

# Universal Convergence Toward an Exponential Law and Geometric Invariance of Distributions via a Quasi-reversible Non-iterative Transformation

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## Abstract

We introduce a reversible, non-iterative transformation applied to signals with a wide variety of statistical distributions, including thirty-five general distributions as well as spectral distributions arising from random matrix theory (GUE, GOE, GSE, Wigner's semicircle law, and the Wigner–Dyson law). In all cases studied, the transformation induces convergence toward an exponential distribution with a unit coefficient of variation, independently of the initial distribution. Applying the inverse transformation makes it possible to reconstruct the original signal with small numerical deviations. The introduction of a single control parameter then allows the coefficient of variation to be varied over a wide range while preserving the shape of the reconstructed distribution, thereby defining a geometric invariance. The results are obtained on signals containing up to points, with computation times of less than one second.

## I. Introduction

The diversity of statistical distributions encountered in natural, artificial, and spectral signals represents a recurring obstacle to unified analysis, modeling, and controlled data transformation. In particular, distributions arising from chaotic, pseudo-random phenomena or from random matrix theory exhibit very different morphologies, making it difficult to apply a single reversible transformational framework.

In this work, we introduce a reversible, non-iterative transformation, constructed as the composition of two elementary operators, capable of projecting a wide variety of signals onto an exponential distribution with a unit coefficient of variation. This property is observed independently of the initial distribution [5],[6] including multimodal distributions, compact-support distributions, highly skewed distributions, and those originating from classical spectral ensembles (GUE, GOE, GSE, Wigner's semicircle law, and the Wigner–Dyson law).

The main originality of the approach lies in the effective reversibility of the transformation, which enables recovery of the original signal through the application of the inverse transform, as well as in the introduction of a simple control parameter that allows continuous variation of the coefficient of variation without altering the geometric shape of the reconstructed distribution. This property defines a robust geometric invariance, independent of scale and of the type of initial distribution.

The results are numerically validated on a set of forty distinct distributions, using signals containing up to points and computation times below one second, demonstrating the universality, stability, and computational efficiency of the method. Seven examples are selected from the thirty-five studied to illustrate the method. Readers who wish to verify the results can do so in just a few clicks; the maximum computation time is 1.77 s (example 21), using a sample size of  $10710^{7107}$  on an HP G9 system.

In the literature, [1] is a fundamental reference on the reversibility and quasi-reversibility of stochastic networks. Reference [4] proposes a general theory of stability and convergence for Markov chains. Reference [2] presents a general framework for the study of Markov chains and invariant measures, while [3] offers a modern synthesis of stochastic networks and queueing systems.

Existing approaches primarily use the exponential law as a statistical model or as a reference for specific phenomena. In contrast, in this work the exponential law with unit coefficient of variation ( $cv = 1$ ) is considered differently: it emerges as a universal convergence point obtained through transformation, independently of the initial signal distribution.

Moreover, a simple modification of the transformation makes it possible to preserve the geometric shape of the distribution while continuously varying its coefficient of variation over several orders of magnitude. This property defines a geometric invariance in the sense that the structure of the distribution is preserved despite significant variations in its statistical parameters.

These two properties—convergence toward an exponential law with  $cv = 1$  and geometric invariance under controlled modification of the coefficient of variation—are illustrated in

the following sections using a set of unimodal, multimodal, and spectral signals of very diverse nature.

**Verification procedure**

To verify the results, proceed as follows:

1. Select the corresponding constants.
2. Select (A) to display the target distribution.
3. Select (A, B, D) for multimodal examples and (A, B) for the rest to display the exponential distribution.
4. Select (A, B, C,) for reconstruction.
5. For invariance, select (A, B, C, D, E) and (D, or E) choose either CVmin or CVma.

The transformation is :

$$Y_T = (y \cdot (1/\text{acos}(\cos(x))))^k$$

Observation : We can also use  $\text{acot}(\cot(x^c))$  or  $\cot(x^c)$ ,  $\cos(x^c)$  instead of  $\text{acos}(\cos(x^c))$ , where  $c$  can take values of 1 or any other value greater than 2 ; however, the constant  $k$  must be modified to achieve the objective.

**II. MULTIMODAL EXAMPLES**

**% CHOISE OF NUMBER OF MODE**

- x=1:1:10^7;
- a=2; b=1; c=6; % 12M
- a=2; b=1; c=5; % 11M
- a=2; b=1; c=4; % 10M
- a=2; b=1; c=3; % 9M
- a=3; b=2; c=3; % 8M
- a=1; b=3; c=1; % 7M
- a=1.5;b=3; c=1.5; % 6M
- a=3; b=3; c=3 ; % 5M
- a=1; b=1; c=0 ; % 4M

```

a=0; b=1; c=1 ;    % 3M
a=0; b=1; c=0 ;    % 2M

% CONTROL PARAMETERS

k=0.416751;        % 12M
k=0.4191001;       % 11M
k=0.4208123;       % 10M
k=0.4220510;       % 9M
k=0.4193810;       % 8M
k=0.4226810;       % 7M
k=0.4222210;       % 6M
k=0.418110;        % 5M
k=0.4240992;       % 4M
k=0.4239451;       % 3M
k=0.4240451;       % 2M

% GENERATING FUNCTION OF MULTIMODAL DE
DISTRIBUTIONS

x=1:1:10^7 ;
y1=1.25*acot(0.43*sin(x.^5))+acot(0.15*cos(x.^15));
y2=acot(2.652+cot(x.^25))+acot(0.1*cos(x.^35));
y3=acot(0.15*cos(x.^37));
y=20+a*y1+b*y2+c*y3;          % A
y= (y.*(1./(acos(cos(x))))).^k; % B
y=(y.^(1/k)).*(acos(cos(x))); % C
y=abs(y); (D)

% GEOMETRICAL INVARIANCE

p=10^-6; y=y+p;    CVmax (E)
p=10^6; y=y+p;    CVmin (F)

figure (1);
histogram(y,200,'Normalization','pdf');mean_y=mean(y);std_y=std(y);
fprintf('Mean:%.5f\n',mean_y);fprintf('standard deviation=%.5f\n',std_y) ;
text(0.65,0.98,sprintf('Mean:%.5f',mean_y),'Units','normalized','FontSize',12,'Color','r');
text(0.34,0.98,sprintf('Std:%.5f',std_y),'Units','normalized','FontSize',12,'Color','r');
cv=std_y/mean_y ;
text(0.01,0.98,sprintf('cv:%.6f',cv),'Units','normalized','FontSize',12,'Color','r');
title(' DISTRIBUTION');xlabel('Observed Value');
ylabel('Probability Density');

```

III. OTHER EXAMLES

%%%%%%%%%% 2 TRIANGLES 32

```

x=1:1:10^7;
y =90+ 10*(acot(cot(x.^35))-3*acot(0.001*cot(x.^33))+acot(cot(x.^32))); %
A
y=(y.*(1./(acos(cos(x))))).^0.4151480251); % B
y=y.^(1/0.4151480251).*(acos(cos(x))); % C
% GEOMETRICAL INVARIANCE

```

```

p=10^-6; y=y+p; CVmax (D)
p=10^6; y=y+p; CVmin (E)

```

%%%%%%%%%% 2 BLOCKS 13

```

x=1:1:10^7;
y1=65+5*(acot(0.3e-1*cos(x.^35))-5*acot(0.1e-2*tan(x.^33)));
y2=-0.126*acot(cot(x.^32))-7*atan(tan(x.^3.2));
y=y1+y2; % A
y=(y.*(1./(acos(cos(x))))).^0.4110358581); % B
y=y.^(1/0.4110358581).*(acos(cos(x))); % C
% GEOMETRICAL INVARIANCE

```

```

p=10^-6; y=y+p; CVmax (D)
p=10^6; y=y+p; CVmin (E)

```

%%%%%%%%%% 2 CORN 14

```

x=1:1:10^7;
y=15+(10*atan(1.6*tan(x.^35)).*acot(7*sin((x+1).^37))).^0.32;
y=abs(y); % A
y=(y.*(1./(acos(cos(x))))).^0.4242345211); % B
y=y.^(1/0.4242345211).*(acos(cos(x))); % C

```

```

% GEOMETRICAL INVARIANCE
p=10^-6; y=y+p; CVmax (E)
p=10^6; y=y+p; CVmin (F)

```

%%%%%%%%%% MAY 15

```

x=1:1:10^7;
y= 0.5*(1+cos(x.^3.7)); % A
y= ((y.*(1./(acos(cos(x))))).^0.39698241); % B
y=(y.^(1/0.39698241).*(acos(cos(x))))); % C

```

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%% UNIFORM 16

y=acot(cot(x.<sup>33</sup>)); % A  
 y=(abs(y).\*(1./(acos(cos(x))))).^(0.4106818311); % B  
 y=(y.(1/0.4106818311)).\*(acos(cos(x))); % C

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%% hedgehog 18  
 x=1:1:10<sup>7</sup>;

y=6.3+acot(2\*cos(x.<sup>9</sup>))+acot(2\*cos(x.<sup>8</sup>))+acot(0.2\*cos(x.<sup>7</sup>))+acot(0.2\*cos(x.<sup>6</sup>));  
 %(A)

y=(y.\*(1./(acos(cos(x))))).^(0.4208702411); %(B)  
 y=(y.(1/0.4208702411)).\*(acos(cos(x))); %(C)

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%% MP 19  
 x=1:1:10<sup>7</sup>;

ya =acot(2.45\*cot(x.<sup>37</sup>)); yb =acot(2.45\*cot(x.<sup>23</sup>));  
 y=0.01\*(103\*(0.1+2.36\*abs(yb.<sup>2</sup>-ya.<sup>2</sup>)).^0.665);%(A)  
 y=(y.\*(1./(acos(cos(x))))).^(0.4076200551); %(B)  
 y=(y.(1/0.4076200551)).\*(acos(cos(x))); %(C)

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%% IMPULSION 20  
 x=1:1:10<sup>7</sup>;

y=10<sup>-2</sup>./(1.6+acot(cot(x.<sup>3</sup>))); % (A)  
 y=((y).\*(1./(acos(cos(x))))).^(0.3923105711); %(B)  
 y=(y.(1/0.3923105711)).\* acos(cos(x)); %(C)

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%%%%%% NOISE AND PICS 21

x=1:1:10<sup>7</sup>;

y1=1.25\*acot(0.143\*sin(x.^5))+acot(0.5\*cos(x.^15));  
 y2=acot(1.652+cot(x.^5))+1\*acot(0.001\*cos(x.^35));  
 y3=acot(0.15\*cos(x.^37))-0.5\*acot(0.15\*cos(x.^31.7));  
 y4=acot(1.652+cot(x.^2.5))+1\*acot(0.001\*cos(x.^3.5));  
 y5=acot(5+cos(x.^2.5))+3\*acot(0.001\*sin(x.^3.5));  
 y=1+0.01\*(a\*y1+b\*y2+c\*y3+d\*y4-0.83\*y5).\*abs(y3).^0.11; %

(A)

y=((y).\*(1./(acos(cos(x))))).^ (0.424124521) ; % (B)  
 y=(y.^(1/0.424124521)).\*(acos(cos(x))); % (C)

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%%%%%% UUU 22

x=1:1:10<sup>7</sup>;

y1=-0.432+abs(acot(0.43\*sin(x.^5)).^-3+2\*acot(0.21\*sin(x.^15)).^-5);  
 y2=0.52\*acot(0.2\* sin(x.^25))+1.52\* acot(0.215\*cos(x.^35))).^0.53);  
 y=y1+y2;y=abs(y); % (A)  
 y= ((y).\*(1./(acos(cos(x))))).^ (0.408344821); % (B)  
 y=(y.^(1/0.408344821)).\*(acos(cos(x))); % (C)

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%%%%%% HAND 23

x=1:1:10<sup>7</sup>;

y=-0.432+abs((acot(0.43\*sin(x.^5)).^-3+2\* acot(0.21\*sin(x.^15)).^-5)+0.52\*acot(0.2\* sin(x.^25))+1.52\* acot(0.215\*cos(x.^35))).^0.53; % A  
 y=abs((y).\*(1./(acos(cos(x))))).^ (0.419968321); % (B)  
 y=(y.^(1/0.419968321)).\*(acos(cos(x))); % (C)

**% GEOMETRICAL INVARIANCE**

p=10<sup>-6</sup>; y=y+p; CVmax (D)  
 p=10<sup>6</sup>; y=y+p; CVmin (E)

%%%%%%%%%%%%%% CLOCHE 24



```

y=4*0+acot(cot(x.^43))+acot(cot(x.^44)); % (A)
y=abs(y) ;
y= ((y.*(1./acos(cos(x))))).^0.40711831); % (B)
y=y.^(1/0.40711831).*(acos(cos(x))); % (C)

% GEOMETRICAL INVARIANCE
p=10^-6; y=y+p; CVmax (D)
p=10^6; y=y+p; CVmin (E)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% PULSEWAVE 29
x=1:1:10^7 ;
y=110+1.52*(acot(0.43*sin(x.^5)).^(-3)+2*acot(.21*sin(x.^15)).^(-5)+0.52*acot(0.2*sin(x.^25))+1.52*acot(0.215*cos(x.^35))).^3;
y=10^-2*y; y=abs(y-1); % A
y=( y.*(1./acos(cos(x))))).^0.40071071); % (B)
y=( y).^0.40071071).*(acos(cos(x))); % (C)

% GEOMETRICAL INVARIANCE
p=10^-6; y=y+p; CVmax (D)
p=10^6; y=y+p; CVmin (E)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% TWO U 30
x=1:1:10^7;
y=41+ 11*acos(.5*cos(x.^35)).*acot(0.1e-2*cot(x.^33));% (A)
y=abs((y.*(1./acos(cos(x))))).^0.406246611);% (B)
y=(y.^0.406246611).*(acos(cos(x))); % (C)

% GEOMETRICAL INVARIANCE
p=10^-6; y=y+p; CVmax (D)
p=10^6; y=y+p; CVmin (E)

```

#### IV.SPECTRAL DISTRIBUTIONS (GUE, GOE, GSE, WSD, QCW)

For the calculation, proceed as follows :

- 1\_ select the corresponding constants,
- 2\_ (A) to display the target distribution,
- 3\_ (A, B, D) to display the exponential distribution,
- 4\_ (A, B, C, D) reconstruction,

5\_invariance (A, B, C, D, (E or F, CV max or CV min)).

**%%%cts +QCW+WSD 31 +32**

x=1:10^-4:10^3;

a1=1;b1=1;p=0.98888589;d2=1;a2=0;b2=0;

T=2.000000000000294 ; a3=1 ; b3=1 ;

d3=2.44123198721;d6=1;d5=-1;d4=0;d7=2;

c2=0.545;f1=37;f2=35;d1=1;w=4;

**%%% cts GSE 33**

x=1:10^-4:10^3;

a1=0.7518;a2=5.65;a3=0.051;b1=1.526;b2=5.6636;

b3=0.051;c1=0.8\*10^(-10.1);c2=9.9705833111;w=0;

d1=0.3678;d2=0.1278;d3=1.55;d4=0.121;d5=0.80655;

d6=0.5483546346745236;f1=45;f2=3;T=0;p=1;d7=1;

**%%% cts GOE 34**

x=1:10^-4:10^3;

a1=1.3952445652;a2=5.5;a3=0.08;d1=0.38;d2=0.2;

c2=8.53265528988517; b1=2;b2=5.5; b3=0.1563771;

d1=0.38;d2=0.2;d3=1;d4=0;d5=1;d6=1;f1=45;f2=3;

T=0;p=1;d7=1;w=0;

**%%%cts GUE 35**

x=1:10^-4:10^3;

a1=1.0815399999999997;a2=5.8;a3=0.08;b1=2;b2=5.8;

b3=0.0819311242;c2=8.66826327010582091;f1=45;f2=3;

d1=0.38;d2=0.2;d3=1;d4=0;d5=1;d6=1;T=0;p=1;d7=1;w=0;

**%% UNIFICATION FUNCTION (U.F)**

y1=a1\*acot(p\*cot(x.^f1)).^d1+a2\*acot(cot(x.^33)).^a3 ;

y2=b1\*acot(p\*cot(x.^f2)).^d2+b2\*acot(cot(x.^47.3)).^b3;

y=w-T+d3\*abs(d6\*y1.^d7+d5\*y2.^d7-d4\*y2.^d7).^c2);

%%%

y=y-4; y=abs(y);k=0.405912321; % QCW A

y=y; k=0.4217574511; % WSD A

y=1.\*10^-9\*y;k=0.4204399611; % GSE A

y=10^-9\*y; k=0.4154124561; % GOE A

y=1.7\*10^-10\*y; k=0.418439891; % GUE A

y=(abs(y).\*(1./(acot(cot(x))))).^k); % B

y=(y.(1/k)) .\*(acot(cot(x))); %C

y=abs(y); % D

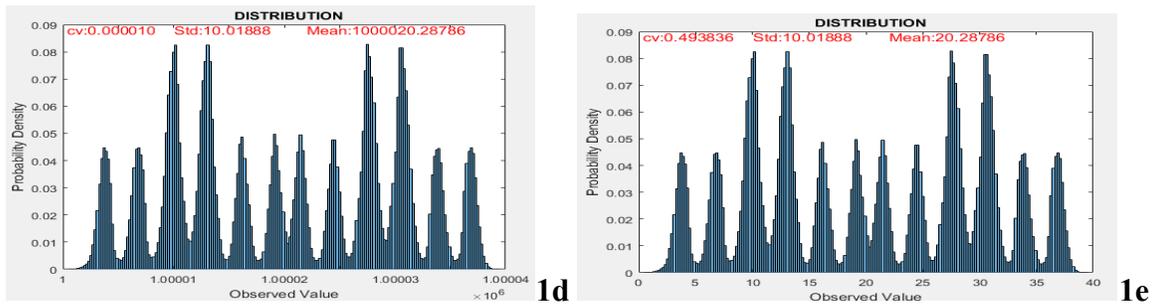
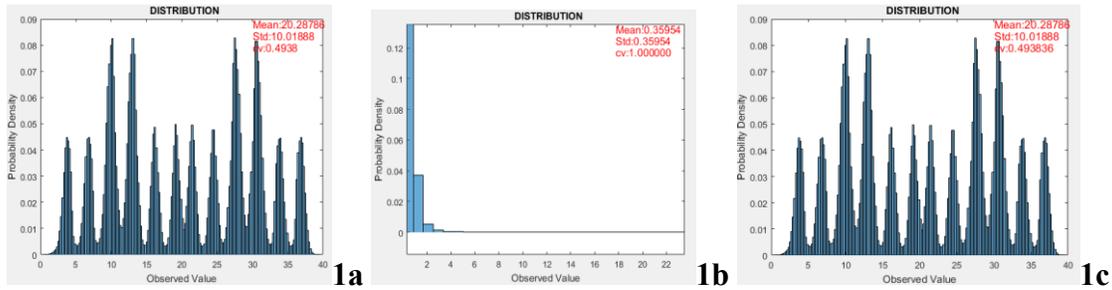
**% GEOMETRICAL INVARIANCE**

p=10^-6; y=y+p; CVmax (E)

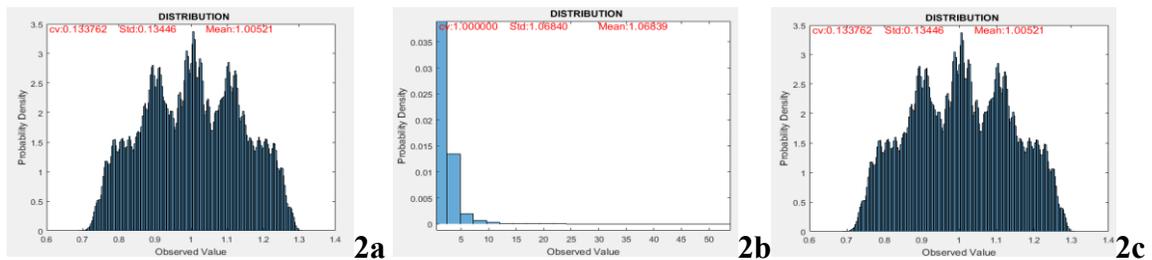
$$p=10^6; \quad y=y+p; \quad \text{CVmin (F)}$$

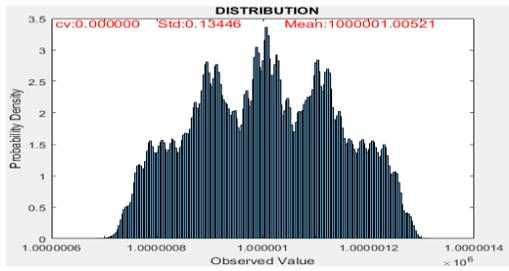
```
figure (1); histogram(y,200,'Normalization','pdf');mean_y=mean(y);std_y=std(y);
fprintf('Mean:%.5f\n',mean_y);fprintf('standard deviation=%.5f\n',std_y) ;
text(0.75,0.98,sprintf('Mean:%.5f',mean_y),'Units','normalized','FontSize',12,'Color','r');
text(0.35,0.98,sprintf('Std:%.7f',std_y),'Units','normalized','FontSize',12,'Color','r');
cv=std_y/mean_y ;
text(0.01,0.98,sprintf('cv:%.7f',cv),'Units','normalized','FontSize',12,'Color','r');
title(' DISTRIBUTION');xlabel('Observed Value');
ylabel('Probability Density');
```

**EX. N=12**

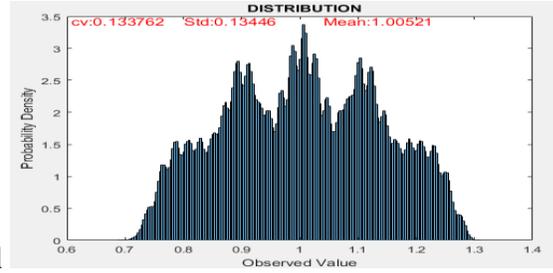


**EX. N=21**



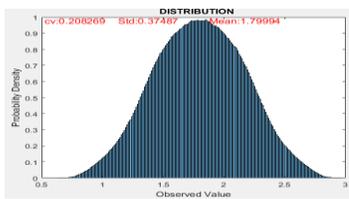


2d

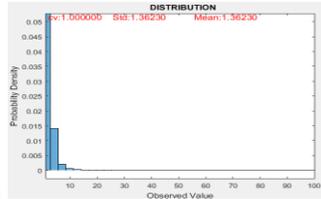


2e

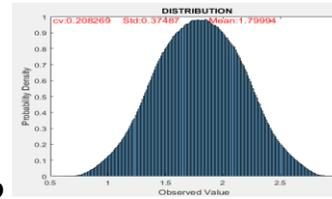
EX. N=24



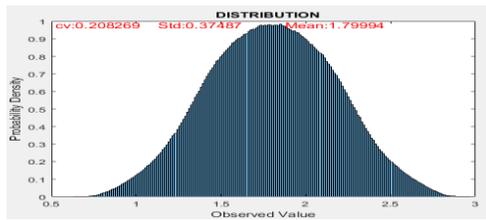
3a



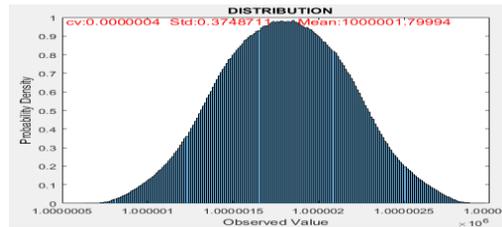
3b



3c

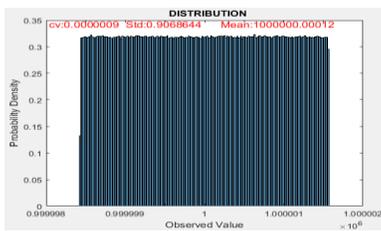


3d

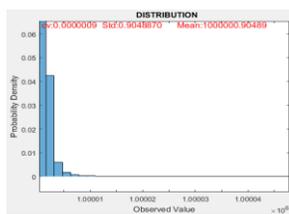


3e

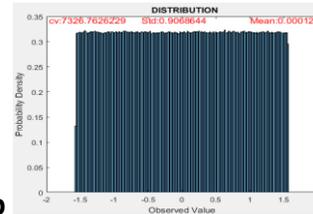
EX. N=16



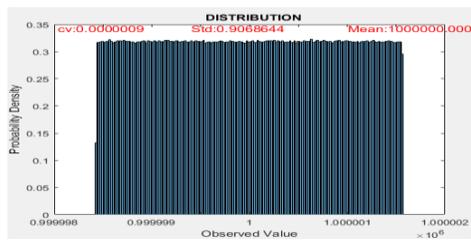
4a



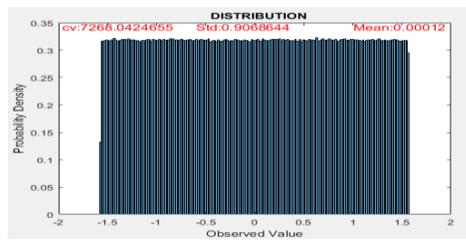
4b



4c

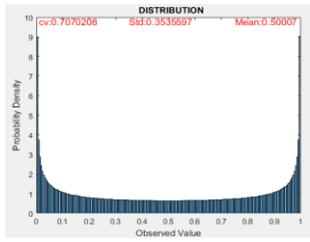


4d

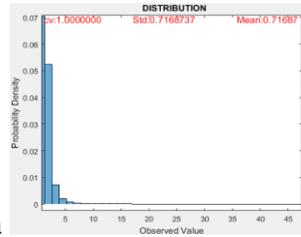


4e

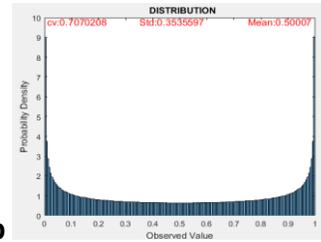
**EX. N=15**



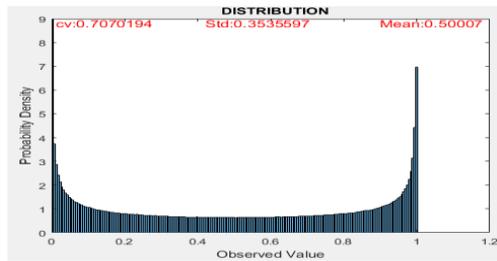
5a



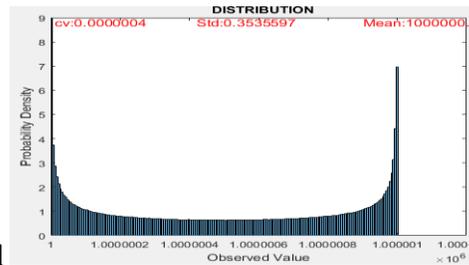
5b



5c

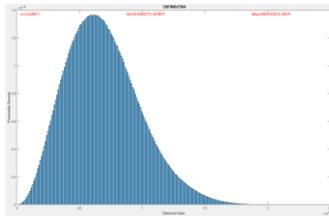


5d

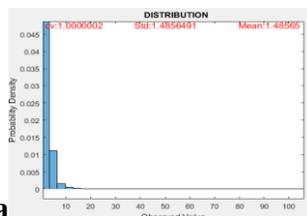


5e

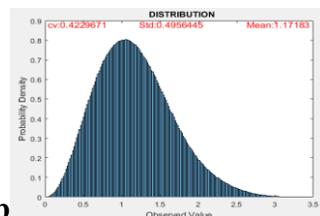
**EX. N=35**



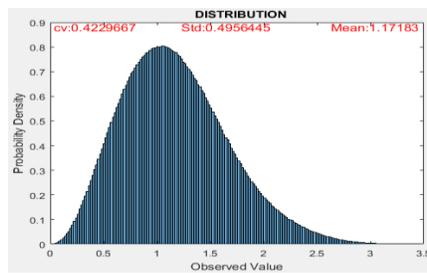
6a



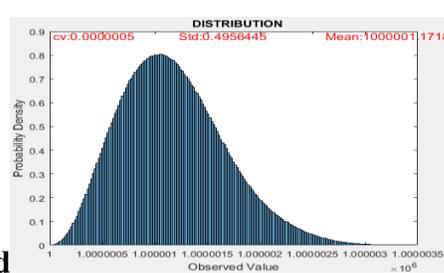
6b



6c

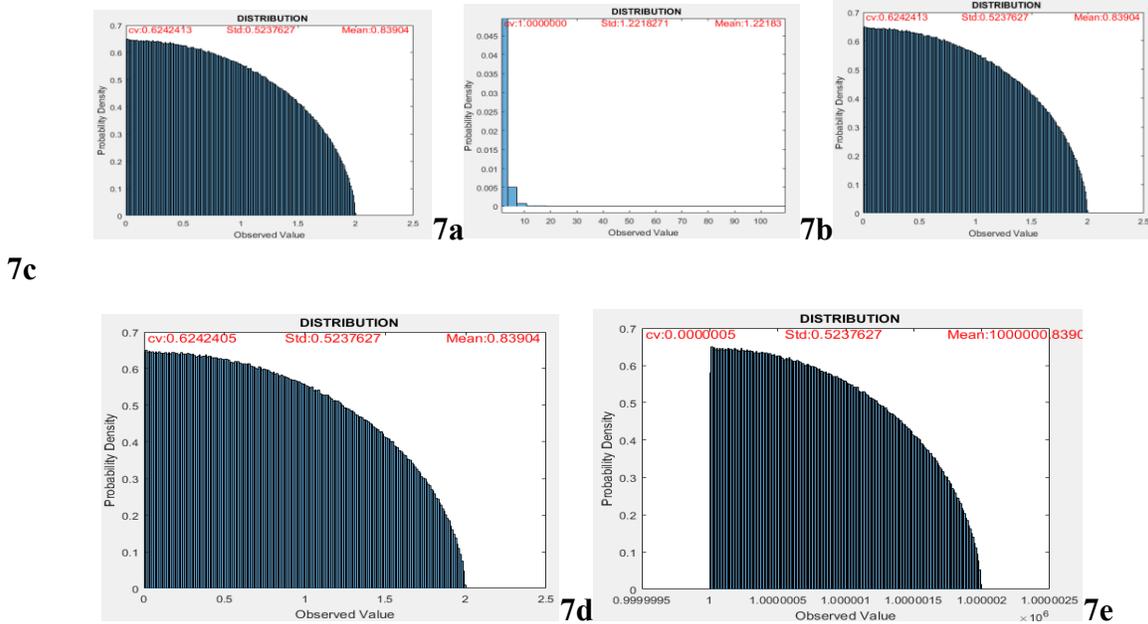


6d



6e

## EX. N=31



**Fig.** Illustration of the distributions associated with the different examples studied.  
 (a) Original distribution.  
 (b) Distribution with a zoom on the high-density region in order to highlight the fine structure.  
 (c) Distribution reconstructed from the model (proposed method).  
 (d–e) Distributions obtained after the invariance test, showing the statistical stability of the distribution with respect to the transformations considered.

## V. Conclusion

We have presented a universal, non-iterative transformation capable of driving a wide class of signals toward an exponential distribution with a unit coefficient of variation, independently of their original distribution. This convergence, observed for both classical distributions and spectral distributions arising from random matrix theory, highlights the existence of a common underlying structure in the signals studied. The ability to reconstruct the original signal via an inverse transformation, with small numerical deviations, establishes a robust quasi-reversibility of the process. The introduction of a single control parameter further allows the coefficient of variation to be adjusted over several orders of magnitude while preserving the geometric shape of the reconstructed distribution, constituting a strong geometric invariance property. From a numerical standpoint, the method stands out for its algorithmic simplicity and efficiency, enabling the processing of large-scale signals without

resorting to iterative techniques, differential equations, or classical spectral tools. These results open up prospects in universal signal analysis, the synthesis and controlled transformation of distributions, as well as potential applications such as data masking, spectrum generation, and the modeling of complex systems. They also suggest the existence of a family of non-iterative generators sharing a common fundamental structure, calling for further theoretical developments.

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**Received: January 7, 2026; Published: February 11, 2026**