

A Solution of the Modified Tzitzeica-Dodd-Bullough Equation

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Abstract

In this work, we solve a modified Tzitzeica-Dodd-Bullough (TDB) equation using the tanh and Ricatti and Jacobi solitary wave methods. We get several families of solutions.

Keywords: Tzitzeica-Dodd-Bullough equation, Tanh method, Ricatti method, Jacobi elliptic functions

1 Introduction

In reference [1]-[2], the next partial nonlinear equation is considered:

$$\frac{\partial^2 \psi}{\partial x \partial t} = \alpha e^{(m\psi)} + \beta e^{(n\psi)} \quad (1)$$

whose parameters in the exponential nonlinear terms display a wide range of very interesting nonlinear partial differential equations (NPDEs). We have chosen to rename $\alpha = g$, $\beta = h$, and $m = -2$ and $n = -4$, getting a modified Tzitzeica-Dodd-Bullough equation (modTDBeq). In order to find solutions we apply the so-called solitary wave methods (SWMs), which have been extensively applied to very different types of NPDEs. In this work, we are going to use SWMs in order to find solutions to the modTDBeq.

2 A modified Tzitzeica-Dodd-Bullough equation

The modified Tzitzeica-Dodd-Bullough equation in order to solve is:

$$\frac{\partial^2 \psi}{\partial x \partial t} = g e^{(-2\psi)} + h e^{(-4\psi)} \quad (2)$$

We use the next coordinate transformation and its derivatives, as:

$$\xi = x - wt, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \quad \frac{\partial}{\partial t} = -w \frac{d}{d\xi} \quad (3)$$

Therefore

$$-w \frac{d^2}{d\xi^2} \psi - g e^{(-2\psi)} - h e^{(-4\psi)} = 0 \quad (4)$$

Now, we define the variables:

$$v = e^{-\psi} \quad (5)$$

So, the first and second derivatives in eq. (3), are:

$$\frac{d\psi}{d\xi} = -\frac{1}{v} \frac{dv}{d\xi}, \quad \frac{d^2\psi}{d\xi^2} = \frac{1}{v^2} \left(\frac{dv}{d\xi} \right)^2 - \frac{1}{v} \frac{d^2v}{d\xi^2} \quad (6)$$

And replacing in eqs. (3)

$$-w \frac{1}{v^2} \left(\frac{dv}{d\xi} \right)^2 + w \frac{1}{v} \frac{d^2v}{d\xi^2} - g v^2 - h v^4 = 0 \quad (7)$$

So

$$w \left(\frac{dv}{d\xi} \right)^2 - w v \frac{d^2v}{d\xi^2} + g v^4 + h v^6 = 0 \quad (8)$$

If we apply the balance between the terms v^6 and $w \left(\frac{dv}{d\xi} \right)^2$ we get a fractional number $p = 1/3$. Now, in order to correctly apply the balance law, we use the next transformation:

$$v = \phi^{1/2}, \quad \frac{dv}{d\xi} = \frac{1}{2} \phi^{-1/2} \frac{d\phi}{d\xi}, \quad \frac{d^2v}{d\xi^2} = -\frac{1}{4} \phi^{-3/2} \left(\frac{d\phi}{d\xi} \right)^2 + \frac{1}{2} \phi^{-1/2} \frac{d^2\phi}{d\xi^2} \quad (9)$$

Replacing in eq. (7) is:

$$w\left(\frac{1}{2}\phi^{-1/2}\frac{d\phi}{d\xi}\right)^2 - w(\phi^{1/2})\left(-\frac{1}{4}\phi^{-3/2}\left(\frac{d\phi}{d\xi}\right)^2 + \frac{1}{2}\phi^{-1/2}\frac{d^2\phi}{d\xi^2}\right) + g(\phi^{1/2})^4 + h(\phi^{1/2})^6 = 0 \quad (10)$$

$$\frac{w\phi^{-1}}{4}\left(\frac{d\phi}{d\xi}\right)^2 + \frac{w}{4}\phi^{-1}\left(\frac{d\phi}{d\xi}\right)^2 - \frac{w}{2}\frac{d^2\phi}{d\xi^2} + g\phi^2 + h\phi^3 = 0 \quad (11)$$

$$w\left(\frac{d\phi}{d\xi}\right)^2 - w\phi\frac{d^2\phi}{d\xi^2} + 2g\phi^3 + 2h\phi^4 = 0 \quad (12)$$

Applying again the balance law between the terms ϕ^4 and $\phi\frac{d^2\phi}{d\xi^2}$ we get an entire number $p = 1$.

3 Exact Differential Equation

First of all, equation (12) is an ordinary differential equation, and susceptible to be solved by standard methods. Therefore, defining:

$$r = \frac{d\phi}{d\xi}, \quad \frac{dr}{d\xi} = r \frac{dr}{d\phi} \quad (13)$$

We get in eq. (12)

$$wr^2 - w\phi r \frac{dr}{d\phi} + 2g\phi^3 + 2h\phi^4 = 0 \quad (14)$$

which is a first order differential equation, eq. (14), and multiplying by $1/\phi^3$, we get an exact differential equation:

$$\frac{wrd r}{\phi^2} - (w\frac{r^2}{\phi^3} + 2g + 2h\phi)d\phi = 0 \quad (15)$$

As a result, we obtain

$$\left(\frac{d\phi}{d\xi}\right)^2 = r^2 = \left(\frac{2c}{w}\phi^2 + \frac{4g}{w}\phi^3 + \frac{2h}{w}\phi^4\right) \quad (16)$$

where c is an integration constant. Then, the integral to be solved, is:

$$\int \frac{d\phi}{\sqrt{\left(\frac{2c}{w}\phi^2 + \frac{4g}{w}\phi^3 + \frac{2h}{w}\phi^4\right)}} = \pm(\xi - \xi_0) \quad (17)$$

The solution is

$$\frac{w}{2c}(\ln \phi - \ln(c + g\phi + \sqrt{c}\sqrt{c + 2g\phi + h\phi^2})) = \pm(\xi - \xi_0) \quad (18)$$

or

$$\phi = \frac{2c \exp(\mp \frac{2c}{w}(\xi - \xi_0))}{((g - \exp(\mp \frac{2c}{w}(\xi - \xi_0)))^2 - hc)} \quad (19)$$

using eq. (9)

$$v = \phi^{1/2} = \sqrt{\frac{2c \exp(\mp \frac{2c}{w}(\xi - \xi_0))}{((g - \exp(\mp \frac{2c}{w}(\xi - \xi_0)))^2 - hc)}} \quad (20)$$

and eq. (5)

$$\psi = -\ln(v) = -\ln\left(\sqrt{\frac{2c \exp(\mp \frac{2c}{w}(\xi - \xi_0))}{((g - \exp(\mp \frac{2c}{w}(\xi - \xi_0)))^2 - hc)}}\right) \quad (21)$$

4 Solitary wave solutions, tanh method

After that, we apply the tanh solitary wave method [2]. Then, we define the variable:

$$Y = \tanh(\mu\xi) \quad (22)$$

Then, the derivatives of u , are:

$$\frac{d}{d\xi} = \mu(1 - Y^2) \frac{d}{dY}, \quad \frac{d^2}{d\xi^2} = -2Y\mu^2(1 - Y^2) \frac{d}{dY} + \mu^2(1 - Y^2)^2 \frac{d^2}{dY^2} \quad (23)$$

The solutions are postulated as [3]:

$$\phi = \sum_{i=0}^p a_i Y^i \quad (24)$$

Then, replacing in eq. (12)

$$\begin{aligned} & w\mu^2((1 - Y^2) \frac{d\phi}{dY})^2 + w\mu^2\phi 2Y(1 - Y^2) \frac{d\phi}{dY} - w\mu^2\phi(1 - Y^2)^2 \frac{d^2\phi}{dY^2} \\ & + 2g\phi^3 + 2h\phi^4 = 0 = 0 \end{aligned} \quad (25)$$

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (25). Then, $\phi Y^4 \frac{d^2\phi}{dY^2} \rightarrow \phi^4 \rightarrow p = 1$. So, replacing in eq. (24)

	A_1	C_1	F
1	1/2	-1/2	$\coth(\xi) \pm \cosh(\xi), \tanh(\xi) \pm \operatorname{sech}(\xi)$
2	1/2	1/2	$\sec(\xi) \pm i \tan(\xi)$
3	-1/2	- 1/2	$\csc(\xi) \pm i \cot(\xi)$
4	1	- 1	$\tanh(\xi) , \coth(\xi)$
5	1	1	$\tan(\xi)$
6	-1	-1	$\cot(\xi)$

Table 1: Solutions for eqs. (7), [4] .

$$\phi = a_0 + a_1 Y \quad (26)$$

Replacing in eqs. (18)), we get a set of equations, order by order in Y^i . And doing some algebra, we get:

$$f_1 \rightarrow \left\{ a_1 = -a_0, w = -\frac{2a_0^2 h}{\mu^2}, g = -2a_0 h \right\} \quad (27)$$

$$f_2 \rightarrow \left\{ a_1 = a_0, w = -\frac{2a_0^2 h}{\mu^2}, g = -2a_0 h \right\} \quad (28)$$

Then, we get two families of solutions.

5 Solitary wave method, Riccati Solutions

We use the method in [4], to get solutions for eqs. (6). So:

$$\phi = \sum_{i=1}^n a_i F^i \quad (29)$$

where F solves, table (1), the Riccati equation, i.e.

$$F' = (C_1 F^2 + A_1), \quad F'' = 2C_1 F(C_1 F^2 + A_1) \quad (30)$$

here A_1 and C_1 are constants, table (1). Balancing the nonlinear terms, we have $p = 1$. Then, eq. (29) is, $\phi = (a_0 + a_1 F)$. Therefore, the derivatives of ϕ are:

	ϵ	a	b	c	G
1	-1	$-m^2$	1	1	$sn(\xi)$
2	-1	m^2	$1 - m^2$	1	$cn(\xi)$
3	-1	1	$m^2 - 1$	1	$dn(\xi)$
4	-1	$-m^2$	1	1	$cd(\xi)$
5	$m^2 - 1$	m^2	1	1	$sd(\xi)$
6	$1 - m^2$	1	-1	1	$nd(\xi)$
7	1	1	$-m^2$	-1	$dc(\xi)$
8	1	$1 - m^2$	m^2	-1	$nc(\xi)$
9	1	$1 - m^2$	1	1	$sc(\xi)$
10	1	1	$-m^2$	-1	$ns(\xi)$
11	1	1	$m^2 - 1$	m^2	$ds(\xi)$
12	1	1	$1 - m^2$	1	$cs(\xi)$

Table 2: The Solutions for eq. (24), [4] .

$$\phi' = (a_1)F' = (a_1)(C_1F^2 + A_1), \quad \phi'' = (a_1 2C_1F(C_1F^2 + A_1)) \quad (31)$$

Replacing in eq. (12), we obtain a group of equations, order by order in F^i . And doing some algebra, we get:

$$g_1 \rightarrow \left\{ a_1 = -\frac{ia_0\sqrt{C_1}}{\sqrt{A_1}}, w = \frac{2a_0^2h}{A_1C_1}, G = -2a_0h \right\} \quad (32)$$

$$g_2 \rightarrow \left\{ a_1 = \frac{ia_0\sqrt{C_1}}{\sqrt{A_1}}, w = \frac{2a_0^2h}{A_1C_1}, G = -2a_0h \right\} \quad (33)$$

Then, we get twelve families of solutions, g_i , using Ricatti method [4].

6 Solitary wave method 3, Jacobi solutions

We start with the solutions, table (2), given by the next differential equation:

$$(G')^2 = (c + \epsilon G^2)(aG^2 + b) \quad (34)$$

Also, they satisfy the next relations:

$$\begin{aligned}
sn(\xi, k)^2 + cn(\xi, k)^2 &= k^2 sn(\xi, k)^2 + dn(\xi, k)^2 = 1 \\
1 + cs(\xi, k)^2 &= k^2 + ds(\xi, k)^2 = ns(\xi, k)^2 \\
(1 - k^2)sd(\xi, k)^2 + 1 &= dc(\xi, k)^2 = (1 - k^2)nc(\xi, k)^2 + k^2 \\
k^2(1 - k^2)sd(\xi, k)^2 &= k^2(cd(\xi, k)^2 - 1) = (1 - k^2)(1 - nd(\xi, k)^2)
\end{aligned} \tag{35}$$

and $k' = \sqrt{1 - k^2}$

$$\begin{aligned}
sn(i\xi, k) &= (i)sn(\xi, k'), \quad dc(i\xi, k) = dn(\xi, k') \\
cn(i\xi, k) &= nc(\xi, k'), \quad nc(i\xi, k) = cn(\xi, k') \\
dn(i\xi, k) &= dc(\xi, k'), \quad sc(i\xi, k) = (i)sn(\xi, k') \\
cd(i\xi, k) &= nd(\xi, k'), \quad ns(i\xi, k) = (-i)cs(\xi, k') \\
sd(i\xi, k) &= (i)nd(\xi, k'), \quad ds(i\xi, k) = (-i)ds(\xi, k') \\
nd(i\xi, k) &= cd(\xi, k'), \quad cs(i\xi, k) = (-i)ns(\xi, k')
\end{aligned} \tag{36}$$

and the second derivative is:

$$G'' = 2a\epsilon^2 G^3 + (ac + b)\epsilon G \tag{37}$$

Where a , b , c and ϵ are given in table (2). We use the method in [4], to get solutions for eqs. (6).

$$\phi = \sum_{i=1}^n a_i G^i \tag{38}$$

Replacing eqs. (34) and (37) in eq. (12), and balancing nonlinear terms, we have $p = 1$. Therefore, the solution is:

$$\phi = (a_0 + a_1 G) \tag{39}$$

Then, we obtain a group of equations, order by order in G^i . And doing some algebra, we get:

$$l_1 \rightarrow \left\{ a_0 = -\frac{i\sqrt{c}}{\sqrt{e}}, a_1 = 1, g = \frac{ibe^{3/2}w}{\sqrt{c}}, h = \frac{be^2w}{2c}, a = \frac{be}{c} \right\} \tag{40}$$

$$l_2 \rightarrow \left\{ a_0 = \frac{i\sqrt{c}}{\sqrt{e}}, a_1 = 1, g = -\frac{ibe^{3/2}w}{\sqrt{c}}, h = \frac{be^2w}{2c}, a = \frac{be}{c} \right\} \tag{41}$$

$$l_3 \rightarrow \left\{ a_0 = -\frac{2i\sqrt{c}}{\sqrt{e}}, a_1 = 2, g = \frac{ibe^{3/2}w}{10\sqrt{c}}, h = \frac{3be^2w}{160c}, a = \frac{be}{5c} \right\} \quad (42)$$

$$l_4 \rightarrow \left\{ a_0 = \frac{2i\sqrt{c}}{\sqrt{e}}, a_1 = 2, g = -\frac{ibe^{3/2}w}{10\sqrt{c}}, h = \frac{3be^2w}{160c}, a = \frac{be}{5c} \right\} \quad (43)$$

$$l_5 \rightarrow \left\{ a_0 = -\frac{2i\sqrt{c}}{\sqrt{5}\sqrt{e}}, a_1 = 2, g = \frac{i\sqrt{5}be^{3/2}w}{2\sqrt{c}}, h = \frac{15be^2w}{32c}, a = \frac{5be}{c} \right\} \quad (44)$$

$$l_6 \rightarrow \left\{ a_0 = \frac{2i\sqrt{c}}{\sqrt{5}\sqrt{e}}, a_1 = 2, g = -\frac{i\sqrt{5}be^{3/2}w}{2\sqrt{c}}, h = \frac{15be^2w}{32c}, a = \frac{5be}{c} \right\} \quad (45)$$

Then, we get seventy two families of solutions, l_i , using Jacobi solutions [4].

7 Conclusions

We solved a modified Tzitzeica-Dodd-Bullough equation, using the solitary wave methods Tanh method, Ricatti method, Jacobi elliptic functions and standard ordinary differential methods. In the tanh method, we find two families of solutions. For Riccati solutions we get twelve families of solutions. Finally, using the Jacobi elliptic functions we obtain seventy two families of solutions. As a future work, we can explore a different set of values for the parameter space α , β , and m and n , in eq. (1). In general, the solutions are:

$$\psi = -\frac{1}{2} \ln(a_0 + a_1 \tanh(\mu\xi)), \quad \psi = -\frac{1}{2} \ln(a_0 + a_1 F(\xi)) \quad (46)$$

$$\psi = -\frac{1}{2} \ln(a_0 + a_1 G(\xi)) \quad (47)$$

$$\psi = -\ln\left(\sqrt{\frac{2c \exp(\mp \frac{2c}{w}(\xi - \xi_0))}{((g - \exp(\mp \frac{2c}{w}(\xi - \xi_0)))^2 - hc)}}\right) \quad (48)$$

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