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Reaching the Betz Limit via the Newton's Method for One (1)-Dimensional Optimisation

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Abstract

The *Betz* theory states that a horizontal axis wind turbine can only extract less than $\frac{16}{27}$ (59.3%) of the kinetic energy of the wind. The value $\frac{16}{27}$ ($0.\overline{592}$) is called the *Betz* limit. In 1919, *Albert Betz* used a method that is analytic to derive the *Betz* theory and henceforth finding the *Betz* limit. In the theory, he derived momentum equations of an actuator disc (AD) in the stream of wind. In this research, a $Newton's$ method scheme is used to reach the *Betz* limit. For the axial induction factor denoted by either a or a_* we used a reasonable initial point guess of 0.259. The results agree well with the analytical results of the *Betz* theory. From the $Newton's$ method scheme results corresponding to a and a_*, the relative errors in comparison with the *Betz* limit are 4% and 0.9%, respectively. The small amount of errors shows the possibility of reaching the *Betz* limit using the $Newton's$ method scheme. If alternate power coefficients [$C_p(a)$ or $C_{p_*}(a_*)$] are found where the (exact) value (a or a_*) cannot exactly be found analytically, the $Newton's$ method scheme would be an ideal way of estimating (or calculating) other proposed limits or the *Betz* limit. This method yields extremely accurate results. Furthermore, for a quadratic equation, the quadratic formula would do. However, if the quadratic equation has floating-point numbers, the $Newton's$ method would be more appropriate.

Keywords: Betz theory, Betz limit, Newton's method, One (1)-dimensional, Optimisation, Scheme

1 Introduction

Researchers are motivated to increase the performance of wind turbines based on demand when they discern substantial growth of power extraction from wind energy. The global installed wind turbine capacity was increased 22 times in 2017 compared to 2001 [1]. Researchers have come up with theories and modified them later. One of the well-known ones is the AD theory. By assumption of ideal and laminar flows, $Rankine$ [2] and $Froude$ [3, 4] used the theory. Based on the AD theory, in 1919, $Betz$ proved the impossibility of energy extraction more than $\frac{16}{27}$ (59.3%) of the available kinetic energy [5]. The same results were found independently by $Lanchester$ [6] and $Joukowsky$ [7] and henceforth the theory was called $Lanchester$-$Betz$-$Joukowsky$ limit or briefly $Betz$ limit. To construct the theory $Betz$ assumed an infinite number of rotor blades under an axial and incompressible flow. He also assumed that a uniform thrust was exerted on the disc with no drag force. Of recent, operational wind turbines can achieve 75% or 80% of the $Betz$ limit value at peak power generation [8].

$Betz$ deduction has been presented in several ways. For example, $Ochieng$ and $Ochieng$ [9] presents a mathematical power series expansion method to obtain the $Betz$ equation functional form to determine the maximum wind power coefficient. In the study, $Okulov$ [10], analytical solutions for optimising wind turbines with specialised rotor models in the case of an infinite number of blades has been presented. In this paper we present the estimation of the $Betz'$ limit using the $Newton's$ method scheme.

One of the most powerful techniques in process integration is optimisation. In the context of optimisation, best can be defined as the optimal option out of several possible choices. By optimising an objective function, the degree of goodness of a solution is determined [11–13]. Constraints and system models are used as a guidance in the search process. In other words, optimising seeks to increase (or decrease) the objective function's value while taking into account a range of limitations (constraints). In terms of equality and inequality, these limits are clear. In numerical optimisation the objective function is an evaluation measure that may be minimised or maximised under a variety of design restrictions. Computational and numerical techniques are used to increase the speed up time it takes to find a solution. Several methods exist to reduce what may be a difficult issue in analytical mathematics to simple algebra and this may involve the integration or resolution of complex differential equations [14–17].

Procedures that are iterative and converge to a solution (for a certain class of problems) and provide approximate answers to specific issues can all be used by researchers to solve problems. An example is $Newton's$ method algorithm. Equations can be effectively solved numerically using $Newton's$ method. It is based on the idea of linear approximation, as is the case with differential calculus (where the idea is used extensively and it's a fundamental concept in differential calculus). Our goal and objective is to use the $Newton's$ method to reach the $Betz$ limit.

This paper is organised as follows. Section 2 presents the idea of the $Betz$ theory and $Newton's$ method scheme in the context of the theory. Section 3 presents the results and discussion. Section 4 presents the conclusions.

2 The Betz Theory and Newton's Method Scheme

One can calculate the maximum theoretical efficiency of a thin rotor by a stationary disc or an AD. Energy is extracted from the stream passing through the disc. The assumptions made are that the flows are frictionless, and there is no rotational velocity in the wake. The disc operates like a drag device that lowers the magnitude of the velocity [18].

The axial induction factor denoted by a can be defined as the fractional decrease in wind velocity between the free stream and the rotor plane [18]. This implies that

$$a = \frac{u_i - u_j}{u_i}, \tag{1}$$

$$u_j = u_i(1-a), \tag{2}$$

and,

$$u_l = u_i(1-2a). \tag{3}$$

u_i and u_j represent the wind speed (or average velocity at the rotor plane) and wake velocity (or the free stream velocity) in the proximity of the disc, respectively. u_l represents the velocity behind the rotor.

The power out, P is equal to the thrust times the velocity at the disc:

$$P = \frac{1}{2}\rho A_d(u_i^2 - u_l^2)u_j. \tag{4}$$

Inserting equations (2) and (3) in (4) yields

$$P = \frac{1}{2}\rho A_d u_i^3 4a(1-a)^2. \tag{5}$$

Replacing the control volume area at the rotor, A_d by A (which will represent the rotor area), and the free stream velocity, u_i by u yields

$$P = \frac{1}{2}\rho A u^3 4a(1-a)^2. \quad (6)$$

Power coefficient denoted by $C_p(a)$ usually characterises the wind turbine rotor performance. This means that $C_p(a)$ is equal to rotor power divided by power in the wind [18]. Mathematically, this can be expressed as

$$C_p(a) = \frac{P}{\frac{1}{2}\rho u^3 A}. \quad (7)$$

This non-dimensional coefficient represents the fraction of the power in the wind that is extracted by the rotor. From equations (6) and (7), one can see that:

$$C_p(a) = 4a(1-a)^2. \quad (8)$$

Now, the maximum $C_p(a)$ is determined as follows:

$$\frac{dC_p(a)}{da} = 4(1-a)(1-3a). \quad (9)$$

Setting $\frac{dC_p(a)}{da} = 0$ implies that either $a = 1$ or $a = \frac{1}{3}$. Using $a = 1$ yields $C_p(a) = 0$ and using $a = \frac{1}{3}$ leads to $C_p(a) = \frac{16}{27} = 0.\overline{592}$. Obviously one can see that the maximum power coefficient corresponds to $a = \frac{1}{3}$ leading to $0.\overline{592}$. This is known as the *Betz* limit [18].

Furthermore, we see that $C_p''\left(a = \frac{1}{3}\right) < 0$ where the primes represent derivatives with respect to a. This further shows that the *Betz* limit is a maximum.

Another way of computing power is [19]

$$P = \rho A_d u_j^2 (u_i - u_l). \quad (10)$$

Equating equations (4) and (10) we have

$$\rho A_d u_j^2 (u_i - u_l) = \frac{1}{2}\rho A_d u_j (u_i^2 - u_l^2), \quad (11)$$

which yields

$$u_j = \frac{1}{2}(u_i + u_l), \quad (12)$$

since the density cannot be zero for any u_j and A_d.

To find the other power function $[C_{p_*}(a_*)]$ used to calculate the $Betz$ limit, we use equation (12) in the first line of equation (4). This results in

$$P_* \;=\; \frac{1}{2}\rho A_d \frac{1}{2}(u_i + u_l)(u_i^2 - u_l^2). \tag{13}$$

Writing this equation in a more suggestive form yields

$$P_* \;=\; \frac{1}{2}\rho A_d u_i^3 \frac{1}{2}\left[1 + \left(\frac{u_l}{u_i}\right) - \left(\frac{u_l}{u_i}\right)^2 - \left(\frac{u_l}{u_i}\right)^3\right]. \tag{14}$$

Again replacing the control volume area at the rotor, A_d by A (which will represent the rotor area), and the free stream velocity u_i by u leads to

$$P_* \;=\; \frac{1}{2}\rho A u^3 \frac{1}{2}\left[1 + \left(\frac{u_l}{u}\right) - \left(\frac{u_l}{u}\right)^2 - \left(\frac{u_l}{u}\right)^3\right]. \tag{15}$$

Now, we let $a_* = \frac{u_l}{u}$. This means that

$$P_* \;=\; \frac{1}{2}\rho A u^3 \frac{1}{2}(1 + a_* - a_*^2 - a_*^3). \tag{16}$$

Henceforth

$$C_{p_*}(a_*) \;=\; \frac{P_*}{\frac{1}{2}\rho A u^3} = \frac{1}{2}(1 + a_* - a_*^2 - a_*^3). \tag{17}$$

The maximum $C_{p_*}(a_*)$ is determined as follows:

$$\frac{dC_{p_*}(a_*)}{da_*} \;=\; \frac{1}{2}(a_* + 1)(3a_* - 1). \tag{18}$$

Setting $\frac{dC_{p_*}(a_*)}{da_*}$ implies that $a_* = -1$ or $a_* = \frac{1}{3}$. Using $a_* = -1$ yields $C_{p_*}(a_*) = 0$ and $a_* = \frac{1}{3}$ leads to $C_{p_*}(a_*) = \frac{16}{27} = 0.\overline{592}$. Obviously one can see that the maximum power coefficient corresponds to $a_* = \frac{1}{3}$ leading to $C_{p_*}(a_*) = 0.\overline{592}$ which is the $Betz$ limit.

Furthermore, we see that $C''_{p_*}\left(a_* = \frac{1}{3}\right) < 0$ where the primes represent derivatives with respect to a_*. This further shows that the $Betz$ limit is a maximum.

Now, we can explore the $Newton's$ method in the context of the $Betz$ theory.

Suppose the function $C_p(a)$ [this can be done for $C_{p_*}(a_*)$ as well] is $k+1$ times differentiable on an open interval I. For any points a and $a+h$ in I

there exists a point w between a and $a+h$ such that

$$C_p(a+h) = C_p(a) + C_p'(a)h + .. + \frac{1}{k!}C_p^{[k]}(a)h^k + \frac{1}{(k+1)!}C_p^{[k+1]}(w)h^{k+1} \tag{19}$$

It is easy to show that as h goes to 0 the higher order terms in equation (19) go to 0 much faster than h goes to 0. This means that (for small values of h)

$$C_p(a+h) \approx C_p(a) + C_p'(a)h. \tag{20}$$

This is referred to as a first order *Taylor* approximation of $C_p(a)$ at a. A more accurate approximation to $C_p(a+h)$ can be constructed for small values of h as:

$$C_p(a+h) \approx C_p(a) + C_p'(a)h + \frac{1}{2}C_p''(a)h^2. \tag{21}$$

This is known as a second order *Taylor* approximation of $C_p(a)$ at a.

Note that the first order *Taylor* approximation can be rewritten as:

$$C_p(a+h) \approx d + eh \tag{22}$$

where $d = C_p(a)$ and $e = C_p'(a)$. This means that the first order Taylor approximation is a linear function in h.

Similarly, the second order *Taylor* approximation can be rewritten as:

$$C_p(a+h) \approx d + eh + \frac{1}{2}gh^2, \tag{23}$$

where $g = C_p''(a)$. This means that the second order *Taylor* approximation is a second order polynomial in h.

Suppose we want to find the value of a that maximises

$$C_p(a) = d + ea + ga^2. \tag{24}$$

First, we calculate the first derivative of $C_p(a)$:

$$C_p'(a) = e + 2ga. \tag{25}$$

We know that $C_p'(\hat{a}) = 0$, where $\hat{a}$ represents the value of a at which $C_p(a)$ attains its maximum and hence yields

$$\hat{a} = -\frac{e}{2g}. \tag{26}$$

The second order condition is $C_p''(a) = 2g < 0$. This means that $C_p\left(-\frac{e}{2g}\right)$ will be a maximum whenever $g < 0$.

Suppose we want to find the value of a that maximises some twice continuously differentiable function $C_p(a)$. Recall that

$$C_p(a+h) \approx d + eh + \frac{1}{2}gh^2, \tag{27}$$

where $d = C_p(a)$, $e = C_p'(a)$, and $g = C_p''(a)$. This means

$$C_p'(a+h) \approx e + gh. \tag{28}$$

The first order condition for the value of h (denoted $\hat{h}$) that maximises $C_p(a+h)$ leads to

$$\hat{h} \;=\; -\frac{e}{g}. \tag{29}$$

In other words, the value that maximises the second order $Taylor$ approximation to $C_p(a)$ at a is

$$a + \hat{h} \;=\; a - \frac{1}{C_p''(a)}C_p'(a). \tag{30}$$

With this in mind we can specify the $Newton's$ method scheme for one (1) dimensional function optimisation such as on equations (8) and (17).

Now, from equation (8),

$$C_p'(a) \;=\; 4 - 16a + 12a^2. \tag{31}$$

Further differentiation of equation (31) yields

$$C_p''(a) \;=\; -16 + 24a. \tag{32}$$

Using equations (31) and (32) in equation (30) leads to

$$a + \hat{h} \;=\; a - \left(\frac{1 - 4a + 3a^2}{6a - 4}\right). \tag{33}$$

This is one of $Newton's$ method for one (1) dimensional optimisation in the context of the $Betz$ theory.

From equation (17),

$$C_{p_*}'(a_*) \;=\; \frac{1}{2}(1 - 2a_* - 3a_*^2). \tag{34}$$

Further differentiation of equation (34) leads to

$$C''_{p_*}(a_*) \quad = \quad -1 - 3a_*. \tag{35}$$

Using equations (34) and (35) in equation (30) yields

$$a_* + h \quad = \quad a_* - \left(\frac{3a_*^2 + 2a_* - 1}{2 + 6a_*} \right). \tag{36}$$

This is also one of $Newton's$ method for one (1) dimensional optimisation in the context of the *Betz* theory. Using a python code we used (or investigated) these methods in order to find $C_p(a)$ and $C_{p_*}(a_*)$ given a suitable initial point (or guess) a and a_*, respectively.

3 Results and Discussion

There is no general criterion to decide the best initial point a or a_* to be used in equation (33) or (36), respectively. However, for equation (33), we can start from a point inside an interval $I = [c_*, d_*]$, such that the function is continuous in I and $C_p(c_*).C_p(d_*) < 0$ where c_* and $d_* \in \mathbb{R}$. For equation (33) we start from a point inside the interval $I = [-1, 2]$, and we see that the function $C_p(a)$ is continuous in I and $C_p(-1).C_p(2) < 0$. That guess or point is 0.259. The guess for equation (36) is $a_* = 0.259$ as well. From equations (9) and (28), the optimum induction factor in generating the highest performance coefficient is $\frac{1}{3}$ $(0.\overline{3})$.

The results of the performance coefficients $[C_p(a)$ and $C_{p_*}(a_*)]$ are 0.568847916 and 0.587272511, respectively. The maximum values of the $C_p(a)$ and $C_{p_*}(a_*)$ is $\frac{16}{27}$ (the *Betz* limit). This is equivalent to $0.\overline{592}$. The average deviations from the *Betz* limit are 4% and 0.9%, respectively. The low errors show that the results from the $Newton's$ method scheme agree well with the *Betz* theory. It is possible to reach the *Betz* limit value via the $Newton's$ method scheme by choosing a suitable initial point or guess for a or a_*.

Furthermore, we see that $C''_p\left(a = \frac{1}{3}\right) < 0$ and $C''_{p_*}\left(a_* = \frac{1}{3}\right) < 0$. This shows that the *Betz* limit is a maximum.

If alternate power coefficients $[C_p(a)$ or $C_{p_*}(a_*)]$ are found [such as in [20] and if the (exact) value (a or a_*) cannot be found for example], the $Newton's$ method would be an ideal way of estimating (or calculating) the other proposed limits (and the *Betz* limit as well). As one can see this method yields very accurate results.

For a quadratic equation, the quadratic formula would do. However, if the quadratic equation has floating-point numbers, the $Newton's$ method would be more appropriate. In the quadratic equation, if the coefficient of the linear dependent variable is large relative to the other coefficients, the quadratic formula can give wrong results when implemented in floating point arithmetic. Hence the need for the $Newton's$ method.

4 Conclusions

$Newton's$ method scheme in the context of the $Betz$ theory was constructed and used. The aim is to reach the $Betz$ limit using the $Newton's$ method scheme. We derived the $Newton's$ method scheme in the context of the $Betz$ theory and used it [on the functions $C_p(a)$ and $C_{p_*}(a_*)$]. The results obtained were in good agreement. The errors in comparison with the $Betz$ limit were 4% and 0.9%. This means that it is possible to reach the $Betz$ limit using the $Newton's$ method scheme by choosing a suitable initial point.

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