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Mean and Stationary Characterization of a Compound Poisson Dam with Uniform Inputs and a Linear Release Rule

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Abstract

A compound Poisson storage model with uniformly distributed jump magnitudes and a linear state-dependent release rule is considered. The empty boundary is treated explicitly, so the transient mean is characterized jointly with the probability of an empty dam rather than by an unsupported closed-form expression. For an infinite-capacity dam, the stationary mean balance is obtained directly from conservation of mean input and output. For the stationary density, the bounded support of the uniform input distribution produces a piecewise structure: a second-order ordinary differential equation on the first unit interval and a delay-differential equation thereafter. For a finite-capacity dam, overflow is defined by truncation at the upper capacity; the same interior balance applies below capacity, while the mean equation contains an explicit overflow-loss term. These results provide a consistent characterization suitable for numerical evaluation and special-case analysis.

Mathematics Subject Classification: 60K30 (Primary); 60G10, 60J25 (Secondary).

Keywords: dam theory; compound Poisson process; uniform inputs; linear release rule; stationary characterization; overflow; finite dam; infinite dam

1. Introduction

Stochastic storage and dam models describe systems in which random inputs accumulate and are removed according to a prescribed release rule. Classical work includes models with general release rules, compound Poisson inputs, finite or infinite capacity, and boundary behavior at zero. Relevant references include Harrison and Resnick (1976), Brockwell, Resnick and Tweedie (1982), Asmussen (1987), Asmussen and Kella (1996), and Kella and Stadje (2001).

Earlier work by Al-Eideh (2002), Al-Zalzalah and Al-Eideh (2002), and Zainal, Al-Eideh and Marafie (2013) considered related dam and reservoir models. The present paper focuses on bounded uniform jump magnitudes. This bounded support is consequential: after one unit of content, the stationary balance involves the density one unit below the current level, producing a delay structure rather than a single elementary density over the whole state space.

The purpose of the paper is therefore a characterization rather than an assertion of a closed-form stationary density. The model, boundary convention, mean balance, finite-capacity overflow rule, and stationary equations are stated explicitly.

1.1 Model formulation

Let $\{N_t, t \geq 0\}$ be a Poisson process of rate $\beta>0$. Let $U_1,U_2,\dots$ be i.i.d. positive random variables, independent of N, and define the compound Poisson input

$$A_t = \Sigma_{n=1}^{N_t} U_n \quad (1)$$

For the infinite-capacity model, let X_t be the content and $X_0=x\geq 0$. Between jumps the content decreases according to the release rule until it reaches zero, where it remains until the next positive input. Equivalently, the dynamics are interpreted with the nonnegativity boundary convention

$$dX_t = dA_t - r(X_t)dt, \;\; X_t \geq 0 \quad (2)$$

The release rule is

$$r(0)=0, \quad r(x)=kx+c \text{ for } x>0, \quad k>0, c\geq 0 \quad (3)$$

Throughout the paper the jump magnitude is uniform on (0,1), so

$$U \sim \text{Uniform}(0,1), \quad m=E[U]=1/2 \quad (4)$$

For a finite dam of capacity V>0, an input jump that would exceed capacity is truncated at V:

$$X_t = \min\{X_{t-}+U, V\} \quad (5)$$

The excess amount is overflow. Since $r(V)>0$, the process leaves V immediately after a jump in the deterministic flow; no positive holding time at V is imposed by this convention.

2. Mean Results

2.1 Infinite-capacity dam

Let $M_x(t)=E_x[X_t]$ and $p_0(t)=P_x(X_t=0)$. Since $E[A_t]=\beta mt$ and $r(X_t)=kX_t+c$ on $\{X_t>0\}$ but $r(0)=0$,

$$E_x[r(X_t)] = kM_x(t)+c[1-p_0(t)] \quad (6)$$

Consequently, whenever differentiation under expectation is justified, the transient mean satisfies

$$M_x'(t)=\beta m-kM_x(t)-c[1-p_0(t)], \quad M_x(0)=x \quad (7)$$

For uniform inputs this becomes

$$M_x'(t)=\beta/2-kM_x(t)-c[1-p_0(t)] \quad (8)$$

Equation (8) is a characterization of the mean; it is not closed in M_x alone because the empty-dam probability is part of the boundary dynamics. This avoids replacing $c[1-p_0(t)]$ by c when the dam can be empty.

If a stationary distribution exists with stationary empty probability ν_0 and finite stationary mean M_∞, stationarity gives the mean flow balance

$$kM_\infty + c(1-\nu_0) = \beta m = \beta/2 \quad (9)$$

2.2 Finite-capacity dam

For capacity V, let $L_V(t)$ denote cumulative overflow lost up to time t. The pathwise content balance is

$$X_t = x + A_t - \int_0^t r(X_s)ds - L_V(t) \quad (10)$$

Hence the finite-dam mean satisfies

$$M_{x,V}'(t)=\beta/2-kM_{x,V}(t)-c[1-p_{0,V}(t)]-\ell_V(t) \quad (11)$$

where $\ell_V(t)=dE[L_V(t)]/dt$ when this derivative exists. Thus overflow must be retained in a correct finite-capacity mean equation. In stationarity, with mean overflow rate ℓ_V, the balance is

$$kM_V + c(1-\pi_0) + \ell_V = \beta/2 \quad (12)$$

3. Stationary Characterization

Let B be the distribution function of U and write $\bar{B}(z)=1-B(z)$. For uniform (0,1) input,

$$\bar{B}(z)=1-z \text{ for } 0\le z\le 1; \quad \bar{B}(z)=0 \text{ for } z>1 \quad (13)$$

3.1 Infinite-capacity dam

Assume a stationary probability distribution exists under the standard stability and accessibility conditions for the storage model. Let ν_0 be its mass at zero and let $g(x)$ be its density on $x>0$. The stationary level-crossing/storage balance is

$$(kx+c)g(x)=\beta\{\nu_0 \bar{B}(x)+\int_0^x \bar{B}(x-y)g(y)dy\} \quad (14)$$

For $0<x<1$, equation (14) becomes

$$(kx+c)g(x)=\beta\{\nu_0(1-x)+\int_0^x(1-x+y)g(y)dy\} \quad (15)$$

Differentiating equation (15) once gives

$$(kx+c)g'(x)+kg(x)=\beta\{-\nu_0+g(x)-\int_0^x g(y)dy\} \quad (16)$$

Differentiating again gives the ordinary differential equation

$$(kx+c)g''(x)+(2k-\beta)g'(x)+\beta g(x)=0, \quad 0<x<1 \quad (17)$$

For $x>1$, bounded jump support changes the balance to

$$(kx+c)g(x)=\beta\int_{x-1}^x(1-x+y)g(y)dy \quad (18)$$

Two differentiations yield the delay-differential equation

$$(kx+c)g''(x)+(2k-\beta)g'(x)+\beta g(x)-\beta g(x-1)=0, \quad x>1 \quad (19)$$

The stationary solution must satisfy nonnegativity, the balance/matching conditions at the unit-interval boundaries, and normalization

$$\nu_0 + \int_0^\infty g(x)dx = 1, \quad g(x)\geq 0 \quad (20)$$

A positive constant density on the whole half-line cannot satisfy (20), so the constant infinite-dam density asserted in the earlier draft is not retained.

3.2 Finite-capacity dam

Let π_0 be the stationary mass at zero for the finite dam and $g_V(x)$ the interior density for $0<x<V$. Under the truncation/overflow convention (5), jumps that overshoot V are lost at the upper boundary. For interior levels $x<V$, such overshoots do not create additional crossings of the level x; consequently the local interior balance has the same form as (14), with ν_0 and g replaced by π_0 and g_V:

$$(kx+c)g_V(x)=\beta\{\pi_0 \bar{B}(x)+\int_0^x \bar{B}(x-y)g_V(y)dy\}, \quad 0<x<V \quad (21)$$

Therefore, on $0<x<\min\{1,V\}$, g_V satisfies

$$(kx+c)g_V''(x)+(2k-\beta)g_V'(x)+\beta g_V(x)=0 \quad (22)$$

If $V>1$, then on $1<x<V$ it satisfies

$$(kx+c)g_V''(x)+(2k-\beta)g_V'(x)+\beta g_V(x)-\beta g_V(x-1)=0 \quad (23)$$

Because the release rate at V is positive and the model imposes no holding at V, the upper boundary is not assigned a stationary atom solely as a consequence of truncation. The stationary probability is normalized by

$$\pi_0 + \int_0^V g_V(x)dx = 1 \quad (24)$$

The overflow rate is determined by stationary jump overshoots. A convenient characterization is

$$\ell_V = \beta\, E_\pi[(X+U-V)^+] \quad (25)$$

where X has the stationary pre-jump content distribution and U is an independent Uniform(0,1) jump. Equation (25), together with (12), provides a consistency check on any numerical stationary solution.

4. Discussion and Conclusion

The corrected formulation separates three issues that must not be conflated: the empty boundary, finite-capacity overflow, and the bounded support of the jump distribution. The empty boundary introduces $p_0(t)$ into the transient mean equation; finite capacity introduces an overflow-loss term; and uniform jump sizes produce a piecewise stationary equation with a one-unit delay beyond the first interval.

For the infinite dam, the stationary mean balance is $kM_\infty+c(1-\nu_0)=\beta/2$. For the finite dam, the corresponding balance includes the mean overflow rate: $kM_V+c(1-\pi_0)+\ell_V=\beta/2$. The stationary density is characterized by an ordinary differential equation on the first unit interval and a delay-differential equation thereafter, supplemented by boundary, matching, nonnegativity, and normalization conditions. Closed-form density claims from the earlier manuscript are therefore replaced by a characterization that is consistent with the stated model.

Further work may solve the interval equations recursively or numerically and study parameter regimes in which additional closed forms are available.

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