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Bitcoin between Equity and Money: A Numéraire-Invariant Riemannian Framework for Diversification and Value Preservation

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Abstract

Whether Bitcoin should be judged by the yardsticks of the capital market or by those of monetary economics is usually answered by two disconnected exercises: a portfolio comparison against an equity benchmark and a nominal comparison against fiat currencies. We show that the two exercises are linked by a change of numéraire, and that most statistics used in practice (correlations, volatility ratios, Sharpe ratios, Euclidean and log-Euclidean distances between covariance matrices) change when the investor's unit of account changes. We therefore work on the manifold of symmetric positive definite covariance matrices with the affine-invariant Riemannian metric. Regime distances, the Karcher

mean regime and geodesic deviations are exactly numéraire invariant; Bitcoin's Euler share of portfolio variance exceeds its capital share if and only if its weight exceeds the global minimum-variance weight; and, in a linearised real mean-variance problem, expected inflation cancels from the Bitcoin versus cash decision, so that a store-of-value role is identified only by an inflation-hedging coefficient. Regime persistence is read off a Matrix Pencil decomposition of the invariant geodesic deviation series. A pilot on annual 2015 to 2025 data (MSCI World Net USD, Bitcoin, ECB EUR/USD, CPI-U, HICP) shows a slightly negative minimum-variance weight, so that a 5% allocation carries about 62% of portfolio variance, and illustrates that the affine-invariant regime distance coincides across numéraires while the Bitcoin to equity correlation halves when moving from dollars to euros.

Mathematics Subject Classification: 53C22, 91G10, 15B48, 91B64

Keywords: Bitcoin, numéraire invariance, Riemannian geometry of covariance matrices, Karcher mean, risk contribution, inflation hedging, Matrix Pencil

1 Introduction

Bitcoin does not confer ownership of a firm, does not entitle its holder to a stream of profits and does not pay interest. It was designed instead as a decentralised system for transferring value, with a supply schedule fixed by protocol and independent of any central bank. Structurally it is therefore closer to money than to equity, yet empirically it has behaved for long stretches like a high-beta speculative asset. This tension is the starting point of a recent practitioner study by INICORBAF, which proposes a twofold analysis: Bitcoin against the MSCI World index as an investment, and Bitcoin against the euro and the dollar as a monetary object, with crossed experiments in which a dollar investor compares Bitcoin with euro cash and a euro investor compares Bitcoin with dollar cash. The study is careful about measurement (annual frequency, eleven observations, end-of-year drawdowns as lower bounds) and closes with a protocol for a daily extension.

The present paper takes that twofold question as its object and supplies what the descriptive exercise lacks: a common mathematical frame in which the equity view and the monetary view are two coordinate charts of the same object. The organising observation is elementary. With B_t the dollar price of one bitcoin and X_t the dollar price of one euro, the euro price of one bitcoin is $E_t = B_t/X_t$, so that in logarithmic returns

$$r_{b,t}^{\mathrm{EUR}} = r_{b,t}^{\mathrm{USD}} - f_t, \qquad f_t = \log(X_t/X_{t-1}). \tag{1}$$

Changing the unit of account is a linear invertible map on the vector of log returns. Statistics that are not equivariant under such maps measure the unit of account as much as the asset. Correlations are a case in point: in our pilot, the annual correlation between Bitcoin and global equities over 2015 to 2019 is about 0.77 for a dollar investor and about 0.39 for a euro investor, although nothing about either asset has changed.

Our methodological answer draws on the Riemannian geometry of positive definite matrices [18, 4, 3]. Covariance regimes are treated as points of the manifold $\mathcal{P}_n$ with the affine-invariant metric, a complete Riemannian manifold of nonpositive curvature on which means are unique and distances are invariant under congruence. The contribution is threefold.

Numéraire invariance (Theorem 3.2). Distances between regimes, the Karcher mean regime and every geodesic deviation from it are identical for the dollar and the euro investor. This gives the first regime-detection statistic for Bitcoin that does not depend on where the investor lives.

Risk concentration (Proposition 3.5). Bitcoin's Euler contribution to portfolio variance exceeds its capital weight if and only if that weight exceeds the global minimum-variance weight, with an explicit closed form for the gap. When the beta of Bitcoin on equities exceeds one, every positive allocation concentrates risk. This turns an empirical remark of the practitioner study (a 5% weight contributes about half of the variance) into a sharp criterion.

Store of value (Proposition 3.7). In a linearised real mean-variance problem, the expected erosion of cash by inflation cancels from the choice between Bitcoin and cash. What remains is a speculative demand and an inflation-hedging demand equal to $\mathrm{Cov}(r_b, \pi)/\mathrm{Var}(r_b)$, which also equals the minimum real-variance weight. Nominal comparisons of Bitcoin with depreciating fiat balances therefore cannot by themselves establish a store-of-value role.

Regime persistence is studied by applying the Matrix Pencil method [11] to the numéraire-invariant geodesic deviation series, which yields interpretable damping rates and half-lives of regime departures. Empirically, we reproduce the practitioner study on its own annual data, compute the new quantities, and release a daily pipeline that implements the full protocol on public sources.

The paper proceeds as follows. Section 2 positions the contribution. Section 3 develops the theory. Section 4 describes data and design. Section 5 reports the annual pilot. Section 6 specifies the daily study and its hypotheses. Section 7 concludes.

2 Related literature

The monetary theory of Bitcoin starts from the indeterminacy of exchange rates between intrinsically useless monies [13]. Schilling and Uhlig [21] embed a fixed-supply cryptocurrency next to a dollar managed by an inflation-

targeting central bank and obtain a fundamental pricing condition under which the Bitcoin price is a martingale in its simplest form. Fernández-Villaverde and Sanches [10] study competition among privately issued fiat monies in a Lagos and Wright environment and show that price stability is possible but not guaranteed. Biais et al. [5] build an overlapping-generations model in which the fundamental value of Bitcoin is the stream of net transactional benefits, which allows large volatility unrelated to fundamentals. These models motivate treating Bitcoin and fiat money as competing units, and hence motivate the numéraire perspective adopted here.

On the asset side, Baur, Hong and Lee [2] document that Bitcoin was largely uncorrelated with traditional assets in its early years and was held mainly for speculation; Liu and Tsyvinski [15] show that crypto returns load on crypto-specific network and momentum factors rather than on standard equity factors. Hedging and safe-haven properties were tested early by Dyhrberg [8] and Bouri et al. [6], and the Covid-19 crash provided a stress test in which Bitcoin increased rather than reduced downside risk [7]. Spillovers between crypto and equity markets increased markedly after 2020 [12], and Marti and Reimat [17] find cointegration between Bitcoin and the MSCI World index over 2018 to 2025. Portfolio studies such as Platanakis and Urquhart [19] report out-of-sample gains from small allocations. None of these works addresses the dependence of the reported statistics on the unit of account, which is the gap this paper fills.

On the methodological side, the affine-invariant geometry of $\mathcal{P}_n$ is developed in Pennec, Fillard and Ayache [18], Bhatia and Holbrook [4] and Bhatia [3], with optimisation tools in Absil, Mahony and Sepulchre [1]. Risk contributions follow the Euler allocation used for risk-parity portfolios [16]. Dynamic correlation benchmarks are provided by the DCC model [9]; inference on Sharpe differences uses the stationary bootstrap [20, 14]. The Matrix Pencil method [11] supplies a parsimonious, interpretable representation of regime dynamics.

3 Theoretical framework

3.1 Setting and change of numéraire

Let $x_t = (r_{b,t}, r_{m,t}, f_t)^\top$ collect, in the dollar numéraire, the log returns of Bitcoin, of the MSCI World Net Total Return index and of the euro (dollar price of one euro). For a euro investor the relevant vector is $x'_t = (r_{b,t} - f_t,\ r_{m,t} - f_t,\ -f_t)^\top$, where the last entry is the log return of holding dollar

cash measured in euros. Hence

$$x'_t = A\,x_t, \qquad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \det A = -1, \tag{2}$$

and for any conditional or rolling covariance Σ_t of x_t, the euro covariance is $\Sigma'_t = A\Sigma_t A^\top$. The crossed experiments of the practitioner study (dollar investor: Bitcoin versus euro cash; euro investor: Bitcoin versus dollar cash) are exactly the first and third coordinates of x_t and x'_t.

Remark 3.1. The identity is exact in log returns. In simple returns $1 + R_b^{\mathrm{EUR}} = (1 + R_b^{\mathrm{USD}})X_{t-1}/X_t$ contains a multiplicative interaction term, which is why differences of simple returns are not additive across numéraires.

3.2 Affine-invariant geometry of covariance regimes

Let $\mathcal{P}_n$ denote the open cone of $n \times n$ symmetric positive definite matrices. With tangent space at Σ identified with symmetric matrices, the affine-invariant metric is $\langle U, V\rangle_\Sigma = \mathrm{tr}(\Sigma^{-1}U\Sigma^{-1}V)$, whose geodesic distance is

$$d_{\mathrm{R}}(\Sigma_1, \Sigma_2) = \big\| \log\big(\Sigma_1^{-1/2}\Sigma_2\Sigma_1^{-1/2}\big)\big\|_F = \Big(\sum_{i=1}^{n} \log^2 \lambda_i\Big)^{1/2}, \tag{3}$$

where $\lambda_1, \ldots, \lambda_n > 0$ are the eigenvalues of $\Sigma_1^{-1}\Sigma_2$ [18, 3]. $(\mathcal{P}_n, d_{\mathrm{R}})$ is a complete, simply connected Riemannian manifold of nonpositive sectional curvature (a Hadamard manifold). For a finite family $\Sigma_1, \ldots, \Sigma_T$, the Karcher (Riemannian) mean

$$\bar{\Sigma} = \arg\min_{M \in \mathcal{P}_n} \sum_{t=1}^{T} d_{\mathrm{R}}^2(M, \Sigma_t) \tag{4}$$

exists and is unique [4]. We call $\bar{\Sigma}$ the *reference regime* and $\delta_t = d_{\mathrm{R}}(\bar{\Sigma}, \Sigma_t)$ the *geodesic deviation* of period t.

Theorem 3.2 (Numéraire invariance of regime geometry)**.** *Let $A \in GL(n)$ and $\Sigma, \Sigma_1, \Sigma_2, \ldots, \Sigma_T \in \mathcal{P}_n$.*

(i) $d_{\mathrm{R}}(A\Sigma_1 A^\top, A\Sigma_2 A^\top) = d_{\mathrm{R}}(\Sigma_1, \Sigma_2)$.

(ii) If $\bar{\Sigma}$ is the Karcher mean of $\{\Sigma_t\}$, then $A\bar{\Sigma}A^\top$ is the Karcher mean of $\{A\Sigma_t A^\top\}$.

(iii) Consequently, for the change of numéraire (2)*, the geodesic deviation series $\{\delta_t\}$ computed by a dollar investor and by a euro investor coincide, and so do all pairwise regime distances.*

(iv) Pearson correlations, ratios of volatilities and the log-Euclidean distance $\|\log\Sigma_1 - \log\Sigma_2\|_F$ *are not invariant under* (2) *in general.*

Proof. (i) Write $\Sigma_i' = A\Sigma_i A^\top$. Then

$$\Sigma_1'^{-1}\Sigma_2' = A^{-\top}\Sigma_1^{-1}A^{-1}A\Sigma_2 A^\top = A^{-\top}(\Sigma_1^{-1}\Sigma_2)A^\top,$$

which is similar to $\Sigma_1^{-1}\Sigma_2$ and therefore has the same eigenvalues. Since (3) depends only on these eigenvalues, the distances coincide. (ii) The map $\phi_A(M) = AMA^\top$ is a bijection of $\mathcal{P}_n$ onto itself with inverse $\phi_{A^{-1}}$. By (i), for every M,

$$F'(\phi_A(M)) := \sum_t d_{\mathrm{R}}^2(\phi_A(M), \phi_A(\Sigma_t)) = \sum_t d_{\mathrm{R}}^2(M, \Sigma_t) =: F(M).$$

Hence M minimises F if and only if $\phi_A(M)$ minimises F', and uniqueness on the Hadamard manifold yields $\bar\Sigma' = A\bar\Sigma A^\top$. (iii) Combining (i) and (ii), $\delta_t' = d_{\mathrm{R}}(A\bar\Sigma A^\top, A\Sigma_t A^\top) = d_{\mathrm{R}}(\bar\Sigma, \Sigma_t) = \delta_t$. (iv) A counterexample suffices, and Table 3 provides one on observed data: the Bitcoin to equity correlation over 2015 to 2019 moves from 0.77 to 0.39 and the log-Euclidean distance between the two subperiod regimes changes, whereas (3) does not. Analytically, correlations are invariant only under diagonal congruences, and the log-Euclidean metric is invariant only under orthogonal ones, while A in (2) is neither. □

Remark 3.3. Invariance requires the full system that contains the exchange rate. The 2×2 block of (r_b, r_m) alone is mapped to the block of $(r_b - f, r_m - f)$, which is not a congruence of the original block, so no statistic computed on the bivariate system can be numéraire invariant in general. The currency must be modelled jointly with the assets; this is the precise sense in which the equity analysis and the monetary analysis of Bitcoin cannot be separated.

Corollary 3.4. *Along the geodesic* $\gamma(s) = \Sigma_1^{1/2}(\Sigma_1^{-1/2}\Sigma_2\Sigma_1^{-1/2})^s\Sigma_1^{1/2}$, $s \in [0,1]$, *the congruence* ϕ_A *maps geodesics to geodesics. Interpolated regimes (for example the regime halfway between a calm and a stressed period) are therefore also numéraire equivariant.*

3.3 When does a small allocation concentrate risk?

Consider an equity index a and Bitcoin b with covariance matrix $S \in \mathcal{P}_2$ with entries σ_a^2, σ_{ab}, σ_b^2, and weights $(1-w, w)$, $w \in (0,1)$. Portfolio variance is

$$\sigma_p^2(w) = (1-w)^2\sigma_a^2 + w^2\sigma_b^2 + 2w(1-w)\sigma_{ab},$$

and the Euler share of Bitcoin [16] is

$$\mathrm{RS}_b(w) = \frac{w\left(w\sigma_b^2 + (1-w)\sigma_{ab}\right)}{\sigma_p^2(w)}, \qquad \mathrm{RS}_a(w) + \mathrm{RS}_b(w) = 1. \tag{5}$$

Proposition 3.5 (Minimum-variance threshold)**.** *Let $D = \sigma_a^2+\sigma_b^2-2\sigma_{ab} > 0$ and let $w_{\mathrm{mv}} = (\sigma_a^2 - \sigma_{ab})/D$ be the global minimum-variance weight of Bitcoin. Then for every $w \in (0,1)$*

$$\mathrm{RS}_b(w) - w = \frac{w(1-w)\,D\,(w - w_{\mathrm{mv}})}{\sigma_p^2(w)}. \tag{6}$$

In particular $\mathrm{RS}_b(w) > w$ if and only if $w > w_{\mathrm{mv}}$, with equality exactly at $w = w_{\mathrm{mv}}$. If $\beta_{b|a} := \sigma_{ab}/\sigma_a^2 \geq 1$, then $w_{\mathrm{mv}} \leq 0$ and every positive allocation to Bitcoin carries a larger share of variance than of capital.

Proof. $D > 0$ because S is positive definite and $D = (1,-1)S(1,-1)^\top$. From (5),

$$\mathrm{RS}_b(w) - w = \frac{w\big[w\sigma_b^2 + (1-w)\sigma_{ab} - \sigma_p^2(w)\big]}{\sigma_p^2(w)}.$$

Expanding

$$\sigma_p^2(w) = (1-w)\big[(1-w)\sigma_a^2 + w\sigma_{ab}\big] + w\big[w\sigma_b^2 + (1-w)\sigma_{ab}\big],$$

the bracket equals

$$\begin{aligned}(1-w)\big[w\sigma_b^2 + (1-w)\sigma_{ab} - (1-w)\sigma_a^2 - w\sigma_{ab}\big]& \\ = (1-w)\big[wD - (\sigma_a^2 - \sigma_{ab})\big] = (1-w)D(w - w_{\mathrm{mv}}),&\end{aligned}$$

which gives (6). The sign statements follow since $w(1-w)D/\sigma_p^2 > 0$ on $(0,1)$. Finally $\sigma_{ab} \geq \sigma_a^2$ is equivalent to $w_{\mathrm{mv}} \leq 0$. □

Remark 3.6. Proposition 3.5 is a statement about a point of $\mathcal{P}_2$; by Theorem 3.2, the regime in which that point lies, and its distance from the reference regime, are numéraire invariant even though w_{mv} itself is not. In the empirical part the threshold is computed regime by regime, which turns a static diversification claim into a time-indexed one.

3.4 Store of value: what inflation can and cannot identify

Fix a numéraire $c \in \{\mathrm{USD}, \mathrm{EUR}\}$ with consumer price inflation π^c. An investor holds a fraction w in Bitcoin and $1-w$ in non-remunerated domestic cash, as in the cash-plus-Bitcoin portfolios of the practitioner study. To first order the real log return of the position is

$$r_p^{\mathrm{real}} \simeq w\, r_b^c - \pi^c. \tag{7}$$

Proposition 3.7 (Speculative and hedging demand)**.** *Let* $\mu_b = \mathbb{E}[r_b^c]$, $\sigma_b^2 = \mathrm{Var}(r_b^c) > 0$, $\sigma_{b\pi} = \mathrm{Cov}(r_b^c, \pi^c)$, *and let the investor maximise* $\mathbb{E}[r_p^{\mathrm{real}}] - \frac{\gamma}{2}\mathrm{Var}(r_p^{\mathrm{real}})$ *with* $\gamma > 0$ *under* (7)*. Then*

$$w^* = \underbrace{\frac{\mu_b}{\gamma\sigma_b^2}}_{\text{speculative}} + \underbrace{\frac{\sigma_{b\pi}}{\sigma_b^2}}_{h^c:\ \text{inflation hedging}} . \tag{8}$$

Moreover: (a) $\mathbb{E}[\pi^c]$ *does not enter* w^**; (b) the weight that minimises real variance is exactly* h^c*; (c) for the euro investor,* $h^{\mathrm{EUR}} = \mathrm{Cov}(r_b - f, \pi^{\mathrm{EUR}})/\mathrm{Var}(r_b - f)$*, which differs from* h^{USD} *through both the exchange-rate covariance and the different price index.*

Proof. The objective is $w\mu_b - \mathbb{E}\pi^c - \frac{\gamma}{2}\big(w^2\sigma_b^2 - 2w\sigma_{b\pi} + \mathrm{Var}\,\pi^c\big)$, strictly concave in w. The first-order condition $\mu_b - \gamma(w\sigma_b^2 - \sigma_{b\pi}) = 0$ gives (8). $\mathbb{E}\pi^c$ is an additive constant, which proves (a). Minimising $w^2\sigma_b^2 - 2w\sigma_{b\pi}$ gives $w = \sigma_{b\pi}/\sigma_b^2$, which proves (b). Part (c) substitutes (1). □

The economic content is a negative result for a common argument. That fiat cash loses purchasing power on average (38.0% cumulative CPI-U and 29.9% HICP inflation between December 2014 and December 2025 in our data) does not by itself favour Bitcoin over cash: the expected erosion hits both legs of (7) identically. A store-of-value role, in the mean-variance sense, requires $h^c > 0$, that is, Bitcoin must pay off when inflation surprises. This is an empirical covariance, estimable on monthly data, and it is numéraire specific.

3.5 Regime dynamics via Matrix Pencil

Let $\{\Sigma_t\}$ be rolling covariances of x_t and δ_t their geodesic deviations from the Karcher mean. We represent the demeaned deviation series as a sum of damped modes,

$$\delta_t - \bar{\delta} = \sum_{m=1}^{M} R_m z_m^t + \varepsilon_t, \qquad z_m = e^{(\alpha_m + i2\pi\nu_m)\Delta}, \tag{9}$$

and estimate poles z_m and amplitudes R_m by the Matrix Pencil method [11]: a Hankel matrix of the series is truncated by singular value decomposition to rank M, and the poles are the eigenvalues of the pencil formed by the shifted right singular subspaces. The damping rates $\alpha_m < 0$ give regime half-lives $\ln 2/|\alpha_m|$, and the frequencies ν_m capture recurrent stress cycles.

Corollary 3.8. *Because* $\{\delta_t\}$ *is numéraire invariant (Theorem 3.2), the Matrix Pencil poles, damping rates, frequencies and half-lives estimated from it are identical for the dollar and the euro investor.*

This is the property that makes the decomposition suitable for comparing the two investors of the crossed experiments: any difference in their outcomes must come from the coordinates (weights, thresholds, hedging coefficients), not from a different reading of when regimes change.

4 Data and empirical design

Pilot sample (annual, 2015 to 2025). We use the series reported in the INICORBAF study, all traceable to primary sources: calendar-year returns of the MSCI World Net Total Return USD index (code 990100, MSCI factsheet), year-end Bitcoin closes in USD (StatMuse compilation), year-end ECB euro reference rates against the dollar, BLS CPI-U (not seasonally adjusted) and Eurostat HICP for the euro area ($2015 = 100$) at December 2014 and December 2025. With eleven annual returns, all statistics in Section 5 are descriptive; they illustrate the algebra and replicate the study, not inference.

Main sample (daily, January 2015 to October 2026). The daily study uses FRED series CBBTCUSD (Coinbase Bitcoin), MSCI World Net Total Return USD daily levels (or, where licensing prevents redistribution, the iShares MSCI World ETF as a disclosed proxy), the ECB daily reference rate EXR.D.USD.EUR.SP00.A, and monthly CPI-U and HICP. Calendars follow the protocol of the original study: returns are measured between consecutive MSCI trading days, with weekend Bitcoin moves accumulated into the Monday return, and sensitivity to a one-day lag is reported.

Estimation steps. (1) Rolling 3×3 covariances of x_t over 252 trading days, advanced monthly, mapped to the euro numéraire by (2). (2) Karcher mean by the fixed-point gradient iteration on $\mathcal{P}_n$ [1], and geodesic deviations δ_t. (3) Matrix Pencil decomposition of δ_t with rank chosen by a singular-value threshold. (4) Regime-specific w_{mv} and Euler shares from the (r_m, r_b) blocks. (5) Monthly hedging coefficients h^{USD} and h^{EUR} with HAC standard errors. (6) Sharpe ratio differences between equity-only and mixed portfolios (weights 1%, 2.5%, 5%, 10%, 20%; monthly, quarterly, annual and threshold rebalancing; costs included) tested with the stationary bootstrap [20, 14]. (7) DCC correlations [9] as a conventional benchmark against which the invariant statistics are compared.

5 Pilot evidence on annual data

Replication. Table 1 reproduces the annual backtest of the practitioner study. All figures match the original to the second decimal or within rounding

of the last digit; the small discrepancies, visible mainly for the all-Bitcoin portfolio (27,347 against 27,330), arise because the study's table reports Bitcoin closes rounded to the dollar.

Table 1: Annual backtest, MSCI World Net USD and Bitcoin, annual rebalancing, 2015 to 2025

Weight	CAGR (%)	Std. dev. (%)	Sharpe	Year-end DD (%)	Value of 100
0%	10.92	15.02	0.79	−18.14	313
1%	12.49	16.80	0.82	−18.60	365
2.5%	14.70	20.66	0.80	−19.29	452
5%	18.03	28.77	0.73	−20.45	619
10%	23.82	47.18	0.63	−22.75	1,048
20%	33.27	86.08	0.56	−27.36	2,356
100%	66.55	404.38	0.48	−73.56	27,347

Eleven annual observations; the year-end drawdown is a lower bound, in absolute value, of the daily maximum drawdown. Gross of costs.

Risk concentration. On the annual covariance, $\sigma_a = 15.02\%$, $\sigma_b = 404.4\%$ and $\rho = 0.373$, so that $\beta_{b|a} = \rho\sigma_b/\sigma_a \approx 10.0$ and, by Proposition 3.5, $w_{\mathrm{mv}} = -1.28\%$. Every positive allocation therefore concentrates risk. Table 2 and Figure 1(a) quantify the effect: a 1% weight carries 13.7% of portfolio variance, a 5% weight 62.4%. The latter equals the sum of the pure Bitcoin term (49.4%) and half the covariance term (13.0%) of the variance decomposition reported in the original study, which provides an independent check.

Table 2: Euler variance share of Bitcoin (Proposition 3.5), annual data

Capital weight w	1%	2.5%	5%	10%	20%
Variance share $\mathrm{RS}_b(w)$	13.7%	36.9%	62.4%	82.6%	93.2%
Ratio $\mathrm{RS}_b(w)/w$	13.7	14.7	12.5	8.3	4.7

Numéraire dependence. Table 3 splits the sample into 2015 to 2019 and 2020 to 2025 and estimates the 3×3 covariance of annual log returns x_t in each window, then maps it to the euro numéraire. The affine-invariant distance between the two regimes is 1.7690 in both numéraires, as Theorem 3.2 requires; the log-Euclidean distance differs, and the Bitcoin to equity correlation in the first window falls from 0.77 to 0.39 when the same data are read in euros (Figure 1(b)). The windows contain five and six observations, so these numbers illustrate the theorem; they are not estimates to be interpreted.

Table 3: Regime statistics in the two numéraires, annual log returns

Statistic	USD numéraire	EUR numéraire
AIRM distance, 2015-19 vs 2020-25	1.7690	1.7690
Log-Euclidean distance	1.3850	1.3811
$\mathrm{Corr}(r_b, r_m)$, 2015-2019	0.768	0.388
$\mathrm{Corr}(r_b, r_m)$, 2020-2025	0.774	0.686

Monetary view. The crossed experiments of the study are reproduced: 100 dollars in Bitcoin at end 2014 become about 27,347 dollars at end 2025, against 96.78 dollars for euro cash; 100 euros in Bitcoin become about 28,257 euros, against 103.33 euros for dollar cash. Deflating by CPI-U and HICP, Bitcoin wealth grows by factors of about 198 (dollar investor) and 218 (euro investor), while cash retains 72.5% and 77.0% of its purchasing power. On annual log returns, $\mathrm{Var}(r_b) = 1.2024$, $\mathrm{Var}(f) = 0.0070$ and $\mathrm{Cov}(r_b, f) = 0.0478$, so that $\mathrm{Var}(r_b - f) = 1.1138$: over this sample the euro tended to strengthen in strong Bitcoin years, which slightly dampened Bitcoin variance for the euro investor. By Proposition 3.7, none of these level comparisons identifies a store-of-value role; that requires the monthly hedging coefficients of the daily study.

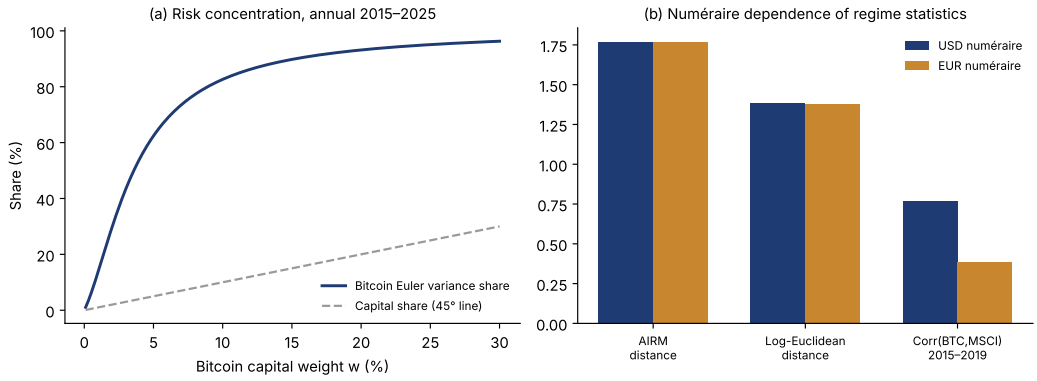


Figure 1: (a) Euler variance share of Bitcoin against its capital weight; the gap is positive everywhere because $w_{\mathrm{mv}} < 0$. (b) The affine-invariant regime distance is identical in the two numéraires, the log-Euclidean distance and the correlation are not.

6 Daily study: hypotheses and protocol

The daily study tests the following hypotheses, each tied to a theoretical result.

H1 (invariant regimes). The geodesic deviation series identifies stress

regimes (2018, March 2020, 2022, 2025 to 2026) identically for both investors, while DCC correlations rank the same regimes differently across numéraires (Theorem 3.2).

H2 (persistence). Regime departures are transient, with Matrix Pencil half-lives shorter after 2020 than before, consistent with tighter integration of Bitcoin with equities (Corollary 3.8).

H3 (risk concentration). In every regime w_{mv} is below the allocations commonly proposed (1% to 5%), so that Bitcoin's variance share exceeds its capital share (Proposition 3.5); the gap (6) is largest in high-correlation regimes.

H4 (store of value). The hedging coefficients h^{USD} and h^{EUR} are not significantly positive, and differ in sign or size across numéraires (Proposition 3.7).

H5 (efficiency). No fixed Bitcoin weight yields a Sharpe improvement over the equity-only portfolio that is robust to the stationary bootstrap, to rebalancing frequency and to costs.

The code released with the paper (`daily_pipeline.py`) implements all steps of Section 4 and includes a self-test that verifies numerically the equivariance of the Karcher mean and the invariance of (3) under (2), and that recovers the damping rate and frequency of a synthetic damped cycle with the Matrix Pencil routine.

7 Conclusion

The question whether Bitcoin is an equity-like investment or a monetary asset has been answered so far with statistics that depend on the investor's unit of account. Once Bitcoin is allowed to compete with fiat currencies, the unit of account itself becomes a variable, and the analysis must be invariant to it. The affine-invariant geometry of covariance matrices provides that invariance exactly: regime distances, the reference regime, geodesic deviations and the regime dynamics extracted from them are the same in dollars and in euros. Within this frame the two halves of the original question receive precise answers. On the investment side, a small allocation concentrates risk whenever its weight exceeds the minimum-variance weight, which on the available evidence it always does. On the monetary side, average fiat depreciation is irrelevant to the choice between Bitcoin and cash, and only an inflation-hedging covariance can support a store-of-value claim.

Two extensions are natural. The first is strategic: the numéraire-invariant regime geometry provides the state space for a game between currency issuers with fixed and discretionary supply, in the spirit of Schilling and Uhlig [21] and Fernández-Villaverde and Sanches [10], in which equilibrium policies can be characterised on the manifold rather than on a single coordinate chart. The second is computational: the Karcher mean and the geodesic deviations extend directly to higher-dimensional systems with gold, bonds and additional currencies, where the invariance under general congruences remains exact.

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