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Effect of Helicity on Turbulent Flow and Millionschikov Hypothesis

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Abstract

In this paper, we present an overview of the role of helicity on the turbulent flow. Governing equations are then derived in terms of correlation functions. Solutions of such equations are obtained by the Smirnov's series solution method. Lastly, role of Millionschikov's closure hypothesis of homogeneous and isotopic turbulence is discussed.

Keywords: Turbulent flow, Homogeneous and isotropic turbulence, Helicity, Correlation functions, Smirnov's series, Millionschikov's hypothesis

1. Introduction

Tsinober and Levich [25] examined the results from a wide variety of experiments and concluded that many turbulent flows possess structures. It is generally believed that coherent structures are dynamically significant. It is important to ascertain whether structures are merely a result of three-dimensional instabilities of a large scale mean flow, or whether they are universal and intrinsic to all turbulent flows i.e., also to homogeneous and isotropic ones [20]. There are problems of homogeneous and isotropic turbulence which are not invariant under plane reflections [22]. The scalar product i.e., of velocity and vorticity in such a flow, say, $\langle \underline{v} \cdot curl\underline{v} \rangle$ is not zero. Stepanov et.al. [23], described that the helicity $H = v^{-1}\int_v \langle \underline{u} \cdot curl\underline{u} \rangle dv$ is the integral of motion with an arbitrary sign for the spectral density of the helicity $H(\kappa)$ and it is possible to indicate the upper bound $|H(\kappa)| \le \kappa E(\kappa)$, where $E(\kappa)$ is the spectral density of the velocity fluctuation energy and κ is the wave number.

In case of two-dimensional turbulence, the vorticity $\underline{\omega} = \nabla \times \underline{v}$ is conserved by non-linear terms of two-dimensional Navier-Stokes equation. There exist a second inviscid integral invariant besides the energy $E = \frac{1}{2}\int_v v^2 d\kappa$ e.g., the enstrophy $\Omega = \frac{1}{2}\int_v \omega^2 d\kappa$. [10] defined the mean helicity by $He = \frac{1}{2}\langle \underline{u} \cdot [\nabla \times \underline{u}] \rangle$.

The correlation tensor $Q_{ij} = \langle u_i'(r',t') u_j'(r'',t'') \rangle$ can be written, following Robertson's theory [21] of homogeneous isotropic turbulence, as

$$Q_{ij} = -\frac{F'}{2r}\xi_i \xi_j + \left(F + \frac{rF'}{2}\right)\delta_{ij} + \frac{H}{r}\varepsilon_{ijn}\xi_n \tag{1.1}$$

where F and H are scalar functions of r and t;

$r = |r'' - r'|;\ t = |t'' - t'|;\ r'' = (x_1'', x_2'', x_3'');\ r' = (x_1', x_2', x_3');\ \xi_i = x_i'' - x_i'$ and $\langle u_2', u_3'' \rangle = H(r,t)$

Now, noting that $2H_e = \lim_{r'' \to r'} \langle u(r',t) \cdot [\nabla \times u(r'',t)] \rangle$ it can be shown that $H_e = -3\lim_{r \to 0} \frac{H(r,t)}{r}$ since $H(0) = 0$ [10].

Since first order solenoidal tensor is identically zero in homogeneous isotropic turbulence possessing helicity, pressure–velocity correlation, temperature–velocity correlation and density–velocity correlation is always zero. The topological invariant H_v is a measure of enlargerlness of velocity lines [18]. In turbulent flow, the appropriate modification of H_v is the mean helicity density $H = \langle \underline{v} \cdot \underline{\omega} \rangle$ [12]. In ordinary homogeneous turbulence $H = 0$, due to reflectional symmetry. The same is true for the mean helicity density.

$$H(\kappa) = \frac{1}{(2\pi)^3} 4\pi\kappa^2 \int \langle \underline{v}(0) \cdot \underline{\omega}(r) \rangle \exp(i\kappa \cdot r)\, dv = 0$$

$H(\kappa)$ helicity spectrum, defined thus, is like mean energy spectral density in ordinary homogeneous isotropic turbulence

$$E(\kappa) = \frac{1}{(2\pi)^3} 4\pi\kappa^2 \int \langle \underline{v}(0), \underline{v}(r) \rangle \exp(i\kappa \cdot r)\, dv$$

We shall mention now, some initial comments of Moffatt [18] on the turbulence with helicity and associated dynamo action, that, "a lack of reflectional symmetry in a random 'background' velocity field $u(x,t)$, and in particular a non-zero value of the mean helicity $\langle u \cdot \nabla \times u \rangle$, is likely to be a crucial factor as far as the effect on large-scale magnetic field evolution is concerned. In these circumstances, it is appropriate to consider the general nature of the dynamics of a turbulent velocity field endowed with non-zero mean helicity. First it must be stated that turbulence exhibiting such a lack of reflexional symmetry has seldom submitted to direct experimental investigation in the laboratory. Nearly, all traditional studies of turbulence (e.g., grid turbulence, boundary layer turbulence, turbulence in waves and jets, channel and pipes turbulence etc.) have been undertaken in conditions that guarantee reflectional symmetry of the turbulence statistics".

2. Quadratic invariants in physical space

We follow Lesieur [11] method and write the Navier-Stokes equations for the velocity and vorticity fields in homogeneous turbulence, as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \tag{1.2}$$

and

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \nabla^2 \omega_i \tag{1.3}$$

We derive from (1.2), imposing the homogeneity condition

$$\frac{d}{dt}\frac{1}{2}\left\langle \vec{u}^2 \right\rangle = \nu\left(\vec{u}\nabla^2\vec{u}\right) \tag{1.4}$$

Noting the variance of vorticity by $D(t) = \frac{1}{2}\left\langle \vec{\omega}^2 \right\rangle$ the enstrophy for homogeneous turbulence, is given by $D(t) = -\frac{1}{2}\left\langle \vec{u}(x,t) \cdot \nabla^2 \vec{u}(x,t) \right\rangle$

The mean kinetic energy evolution equation is $\frac{d}{dt}\frac{1}{2}\left\langle \vec{u}^2 \right\rangle = -2\nu D(t)$ (1.5)

Now, multiplying equation (1.2) by $\frac{\omega_i}{2}$ and equation (1.3) by $\frac{u_i}{2}$, adding them and then averaging, we obtain

$$\frac{dH_e}{dt} + \frac{1}{2}\left\langle u_j \frac{\partial}{\partial x_j}(\omega_i u_i) \right\rangle = -\frac{1}{2}\left\langle \omega_i \frac{\partial p}{\partial x_i} \right\rangle + \frac{1}{2}\left\langle \omega_j u_i \frac{\partial u_i}{\partial x_j} \right\rangle = -\nu\left\langle \vec{\omega} \cdot \left(\vec{\nabla} \times \vec{\omega}\right) \right\rangle \tag{1.6}$$

Since $H_e = \frac{1}{2}\langle \vec{u} \cdot \vec{\omega} \rangle$ we may transform, in view of the conditions

$\left\langle \vec{A} \cdot \left(\vec{\nabla} \times \vec{B}\right) \right\rangle = \left\langle \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) \right\rangle$ and $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{u}\right) = -\nabla^2 \vec{u}$

where $\vec{A}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are two vector fields depending on the velocity field, the mean helicity evolution equation, as

$$\frac{dH_e}{dt} = \frac{\nu}{2}\left(\left\langle \vec{\omega}\cdot\nabla^2\vec{u}\right\rangle + \left\langle \vec{u}\cdot\nabla^2\vec{\omega}\right\rangle\right) \tag{1.7}$$

Now, since the operators ∇^2 and $\vec{\nabla}\times$ commute, the latter term is equal to

$$\left\langle \vec{u}\cdot\left(\vec{\nabla}\times\nabla^2\vec{u}\right)\right\rangle = \left\langle \vec{\omega}\cdot\nabla^2\vec{u}\right\rangle \text{, which gives } \frac{dH_e}{dt} = \nu\left\langle \vec{\omega}\cdot\nabla^2\vec{u}\right\rangle \tag{1.8}$$

Let us now note down the relation $\left\langle \vec{\omega}\cdot\nabla^2\vec{u}\right\rangle = -\int d\vec{\kappa}\, d\vec{\kappa}'\cdot e^{i(\vec{\kappa}+\vec{\kappa}')\cdot\vec{x}}\kappa^2\left\langle \hat{\omega}(\vec{\kappa}',t)\hat{u}_i(\vec{\kappa},t)\right\rangle$. The helical spectral tensor $\hat{H}_{ij}(\vec{\kappa},t)$ which is the Fourier transform of $\left\langle \omega_i(\vec{x},t)u_j(\vec{x}+\vec{r},t)\right\rangle$.

Accordingly, we write $\left\langle \hat{\omega}_i(\vec{\kappa}',t)\hat{u}_i(\vec{\kappa},t)\right\rangle = \hat{H}_{ii}(\vec{\kappa},t)\,\delta(\vec{\kappa}+\vec{\kappa}')$ with, for the case of isotropic turbulence $H_e = \frac{1}{2}\int \hat{H}_{ii}(\vec{\kappa},t)\,d\vec{\kappa} = \int_0^{+\infty} H(\kappa,t)\,d\kappa$. The helicity dissipation rate, is given by $\nu\left\langle \vec{\omega}\cdot\nabla^2\vec{u}\right\rangle = -\nu\int \kappa^2\hat{H}_{ii}(\vec{\kappa},t)\,d\vec{\kappa} = \nu\int_0^{+\infty}\kappa^2 H(\kappa,t)\,d\kappa$ and the final helicity dissipation equation is written, as

$$\frac{d}{dt}\int_0^{+\infty} H(\kappa,t)\,d\kappa + 2\nu\int_0^{+\infty}\kappa^2 H(\kappa,t)\,d\kappa = 0 \tag{1.9}$$

For passive scalar $\Theta(\vec{\kappa},t)$, the corresponding spectral dissipation equation can be derived [11], as

$$\frac{d}{dt}\int_0^{+\infty} E_\theta(\kappa,t)\,d\kappa + 2\kappa\int_0^{+\infty}\kappa^2 E_\theta(\kappa,t)\,d\kappa = 0 \tag{1.10}$$

The equations (1.9) and (1.10) are termed as quadratic invariants of turbulence. But they are not invariant, since the viscous dissipation will be seen as of prior importance in three-dimensional turbulence. The viscous terms in these equations can both dissipate and generate helicity. The helicity is thus pseudo-scalar and $H(\kappa)$ can either be positive or negative. Helicity can play an important role in the evolution and stability of turbulent and laminar flows [4], [5]. Brissaud et.al. [1] introduced the concept of helical cascade and discussed the limiting cases of parallel energy and helicity flows along the spectrum $E(\kappa)\approx\bar{\varepsilon}^{\frac{2}{3}}\kappa^{-\frac{5}{3}}$, $H(\kappa)\approx\bar{\eta}\,\bar{\varepsilon}^{\frac{1}{3}}\kappa^{-\frac{5}{3}}$ corresponding to a Kolmogorov cascade, and a helicity flux with no energy flux $E(\kappa)\approx\bar{\eta}^{\frac{2}{3}}\kappa^{-\frac{7}{3}}$ and $H(\kappa)\approx\bar{\eta}^{\frac{2}{3}}\kappa^{-\frac{4}{3}}$ –a helicity cascade ($\bar{\varepsilon}$, $\bar{\eta}$ are the dissipation of energy and helicity respectively) – were examined. Kraichnan [9], denoted respectively, the total rate of energy and helicity transfer by $\pi(\kappa)$ and $\Sigma(\kappa)$ from all wave-numbers lower than κ to all wave-numbers greater than κ and put, $\pi(\kappa) \approx \kappa \dfrac{E(\kappa)}{\tau(\kappa)}$; $\Sigma(\kappa) \approx \kappa \dfrac{H(\kappa)}{\tau(\kappa)}$ where, $\tau(\kappa) \approx \left(\int_0^\infty p^2 E(p)\, dp \right)^{-\frac{1}{2}}$ (1.11)

In this paper, we would construct the deductive theory of homogeneous and isotropic turbulence possessing helicity as a basic task.

3. Correlation Method for the Homogeneous Turbulence Possessing Helicity

We attempt to develop a deductive theory of stationary homogeneous turbulent flow whose statistical characteristics are invariant under translations and rotations but lack reflectional symmetry. Firstly, we will derive the correlation tensors and their forms [15]

We consider two different points $P'(r'')$ and $P''(r'')$, respectively, at two different times t' and t'' and construct the following correlation tensors

$$Q_{ij} = \left\langle u_i'(r',t') u_j'(r'',t'') \right\rangle,$$
$$T_{ij,k} = \left\langle u_i'(r',t') u_j'(r',t') u_k''(r'',t'') \right\rangle,$$
$$Q_{ij,kl} = \left\langle u_i'(r',t') u_j'(r',t') u_k''(r'',t'') u_l''(r'',t'') \right\rangle,$$
$$P_{ij} = \left\langle \varpi''(r'',t'') u_i'(r',t') u_j'(r',t') \right\rangle \tag{1.12}$$

where $\varpi'' = \dfrac{p''}{\rho}$, $r' = (x_1', x_2', x_3')$ and $r'' = (x_1'', x_2'', x_3'')$, ρ is the density of the fluid.

Now, imposition of homogeneity and stationarity conditions would imply that all these tensors are to be functions of the distance $r = |r'' - r'|$ between the points and the interval of time $t = |t'' - t'|$.

We note that $\dfrac{\partial}{\partial t'} = +\dfrac{\partial}{\partial t}$ or $-\dfrac{\partial}{\partial t}$ according as, $t'' < t'$ or $t'' > t'$.

The opposite is the case for $\dfrac{\partial}{\partial t''}$, that is,

$\dfrac{\partial}{\partial t''} = -\dfrac{\partial}{\partial t}$ or $+\dfrac{\partial}{\partial t}$ according to as, $t'' < t'$ or $t'' > t'$.

Furthermore,

$$\frac{\partial}{\partial x_k'} = -\frac{\partial}{\partial \xi_k}, \quad \frac{\partial}{\partial x_k''} = \frac{\partial}{\partial \xi_k} \quad \text{and} \quad \nabla_{r'}^2 = \nabla_{r''}^2 = \nabla_\xi^2 = \frac{\partial^2}{\partial \xi_i \partial \xi_i} \quad \text{where} \quad |\xi| = r \tag{1.13}$$

The second and third-order correlation tensors in homogeneous isotropic stationary turbulence with lack of reflectional symmetry can be written as

$$
\begin{aligned}
&Q_{ij} = -\frac{F'}{2r}\xi_i\xi_j + \left(F + \frac{rF'}{2}\right)\delta_{ij} + \frac{H}{r}\varepsilon_{ijn}\xi_n, \\
&T_{ij,\kappa} = \frac{T'}{r}\xi_i\xi_j\xi_\kappa - \frac{1}{2}\left(rT' + 3T\right)\left(\xi_i\delta_{j\kappa} + \xi_j\delta_{\kappa i}\right) + T\xi_\kappa\delta_{ij} + M\left(\varepsilon_{j\kappa m}\xi_m\xi_i - \varepsilon_{\kappa im}\xi_m\xi_j\right), \\
&P_{ij} = P_1\xi_i\xi_j + P_2\delta_{ij}
\end{aligned}
\qquad (1.14)
$$

where F, H, T, M, P_1 and P_2 are scalar functions of r and t and the primes denote differentiation with respect to r.

It is to be noted that Q_{ij} is not symmetrical in the indices i and j but is solenoidal in both indices, while $T_{ij,\kappa}$ is symmetrical in the indices i and j and solenoidal in the index κ. P_{ij} is symmetrical in the indices i and j. No term involving $\varepsilon_{ij\eta}\xi_\eta$ would therefore appear in P_{ij}.

We continue now with the following provable mathematical results [15]. If A is a scalar function of r and t, then

$$
\begin{aligned}
&Curl\left[A\varepsilon_{iqn}\xi_n\right] = \frac{A'}{r}\xi_i\xi_j - \left(rA' + 2A\right)\delta_{ij}, \\
&Curl\left[A\left(\varepsilon_{jqm}\xi_m\xi_i - \varepsilon_{qim}\xi_m\xi_j\right)\right] = \frac{2A'}{r}\xi_i\xi_j\xi_k + 2A\xi_k\delta_{ij} - \left(rA' + 3A\right)\left(\xi_i\delta_{jk} + \xi_j\delta_{ki}\right) \\
&\frac{\partial}{\partial\xi_k}\left[A\left(\varepsilon_{kqm}\xi_m\xi_i - \varepsilon_{qim}\xi_m\xi_k\right)\right] = \left(r\frac{\partial}{\partial r} + 5\right)A\varepsilon_{iqm}\xi_m, \\
&\nabla_\xi^2\left[A\varepsilon_{iqm}\xi_m\right] = D_5 A\varepsilon_{iqm}\xi_m, \qquad \text{where, } D_5 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right), \\
&\nabla_\xi^2\left[A\left(\varepsilon_{lim}\xi_m\xi_j - \varepsilon_{ijm}\xi_m\xi_l\right)\right] = D_7 A\left(\varepsilon_{lim}\xi_m\xi_j - \varepsilon_{ijm}\xi_m\xi_l\right), \text{ where } D_7 = \left(\frac{\partial^2}{\partial r^2} + \frac{6}{r}\frac{\partial}{\partial r}\right).
\end{aligned}
\qquad (1.15)
$$

These relations are used here in the subsequent analysis of the problem.

4. Governing Equations of the Problem: Smirnov's Method of Solution

Next, we can derive the following equation $\pm\frac{\partial}{\partial t}Q_{ij} = \frac{\partial}{\partial \xi_k}T_{ik,j} + \nu\nabla_\xi^2 Q_{ij}$

where the plus and minus sign is to be taken according as, $t'' < t'$ or $t'' > t'$.

This equation is transformed using the forms (1.14) for Q_{ij} and T_{ij} and the mathematical relations (1.15) to

$$\pm\frac{1}{2}\left[\frac{1}{r}\frac{\partial F'}{\partial t}\xi_i\xi_j-\left(r\frac{\partial F'}{\partial t}+2\frac{\partial F}{\partial t}\right)\delta_{ij}\right]+\left[\pm\frac{\partial H_1}{\partial t}\varepsilon_{ijn}\xi_n\right]=$$

$$=\frac{1}{2}\left[\frac{T_1'}{r}\xi_i\xi_j-\left(rT_1'+2T_1\right)\delta_{ij}\right]+\left(r\frac{\partial}{\partial r}+5\right)M\varepsilon_{ijm}\xi_m-\frac{1}{2}\nu\left[\frac{F_1'}{r}\xi_i\xi_j-\left(rF_1'+2F_1\right)\delta_{ij}\right]+\nu D_5H_1\varepsilon_{ijn}\xi_n$$

(1.16)

Here $\left(r\frac{\partial}{\partial r}+5\right)T=T_1,\quad D_5F=F_1,\quad H_1=\frac{H}{r}$

Equation (1.16) can be split up to the three equations

$$\pm\left(\frac{1}{r}\frac{\partial F'}{\partial t}\right)=\frac{T_1'}{r}-\nu\frac{F_1'}{r},$$
$$\pm\left(r\frac{\partial F'}{\partial t}+2\frac{\partial F}{\partial t}\right)=\left(rT'+2T_1\right)-\nu\left(rF'+2F_1\right),$$
$$\pm\frac{\partial H_1}{\partial t}=\left(r\frac{\partial}{\partial r}+5\right)M+\nu D_5H_1.$$

Combining the first two equations, we may write

$$\pm\frac{\partial F}{\partial t}=\left(r\frac{\partial}{\partial r}+5\right)T-\nu D_5F\ .$$

Now, the equation of motion at the point $P''(r'',t)$ is

$$\frac{\partial u_i''}{\partial t''}+\frac{\partial}{\partial x_k''}u_i''u_k''=-\frac{\partial\varpi''}{\partial x_i''}+\nu\nabla_{r''}^2u_i''\quad\text{where}\quad\varpi''=\frac{p''}{\rho}.$$

Multiplying this equation by $u_j'u_l'$ and averaging, we obtain

$$\pm\frac{\partial}{\partial t}T_{jl,i}+\nu\nabla_\xi^2T_{jl,i}=\frac{\partial}{\partial\xi_k}Q_{jl,ik}+\frac{\partial}{\partial\xi_i}P_{jl}=X_{jl,i}$$

where the plus and minus signs are according as, t'' < t' or t'' > t'. The above equation can be put into a set of two equations

$$\left(\pm\frac{\partial}{\partial t}+\nu D_7\right)T=S\ ,\quad\left(\pm\frac{\partial}{\partial t}+\nu D_7\right)M=R\ ,$$

where S and R are the appropriate functions of r and t. Another pair of scalar equations which are equivalent to them are derived as,

$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}+5\right)S=-F\frac{\partial}{\partial r}D_5F-4H_1\frac{\partial}{\partial r}H_1,$$

$$\left(r\frac{\partial}{\partial r}+5\right)R=FD_5H_1-H_1D_5F.$$

In obtaining these equations, we have employed that the fourth order moment $Q_{ij,kl}$ is related to the second order moment Q_{ij} as in a normal distribution $Q_{ij,kl}=Q_{ik}Q_{jl}+Q_{il}Q_{jk}+Q_{ij}\left(0,0\right)Q_{kl}\left(0,0\right)$ [16].

Through some algebraic elimination process, we obtain the equations for F and H_1 as

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5^2\right)F = F\frac{\partial}{\partial r}D_5 F + 4H_1\frac{\partial}{\partial r}H_1,$$
$$\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5^2\right)H_1 = FD_5H_1 - H_1D_5F. \tag{1.17}$$

If $H = 0$, that is, $H_1 = 0\ (r \neq 0)$ on putting $F = -2Q$, we derive from last two equations, the single equation

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5^2\right)Q = -2Q\frac{\partial}{\partial r}D_5 Q$$

This is Chandrasekhar's equation of helicity-free isotropic turbulent flow.

5. Smirnov’s Method of Solution

For the sake of convenience, we introduce the longitudinal correlation function $f(r,t) = \frac{F}{\langle u^2\rangle}$ and the correlation function $h(r,t) = \frac{H_1}{\langle u^2\rangle}$ for helicity and writing $\langle u^2\rangle = \beta$, we obtain from the last two equations (1.17),

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5^2\right)f = \beta\left(f\frac{\partial}{\partial r}D_5 f + 4h\frac{\partial h}{\partial r}\right) \text{ and}$$
$$\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5^2\right)h = \beta\left(f D_5 h - h D_5 f\right) \tag{1.18}$$

We take power series for f and h as

$$f = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{i,k} r^{2i} t^{2k}, \quad (a_{0,0} = 1) \text{ and}$$
$$h = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{i,k} r^{2i} t^{2k}, \quad (b_{0,0} = 1) \tag{1.19}$$

Substituting last two power series for f and h in the last two equations, we find

$$a_{i,k+1} = \frac{1}{i(k+1)(2k+1)}\Big[2\nu^2 i(i+1)(i+2)(2i+5)(2i+7)a_{i+2,k} + $$
$$+\beta\left\{\sum_{p=1}^{i}\sum_{q=0}^{k} p(p+1)(2p+5)a_{i-p,k-q}a_{p+1,q} + \sum_{p=1}^{i}\sum_{q=0}^{k} 2pb_{i-p,k-q}b_{p,q}\right\}\Big] \tag{1.20}$$

where $i \geq 0,\ k \geq 0.$

$$b_{i,k+1} = \frac{1}{(k+1)(2k+1)}\Big[2\nu^2(i+1)(i+2)(2i+5)(2i+7)b_{i+2,k} + \\ + \beta\sum_{p=1}^{i}\sum_{q=0}^{k}(p+1)(2p+5)\left\{a_{i-p,k-q}b_{p+1,q} - b_{i-p,k-2}a_{p+1,q}\right\}\Big] \tag{1.21}$$

Now, if $a_{i,0}$ $i = 0,1,2,3,...$ are known, then $a_{i,k}$ can be determined from equation (1.20). If $b_{i,0}$ $i = 0,1,2,3,...$ are known, then $b_{i,k}$ can be determined from equation (1.21).

6. Millionschkikov's Hypothesis

The dynamic equation due to von Karman–Howarth [26] is given by

$$\frac{\partial}{\partial t}\left(\bar{u}^2 f\right) - \frac{(\bar{u})^{\frac{3}{2}}}{r^4} - \frac{\partial}{\partial r}\left(r^4\kappa\right) = 2\nu\frac{\bar{u}^2}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \tag{1.22}$$

Among many measurements the one by Stewart [24] measured each term of equation (1.22) supported that it is satisfied within the limits of experimental errors. From (1.22) we can find

$$\frac{\partial}{\partial t}\int_0^\infty \overline{u^2} r^4 f(r)\,dr = \left(\overline{u^2}\right)^{\frac{3}{2}}\left[r^4 k(r)\right]_0^\infty + 2\nu\overline{u^2}\left[r^4\frac{\partial f}{\partial r}\right]_0^\infty \tag{1.23}$$

Now, if $\lim_{r\to\infty} r^4 k(r) = 0$ and $\lim_{r\to\infty} r^4\frac{\partial f}{\partial r} = 0$ we obtain $\frac{\partial}{\partial t}\int_0^\infty \overline{u^2} r^4 f(r)\,dr = 0$

and accordingly, $\int_c^\infty \overline{u^2} r^4 f(r)\,dr = \Lambda$ where Λ is constant. (1.24)

The above equation (1.24) is known as Loitsianskii's integral invariant [13]. In the final period decay, $\frac{u\,l}{v} \to 0$ and the non-linear term becomes unimportant, as we may put $k(r) = 0$ and then equation (1.22) reduces to

$$\frac{\partial f}{\partial t} = \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial f}{\partial r}\right) \tag{1.25}$$

and we have the solution $f(r,t) = \exp\left(-\frac{r^2}{8\nu t}\right)$

Substitution of this result into equation (1.24) we obtain $\Lambda = 48\sqrt{2\pi}\,(\nu t)^{\frac{5}{2}}\,\overline{u^2}$

from which we see that during the late stages of decay $\overline{u^2} \approx t^{-\frac{5}{2}}$.

Let us consider the equation for the double correlation of fluctuating pressures at points A and B in homogeneous and isotropic turbulence as [14]

$$\frac{\partial^4 \overline{p_A p_B}}{\partial\xi_i \partial\xi_i \partial\xi_j \partial\xi_j} = \frac{\partial^4 \overline{(u_i u_j)_A (u_k u_l)_B}}{\partial\xi_i \partial\xi_j \partial\xi_k \partial\xi_l}$$ where $\xi_i = (x_i)_B - (x_i)_A$ and $r^2 = \xi_i \xi_i$

Under isotropic conditions, we may write $\frac{\partial^4 \overline{(u_i u_j)_A (u_k u_l)_B}}{\partial\xi_i \partial\xi_j \partial\xi_k \partial\xi_l} = M$

It can be derived now $\frac{1}{r}\frac{\partial^4 (rQ_{p,p})}{\partial r^2} = M(r,t)$

Now, $Q_{p,p}(r,t)$ is assumed to decrease rapidly to zero with the indefinite increase of r, as such for any value of n, we have $\lim_{r\to\infty} \frac{d^n}{dr^n}\left[rQ_{p,p}(r,t)\right] = 0$

Finally, $Q_{p,p}(r,t)$ can be calculated from $M(r,t)$ by a single integration process, as

$$Q_{p,p}(r,t) = \frac{1}{3!r}\int_r^\infty dr'(r'-r)^3 M(r',t)$$

At this stage, we make the simplifying assumption due to Millionschikov [16] e.g., about the quadruple correlation $\overline{(u_i u_j)_A (u_k u_l)_B}$ that the joint probability distribution of turbulent velocities is normal. In effect, we have the relation

$$\overline{(u_i u_j)_A (u_k u_l)_B} = \overline{(u_i u_j)_A} \cdot \overline{(u_i u_j)_A} + \overline{(u_i)_A (u_k)_B} \cdot \overline{(u_j)_A (u_l)_B} +$$
$$+ \overline{(u_i)_A (u_l)_B} \cdot \overline{(u_j)_A (u_k)_B} = Q_{i,j}(0)Q_{k,l}(0) + Q_{i,k}(r)Q_{j,l}(r) + Q_{i,l}(r)Q_{j,k}(r)$$

Physically, the hypothesis is explained, as Hinze [8]. Though the probability density of turbulent velocity may be Gaussian, the joint probability density cannot be Gaussian since a zero value of fourth-order cumulant would lead to non positive parts of the energy spectrum.

Remarks

i) Millionschikov's quasi-normality hypothesis is very useful for closure of homogeneous and isotropic turbulence as such a hypothesis has been proved to be valid within the limits of experimental errors.
ii) To gain more insight into the closure problems of turbulence, data from recent and advanced level measurements are to be brought into account in developing the appropriate models.
iii) As and when necessary, modification of this hypothesis is welcomed, as Ogura [19] has pointed that tentatively, the errors that arise from finite difference approximations in numerical integration of the main equation of turbulence are not responsible for the generation of the negative energy spectrum but are the consequences of the quasi-normality hypothesis itself.

iv) Third-order moments of the velocity field should not be assumed zero and it should be considered into the calculation even at the initial evolution of spectral energy of turbulence.

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