 16 cycle

$$C : 1 - 7 - 10 - 13 - 8 - 2 - 11 - 14 - 4 - 6 - 15 - 9 - 3 - 16 - 5 - 12 - 1.$$

We now determine the number of maximum matchings in graph G_2, or equivalently the number of one to one functions from $A = \{13, 14, 15, 16\}$ to

$B = \{1, 2, \ldots, 12\}$ under the specified constraints. We say a function f from A to B satisfies property p_i, $1 \le i \le 12$, if i appears at least twice in the range of f, that is $|f^{-1}(i)| > 1$. Using the introduced notation we have $|S| = 6^4$, $N(i) = 6^2$, $1 \le i \le 12$, and

$$\begin{cases} N(1,3) = N(1,4) = N(1,7) = 1, N(1,j) = 0 \text{ for } j \notin \{3,4,7\}; \\ N(2,11) = N(2,12) = 1, N(2,j) = 0 \text{ for } j \notin \{11,12\} \text{ and } j > 2; \\ N(3,9) = N(3,10) = 1, N(3,j) = 0 \text{ for } j \notin \{9,10\} \text{ and } j > 3; \\ N(4,9) = N(4,10) = 1, N(4,j) = 0 \text{ for } j \notin \{9,10\} \text{ and } j > 4; \\ N(5,11) = N(5,12) = 1, N(5,j) = 0 \text{ for } j \notin \{11,12\} \text{ and } j > 5; \\ N(6,8) = 1, N(6,j) = 0 \text{ for } j > 6 \text{ and } j \ne 8; \\ N(7,9) = N(7,10) = 1, N(7,j) = 0 \text{ for } j > 7 \text{ and } j \ne 9, 10; \\ N(i,j) = 0 \text{ for } 7 < i < j. \end{cases}$$

Thus $\sum_i N(i) = 12 \times 6^2$ and $\sum_{i<j} N(i,j) = 14$. Hence the number of 4-level Menage-type length 16 Hamiltonian cycles derived from the given length 12 3-level Hamiltonian cycle is

$$\alpha = |S| - \sum_i N(i) + \sum_{i<j} N(i,j) = 6^4 - 12 \times 6^2 + 14 = 878.$$

According to these two examples we have constructed 134 3-level Menage-type length 12 and 878 4-level Menage-type length 16 Hamiltonian cycles by applying the introduced cycle construction method on a given 2-level Menage-type length 8 Hamiltonian cycle. This proves the efficiency of the given technique.

4 Summary

We showed that the traditional Menage problem is equivalent to determining Hamiltonian cycles on the entries of a $2 \times n$ array formed by moving diagonally on them. Referring to this cycle as a 2-level Menage-type length $2n$ Hamiltonian cycle, we generalized it to k-level Menage-type length kn Hamiltonian cycle. An efficient recursive method was given to construct k-level Menage-type length kn Hamiltonian cycles from a given $(k-1)$-level Menage-type cycle.

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