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Finslerian Hypersurfaces and

Randers Change of Finsler Metric

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Abstract

In the present paper we have studied the Finslerian Hypersurface and Randers Change of Finsler metric. We have also proved that Randers change makes three types of hypersurfaces invariant under certain condition. These three type of hypersurfaces are hyperplanes of first, second and third kind.

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1. Introduction

Let $F^n = (M^n, L)$ be an n – dimensional Finsler Space on a differentiable Manifold M^n, equipped with the fundamental function L(x ,y) . In 1984 C. Shibta [3] introduced the transformation of Finsler metric

(1.1) $L^*(x,y) = f(L,\beta)$,

where $\beta = b_i(x)y^i$, $b_i(x)$ are the components of a covariant vector in F^n and f be a positively homogeneous function of degree one in L and β . This change of metric is called a β-change.

In the present paper the metric function is defined as

(1.2) $L^*(x,y) = L(x,y) + A(x,y)$,

where A(x,y) $= a_i(x)y^i$ is a 1-form on the manifold M^n. The change of metric is called a Randers change.
If L(x,y) reduces to the metric function of Riemannian space then $L^*(x,y)$ reduces to the metric function of space generated by Randers metric.

On the other hand in 1985, M.Matsumoto investigated the theorey of Finslerian hypersurface [5] . In the year 2018, G. Shanker and R.S. Kushwaha [4] studied the Finslerian hypersurface and Quartic Change of Finsler metric . Again in the year 2005, Prasad and Tripathi [2] studied the Finslerian hypersurfaces and Kropina Change of Finsler metric and obtained different results in this paper.

2. Preliminaries

The hypersurface $F^{n-1} = \left(M^{n-1}, \bar{L}(u,v)\right)$ of the Finsler space $F^n = \left(M^n, L(x,y)\right)$ may be represented by the equations $x^i = x^i(u^\alpha)\,, i = 1,2 \ldots\ldots, n\,;\ \alpha = 1,2,\ldots, n-1;$ where u^α are Gaussian coordinates on F^{n-1} . We suppose that the matrix consisting of projection factors $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ is of rank (n-1). Then B_α^i may be regarded as (n-1) linearly independent vectors tangent to F^{n-1} at a point u^α. The supporting element y^i at a point u^α of F^{n-1} is assumed to be tangential to F^{n-1} so that

$$y^i = B_\alpha^i v^\alpha \ .$$

Thus v^α is the supporting element of F^{n-1} at a point u^α. The metric tensor $g_{\alpha\beta}(u,v)$of F^{n-1} is given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \quad ,$$

where g_{ij} are components of the fundamental metric tensor of F^n . The components $N^i(u, v)$ of unit normal vector at a point u^α of F^{n-1} are given by

(2.1) (a) $g_{ij} B^i_\alpha N^j = 0$,
(b) $g_{ij} N^i N^j = 1$.

If (B^α_i , N_i) be the inverse matrix of (B^i_α , N^i) , then we have

(2.2) (a) $B^\alpha_i B^i_\beta = \delta^\alpha_\beta$,
(b) $B^i_\alpha N_i = 0$,
(c) $B^\alpha_i N^i = 0$,
(d) $N^i N_i = 1$,
(e) $B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j$.

We introduce the following important tensors from Cartan's C-tensor $\left(C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}\right)$:

(2.3) (a) $M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k$,
(b) $M_\alpha = C_{ijk} B^i_\alpha N^j N^k$.

The induced Finsler connection $\mathrm{IF}\Gamma = \left(\mathrm{F}^\alpha_{\beta\gamma}, N^\alpha_\beta, C^\alpha_{\beta\gamma}\right)$ of the Finsler connection $\mathrm{F}\Gamma = \left(\mathrm{F}^i_{jk}, N^i_j, C^i_{jk}\right)$ is given by [5]

(2.4) (a) $F^\alpha_{\beta\gamma} = B^\alpha_i \left\{ B^i_{\beta\gamma} + B^j_\beta \left(F^i_{jk} B^k_\gamma + C^i_{jk} N^k H_\gamma \right) \right\}$,
(b) $N^\alpha_\beta = B^\alpha_i \left(B^i_{0\beta} + N^i_j B^j_\beta \right)$
(c) $C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} B^j_\beta B^k_\gamma$,

where

$$B^i_{\beta\gamma} = \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} \;,\quad B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha$$

and

$$H_\gamma = N_i \left(B^i_{0\gamma} + N^i_j B^j_\gamma \right) \; .$$

The relative h $-$ covariant and v $-$covariant derivative of the projection factor B^i_α are given by [5] :

(2.5) (a) $B^i_{\alpha|\beta} = H_{\alpha\beta}N^i$,

(b) $B^i_{\alpha|\beta} = K_{\alpha\beta}N^i$,

where

$$H_{\alpha\beta} = N_i\left(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta\right) + M_\alpha H_\beta$$

and

$$K_{\alpha\beta} = N_i C^i_{jk}B^j_\alpha B^k_\beta \ ,$$

are called the second fundamental h −tensor and v −tensor respectively.

3. Randers Change of Finsler Metric

Let F^n be a given Finsler space and let $a_i(x)dx^i$ be 1-form on M^n. We shall define on M^n a function $L^*(x,y)\ (>0)$ by the equation (1.2) where we put $A(x,y) = a_i(x)y^i$.

To find the metric tensor g^*_{hk} , the angular metric tensor h^*_{hk} , the Cartan C-tensor C^*_{hkj} of $F^{*n} = (M^n, L^*)$, we use the following results :

(3.1) (a) $\frac{\partial A}{\partial y^i} = a_i$,

(b) $\frac{\partial L}{\partial y^i} = l_i$,

(c) $\frac{\partial l_j}{\partial y^i} = \frac{1}{L}h_{ij}$,

where h_{ij} are the components of the angular metric tensor of F^n given by

$$h_{ij} = g_{ij} - l_i l_j$$
$$= L\frac{\partial^2 L}{\partial y^i \partial y^j} \ .$$

In view of (3.1) the successive differentiation of (1.2) with respect to y^h and y^k gives :

(3.2) $l^*_h = l_h + a_h$.

(3.3) $h^*_{hk} = \tau h_{hk}$,

where $\tau = \frac{L^*}{L}$.

The fundamental metric tensor $g^*_{hk} = h^*_{hk} + l^*_h l^*_k$ is given by :

(3.4) $$g^*_{hk} = \tau g_{hk} + a_h a_k + (a_h l_k + a_k l_h) - \frac{A}{L} l_h l_k \ .$$

Differentiating (3.4) with respect to y^j and using (3.1) we get

(3.5) $C^*_{hkj} = \tau C_{hkj} - \frac{1}{2L}\left(h_{hk}m_j + h_{hj}m_k + h_{kj}m_h\right)$,

where $m_i = \frac{A}{L} l_i - a_i$.

It is to be noted that

(3.6) $m_i l^i = 0$, $m_i a^i = \frac{A^2}{L^2} - a^2$, $h_{ij} l^j = 0$, $h_{ij} m^j = h_{ij} a^j = m_i$,

where

$$m^i = g^{ij} m_j = \frac{A}{L} l^i - a^i .$$

Proposition (3.1): In view of (1.2) Randers change of Finsler metric the Cartan's C-tensor is given by (3.5) .

4. Hypersurface Given by a Randers Change

Suppose a Finsler hypersurface $F^{n-1} = (M^{n-1}, \bar{L})$ of the Finsler Space F^n and another Finsler hypersurface $F^{*n-1} = (M^{n-1}, \bar{L}^*)$ of the Randers change of the Finsler Space F^{*n} . Thus we shall show that a unit normal vector N^{*i} of F^{*n-1} is uniquely determined by

(4.1) (a) $g^*_{hk} B^h_\alpha N^{*k} = 0$

(b) $g^*_{hk} N^{*h} N^{*k} = 1$.

Contracting (3.4) by $N^h N^k$ and using (2.1) (a) , (b) , $l_i N^i = 0$, we get :

$g^*_{hk} N^h N^k = \tau + (a_h N^h)^2$.

Thus we obtain :

$$g^*_{hk}\left(\pm\frac{N^h}{\sqrt{\tau+\left(a_h N^h\right)^2}}\right)\left(\pm\frac{N^k}{\sqrt{\tau+\left(a_h N^h\right)^2}}\right) = 1 .$$

Hence we can put

(4.2) $N^{*h} = \dfrac{N^h}{\sqrt{\tau+\left(a_h N^h\right)^2}}$,

where we have chosen the positive sign in order to fix an orientation .

Using (2.1) , (3.4) , (4.1) and (4.2) , we obtain from (4.1) (a) , we get

$$(a_h B^h_\alpha + l_h B^h_\alpha) a_k N^k = 0 .$$

Contracting it by v^α , we get $L^* = 0$, which is a contradiction with assumption that

$L^* > 0$.

Hence $a_h N^h = 0$.
Therefore from (4.2) is written as

(4.3) $N^{*h} = \frac{N^h}{\sqrt{\tau}}$.

Thus we have

Proposition (4.1):
For a field of linear frame $\left(B_1^i, B_2^i, \dots \dots, B_{n-1}^i, N^i\right)$ of F^n there exists a field of linear frame
$\left(B_1^i, B_2^i, \dots \dots, B_{n-1}^i, N^{*i} = \frac{N^i}{\sqrt{\tau}}\right)$ such that (4.1) is satisfied along F^{*n-1} and then a_i is tangential to both the hypersurface F^{n-1} and F^{*n-1} .

The quantities $B_i^{*\alpha}$ are uniquely defined along F^{*n-1} by

$$B_i^{*\alpha} = g^{*\alpha\beta} g_{ij}^* B_\beta^j \quad ,$$

where ($g^{*\alpha\beta}$) is the inverse matrix of $(g_{\alpha\beta}^*)$.
Let $(B_i^{*\alpha}, N_i^*)$ be the inverse of (B_α^i, N^{*i}) , then we have

$$B_\alpha^i B_i^{*\beta} = \delta_\alpha^\beta \quad ,$$

$$B_\alpha^i N_i^* = 0 \, ,$$

$$N^{*i} N_i^* = 1 \, ,$$

and

$$B_\alpha^i B_j^{*\alpha} + N^{*i} N_j^* = \delta_j^i \quad .$$

We also get $N_i^* = g_{ij}^* N^{*j}$ which in view of (3.4) , (2.2) and (4.3) gives :

(4.4) $N_i^* = \sqrt{\tau}\, N_i$.

We denote the Cartan's connection of F^n and F^{*n} by $\left(F_{jk}^i, N_j^i, C_{jk}^i\right)$ and $\left(F_{jk}^{*i}, N_j^{*i}, C_{jk}^{*i}\right)$ respectively and put $D_{jk}^i = F_{jk}^{*i} - F_{jk}^i$, which will be called the difference tensor . We chose that the vector field a_i in F^n such that

(4.5) $D_{jk}^i = A_{jk} a^i - B_{jk} l^i$,

where A_{jk} and B_{jk} are components of a symmetric covariant tensor of second order .
Now $N_i a^i = 0$ and $N_i l^i = 0$, from (4.5) we get

(4.6) $N_i D_{jk}^i = 0$, $N_i F_{jk}^{*i} = N_i F_{jk}^i$

and

$$N^i \, D_{ok}^i = 0 \quad .$$

Thus from (2.4) and (4.3) , we get

(4.7) $H^*_\alpha = \sqrt{\tau}\, H_\alpha$.

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of enveloping space F^n , then F^{n-1} is called a hyperplane of the first kind [1] . A hyperplane of the first kind is characterized by $H_\alpha = 0$.
Hence we have from (4.7)

Theorem (4.1):
If $a_i(x)$ be a vector field in F^n satisfying (4.5) , then a hypersurface F^{n-1} is a hyperplane of the first kind iff the hypersurface F^{*n-1} is a hyperplane of the first kind .
Again contracting (3.5) by $B^h_\alpha N^{*k} N^{*j}$ and paying attention to (4.4) , $m_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$, we get

(4.8) $M^*_\alpha = M_\alpha - \frac{1}{2L\tau} m_i B^i_\alpha$.

From (2.3) , (4.4) , (4.5) , (4.6) , (4.7) and (4.8) , we have

(4.9) $H^*_{\alpha\beta} = \sqrt{\tau}\left(H_{\alpha\beta} - \frac{1}{2L\tau} m_i B^i_\alpha H_\beta\right)$.

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the enveloping space F^n , then F^{n-1} is called a hyperplane of the second kind [1] . A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$.
Now $H_{\alpha\beta} = 0$ implies $H_\alpha = 0$,
Thus we have from (4.7) and (4.9) :

Theorem (4.2):
If $a_i(x)$ be a vector field in F^n satisfying (4.5) , then a hypersurface F^{n-1} is a hyperplane of the second kind iff the hypersurface F^{*n-1} is a hyperplane of the second kind .
Lastly contracting (3.5) by $B^h_\alpha B^k_\beta N^{*j}$ and paying attention to (4.3) , we have
(4.10) $M^*_{\alpha\beta} = \sqrt{\tau}\, M_{\alpha\beta}$.
If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1} , then F^{n-1} is called a hyperplane of the third kind [1] . A hyperplane of third kind is characterized by $H_{\alpha\beta} = 0$,
$M_{\alpha\beta} = 0$.
Thus we have from (4.7), (4.9) and (4.10) :

Theorem (4.3):
If $a_i(x)$ be a vector field in F^n satisfying (4.5) , then a hypersurface F^{n-1} is a hyperplane of the third kind iff the hypersurface F^{*n-1} is a hyperplane of the third kind.

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