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Optimistic Value Based Optimization Problem for Investment and Consumption with Bankruptcy

Zongzhen Fan and Yuanguo Zhu

School of Mathematics and Statistics
Nanjing University of Science and Technology
Nanjing 210094 Jiangsu, China

Abstract

The investment and consumption problem is an important branch of portfolio theory, focusing on how to achieve a dynamic balance between investment and consumption to optimize wealth and utility. Considering the significant impact that extreme scenarios may have on the investment-consumption problem, this paper constructs an optimistic value model under uncertain environments and provides an analytical solution for the optimal investment strategy and consumption rate. Furthermore, the paper explores how to adjust investment and consumption behavior under bankruptcy constraints to maximize an objection function and provides numerical solutions. An empirical analysis ensures that the proposed model and method are effective and realistic.

Mathematics Subject Classification: 91G10, 90C70

Keywords: Uncertain differential equation, Optimistic value, Portfolio, Consumption, Bankruptcy

1 Introduction

Investment is a key branch of finance, and portfolio theory is a popular research topic within the discipline. Markowitz established modern portfolio theory, while Sharpe [8] introduced the capital asset pricing model. Subsequent studies explored new investment criteria extended from single-cycle to multi-cycle frameworks. Merton [6] pioneered the study of multi-period dynamic portfolio selection problems and Bellman [1] proposed the stochastic dynamic programming method. Bielecki et al. [2] established a bankruptcy injunction that permits.

Due to emergencies, economic factors and technical limitations, we may have little or no useful sample data available to estimate a distribution function. Based on this, Liu proposed the uncertainty theory [4]. Zhu [14] put forward an uncertain optimal control problem and applied it to the portfolio selection model. Yao & Chen [11] proposed to use α-path to find the numerical solution of an uncertain differential equation. Sheng & Zhu [9] provided an explicit solution for optimistic value model by the uncertain optimal control. Qin et al. [7] proposed different portfolio model. Liu [5] established the concept of first hitting time, and then Yao [10] provided the uncertainty distribution of the first hitting time through the *alpha*-path. Jin et al. [3] optimized the objective function about the first hitting time.

This paper explores the optimal investment and consumption problem in uncertain environments without or with bankruptcy constraint. The paper is structured as follows: Section 2 recalls the key results in uncertain theory. Section 3 develops optimal investment and consumption strategies, offering both analytical and numerical solutions under unconstrained and bankruptcy constraint. Section 4 applies the optimistic value model to real-world scenarios to demonstrate its effectiveness. Finally, Section 5 concludes findings and suggestions for future research.

2 Preliminary

This section recalls basic definitions and lemmas in uncertainty theory [4].

Definition 2.1. *An uncertain process C_t is said to be a Liu process if* (i) $C_0 = 0$ *and almost all sample paths are Lipschitz continuous,* (ii) C_t *has stationary and independent increments,* (iii) *every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value* 0 *and variance* t^2.

Definition 2.2. *Assume that C_t is a Liu process, f and g are two continuous functions. Then*

$$\mathrm{d}X_t = f\left(t, X_t\right)\mathrm{d}t + g\left(t, X_t\right)\mathrm{d}C_t$$

is called an uncertain differential equation (UDE).

Definition 2.3. *Assume that $\alpha \in (0,1)$, an UDE like Definition 2.2 is said to have an α-path X_t^α, if it solves the corresponding ordinary differential equation*

$$\mathrm{d}X_t^\alpha = f\left(t, X_t^\alpha\right)\mathrm{d}t + \left|g\left(t, X_t^\alpha\right)\right|\Phi^{-1}\left(\alpha\right)\mathrm{d}t,$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e., $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$.

Definition 2.4. *Let X_t be an uncertain process and z be a given level. Then the uncertain variable*

$$\tau_z = \begin{cases} \inf\{t \geq 0 \mid X_t \geq z\}, & \text{if } z > X_0, \\ \inf\{t \geq 0 \mid X_t \leq z\}, & \text{if } z < X_0. \end{cases}$$

is called the first hitting time that X_t reaches the level z.

Lemma 2.5. *Let X_t and X_t^α be the solution and α-path of the UDE like Definition 2.2 with an initial value X_0, respectively. Given a strictly increasing function $J(x_t)$, the first hitting time τ_z that $J(x_t)$ reaches z value has an uncertainty distribution*

$$\Psi(s) = \begin{cases} 1 - \inf\left\{\alpha \in (0,1) \mid \sup\limits_{0 \leq t \leq s} J\left(X_t^\alpha\right) \geq z\right\}, & \text{if } z > J(x_0), \\ \sup\left\{\alpha \in (0,1) \mid \inf\limits_{0 \leq t \leq s} J\left(X_t^\alpha\right) \leq z\right\}, & \text{if } z < J(x_0). \end{cases}$$

Consider an uncertain optimistic value optimal control problem

$$\begin{cases} J(t,x) = \sup\limits_{u_t \in U} F_{\sup}(\alpha) \\ \text{subject to} \\ \quad \mathrm{d}X_s = \mu(s, u_s, X_s)\,\mathrm{d}t + \sigma(s, u_s, X_s)\,\mathrm{d}C_s, \quad X_t = x, \end{cases} \tag{1}$$

where $F_{\sup}(\alpha)$ is the β-optimistic value of uncertain variable F provided by

$$F_{\sup}(\alpha) = \sup\{r \mid \mathcal{M}\{F \geq r\} \geq \alpha\}, \quad \alpha \in (0,1).$$

Lemma 2.6. *[9] For Problem (1), if $J(t,x)$ is twice differentiable, then*

$$-J_t(t,x) = \sup_{u \in U}\left\{f(t,u_t,x) + J_x(t,x)\mu(t,u_t,x) + \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\left|J_x(t,x)\sigma(t,x,u)\right|\right\},$$

where $J_t(t,x)$ and $J_x(t,x)$ are the partial derivatives of the function $J(t,x)$ with respect to t and x, respectively.

3 Portfolio Model

In this section, we focus on the optimal portfolio strategy and optimal consumption of investment and consumption problems. Assume that the investor allocates the wealth into a riskless asset and n risky assets with initial wealth x_0 in a interval $[0,T]$. The riskless asset B_t at time t has a fixed rate r satisfying the following differential equation $\mathrm{d}B_t = rB_t\mathrm{d}t$. And the i-th risky asset S_{it} at time t satisfies the UDE with draft coefficient μ_i and diffusion coefficient σ_i, i.e. $\mathrm{d}S_{it} = \mu_i S_{it}\mathrm{d}t + \sigma_i S_{it}\mathrm{d}C_t$. Assume that consumption c_t per unit of time is the linear function of the total wealth X_t,

$$c_t = k_t X_t, \tag{2}$$

where $k_t \in (0,1)$. The wealth X_t satisfies

$$\begin{aligned}\mathrm{d}X_t =& \mathrm{d}B_t + \sum_{i=1}^{n} \mathrm{d}S_{it} - c_t\mathrm{d}t\\ =& \sum_{i=1}^{n} (\mu_i - r)\,\omega_i X_t\mathrm{d}t + (r - k_t)\,X_t\mathrm{d}t + \left(\sum_{i=1}^{n} \omega_i\sigma_i X_t\right)\mathrm{d}C_t, \quad X_0 = x_0,\end{aligned} \tag{3}$$

where ω_i is the proportion invested in the i-th risky asset, $1 - \sum\limits_{i=1}^{n} \omega_i$ is the proportion invested in the riskless asset.

In financial markets, investor's returns and risks exhibit significant volatility, driven by dynamic economic conditions, market sentiment shifts and so on. Given this inherent unpredictability, assuming that the investor accounts for extreme scenarios, an optimistic value model can be constructed as follows:

$$\begin{cases} J = \max\limits_{k_t,\omega_i} \left[\displaystyle\int_0^T e^{-bt}\frac{\sum\limits_{i=1}^{n}(\omega_i X_t)^\lambda + (k_t X_t)^\lambda}{\lambda}\mathrm{d}t\right]_{\sup} \quad (\alpha)\\ \text{subject to}\\ \quad \mathrm{d}X_t = \sum\limits_{i=1}^{n} (\mu_i - r)\,\omega_i X_t\mathrm{d}t + (r - k_t)\,X_t\mathrm{d}t + \left(\sum\limits_{i=1}^{n} \omega_i\sigma_i X_t\right)\mathrm{d}C_t, \quad X_0 = x_0, \end{cases} \tag{4}$$

where $b > 0$, $\lambda \in (0,1)$. According to Lemma 2.6, the extremum value satisfies

$$\begin{aligned}-J_t =& \max_{k_t,\omega_i} \left\{ e^{-bt}\frac{\sum\limits_{i=1}^{n}(\omega_i x)^\lambda + (k_t x)^\lambda}{\lambda} + J_x\left[\sum_{i=1}^{n} (\mu_i - r)\,\omega_i x + (r - k_t)\,x\right]\right.\\ &\left. + \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}J_x\sum_{i=1}^{n}\omega_i\sigma_i x\right\}\\ =& \max_{k_t,\omega_i} L\left(k_t, \omega_1, \cdots, \omega_n\right).\end{aligned} \tag{5}$$

Taking partial derivatives of the Lagrange function $L\left(k_t, \omega_1, \cdots, \omega_n\right)$ in (5), let the partial derivative equal to zero. The optimal solution ω_i^* and k_t^* can be written as

$$\begin{cases} k_t^* = \dfrac{\left(J_x e^{bt}\right)^{\frac{1}{\lambda-1}}}{x}, \\ \omega_i^* = \dfrac{\left[J_x e^{bt}\left(r-\mu_i-\frac{\sqrt{3}}{\pi}\sigma_i \ln\frac{1-\alpha}{\alpha}\right)\right]^{\frac{1}{\lambda-1}}}{x}, \quad i = 1, 2, \cdots, n. \end{cases} \tag{6}$$

Assume that $J(t, x) = kx^\lambda e^{-bt}$. Then

$$J_t = -kbx^\lambda e^{-bt}, \; J_x = k\lambda x^{\lambda-1} e^{-bt}. \tag{7}$$

Substituting Eq.(6) and Eq.(7) into Eq.(5) yields

$$(k\lambda)^{\frac{1}{\lambda-1}} = \left(\frac{b-r\lambda}{1-\lambda}\right)\frac{1}{\sum\limits_{i=1}^{n}\left(r-\mu_i-\frac{\sqrt{3}}{\pi}\sigma_i \ln\frac{1-\alpha}{\alpha}\right)^{\frac{\lambda}{\lambda-1}}+1}. \tag{8}$$

At the end, the optimal values k_t^* and ω_i^* are obtained as follows:

$$\begin{cases} k_t^* = \left(\dfrac{b-r\lambda}{1-\lambda}\right)\dfrac{1}{\sum\limits_{i=1}^{n}\left(r-\mu_i-\frac{\sqrt{3}}{\pi}\sigma_i \ln\frac{1-\alpha}{\alpha}\right)^{\frac{\lambda}{\lambda-1}}+1}, \\ \omega_i^* = \left(\dfrac{b-r\lambda}{1-\lambda}\right)\dfrac{\left(r-\mu_i-\frac{\sqrt{3}}{\pi}\sigma_i \ln\frac{1-\alpha}{\alpha}\right)^{\frac{1}{\lambda-1}}}{\sum\limits_{i=1}^{n}\left(r-\mu_i-\frac{\sqrt{3}}{\pi}\sigma_i \ln\frac{1-\alpha}{\alpha}\right)^{\frac{\lambda}{\lambda-1}}+1}, \quad i = 1, 2, \cdots, n. \end{cases} \tag{9}$$

Incorporating bankruptcy constraints helps investors manage risks and avoid financial crises while pursuing wealth growth. The first hitting time (bankruptcy time) τ_z serves as a key constraint for evaluating bankruptcy. If the wealth X_t drops to a certain level z, the investor is deemed bankrupt, i.e. $\tau_z = \inf\{t \geq 0 \mid X_t \leq z\}$. For any t, if $X_t \geq z$, then $\tau_z = +\infty$. As for the time $t \in [0, T]$, if investors avoid bankruptcy, we have $\tau_z > T$. Then, the risk of bankruptcy can be restricted by following inequality $\mathcal{M}\{\tau_z > T\} \geq \beta$ for a suitable belief degree β. The distribution function of the bankruptcy time τ_z can be given by the following theorem.

Theorem 3.1. *Assume that the wealth X_t satisfies Eq.(3). When X_t reaches a certain value z with $z < x_0$, the distribution function of the bankruptcy time* *can be given by*

$$U\left(s\right)=\frac{\exp\left\{\frac{\pi\left[\ln\frac{z}{x_0}-\sum_{i=1}^{n}(\mu_i-r)\omega_i s-(r-k_t)s\right]}{\sqrt{3}\sum_{i=1}^{n}\omega_i\sigma_i s}\right\}}{1+\exp\left\{\frac{\pi\left[\ln\frac{z}{x_0}-\sum_{i=1}^{n}(\mu_i-r)\omega_i s-(r-k_t)s\right]}{\sqrt{3}\sum_{i=1}^{n}\omega_i\sigma_i s}\right\}}. \tag{10}$$

Proof. From Definition 2.3, α-path X_t^α satisfies

$$X_t^\alpha=x_0\exp\left\{\sum_{i=1}^{n}\left(\mu_i-r\right)\omega_i t+\left(r-k_t\right)t+\sum_{i=1}^{n}\omega_i\sigma_i t\Phi^{-1}\left(\alpha\right)\right\}=x_0\exp\left\{th\right\}, \tag{11}$$

where $h=\sum_{i=1}^{n}\left(\mu_i-r\right)\omega_i+\left(r-k_t\right)+\sum_{i=1}^{n}\omega_i\sigma_i\Phi^{-1}\left(\alpha\right)$. A suitable value α can be selected in accordance with Lemma 2.5, such as meeting the requirement that $\inf_{0\le t\le s}X_t^\alpha\le z$. If $h\ge 0$, then $\tau_z=+\infty$. For $h<0$, we have

$$\begin{aligned}
\alpha_0&=\sup\left\{\alpha\mid\inf_{0\le t\le s}X_t^\alpha\le z\right\}\\
&=\sup\left\{\alpha\mid\inf_{0\le t\le s}x_0\exp\left\{\sum_{i=1}^{n}\left(\mu_i-r\right)\omega_i t+\left(r-k_t\right)t+\sum_{i=1}^{n}\omega_i\sigma_i t\Phi^{-1}\left(\alpha\right)\right\}\le z\right\}\\
&=\sup\left\{\alpha\mid\inf_{0\le t\le s}x_0\exp\left\{\sum_{i=1}^{n}\left(\mu_i-r\right)\omega_i t+\left(r-k_t\right)t+\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\sum_{i=1}^{n}\omega_i\sigma_i t\right\}\le z\right\}\\
&=\sup\left\{\alpha\mid x_0\exp\left\{\sum_{i=1}^{n}\left(\mu_i-r\right)\omega_i s+\left(r-k_t\right)s+\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\sum_{i=1}^{n}\omega_i\sigma_i s\right\}\le z\right\}\\
&=\frac{\exp\left\{\frac{\pi\left(\ln\frac{z}{x_0}-\sum_{i=1}^{n}(\mu_i-r)\omega_i s-(r-k_t)s\right)}{\sqrt{3}\sum_{i=1}^{n}\omega_i\sigma_i s}\right\}}{1+\exp\left\{\frac{\pi\left(\ln\frac{z}{x_0}-\sum_{i=1}^{n}(\mu_i-r)\omega_i s-(r-k_t)s\right)}{\sqrt{3}\sum_{i=1}^{n}\omega_i\sigma_i s}\right\}}.
\end{aligned}$$

According to Lemma 2.5, we have

$$U(s)=\mathcal{M}\left\{\tau_z\le s\right\}=\alpha_0.$$

The proof ends. □

Theorem 3.2. *Assume that the wealth X_t satisfies Eq.(3). Then bankruptcy constraint $\mathcal{M}\{\tau_z > T\} \geq \beta$ is equivalent to*

$$\pi \ln \frac{z}{x_0} - \sqrt{3} \ln \frac{1-\beta}{\beta} \sum_{i=1}^{n} \omega_i \sigma_i T - \pi \sum_{i=1}^{n} (\mu_i - r) \omega_i T - \pi (r - k_t) T \leq 0. \quad (12)$$

Proof. The conclusion follows from Theorem 3.1. □

Denote the function in the left hand side of Eq.(12) as $g(k_t, \omega_1, \cdots, \omega_n)$. Another problem should be considered is as follows

$$\begin{cases} J = \max\limits_{k_t, \omega_i} \left[\int_0^T e^{-bt} \frac{\sum\limits_{i=1}^{n} (\omega_i X_t)^\lambda + (k_t X_t)^\lambda}{\lambda} \mathrm{d}t \right]_{\sup} (\alpha), \\ \text{subject to} \\ \quad \mathrm{d}X_t = \sum\limits_{i=1}^{n} (\mu_i - r) \omega_i X_t \mathrm{d}t + (r - k_t) X_t \mathrm{d}t + \left(\sum\limits_{i=1}^{n} \omega_i \sigma_i X_t \right) \mathrm{d}C_t, \ X_0 = x_0, \\ \quad g(k_t, \omega_1, \cdots, \omega_n) \leq 0, \\ \quad \sum\limits_{i=1}^{n} \omega_i - 1 \leq 0, \\ \quad -\omega_i \leq 0, \\ \quad -k_t \leq 0. \end{cases} \quad (13)$$

We consider setting a penalty function to find its numerical solution. Problem (13) with inequality constraints can be organized into the following optimization problem without portfolio constraint,

$$\max_{k_t, \omega_i} F(k_t, \omega_i) = \left[\int_0^T e^{-bt} \frac{\sum\limits_{i=1}^{n} (\omega_i X_t)^\lambda + (k_t X_t)^\lambda}{\lambda} \mathrm{d}t \right]_{\sup} (\alpha) - \phi P(k_t, \omega_i), \quad (14)$$

where X_t satisfies Eq.(3), weighting parameter ϕ is suitable large, and

$$P(k_t, \omega_i) = \sum_{j=1}^{4} \left[\max\{0, g_j(k_t, \omega_i)\} \right]^2,$$

$g_1(k_t, \omega_i) = g(k_t, \omega_i, \cdots, \omega_n)$, $g_2(k_t, \omega_i) = \sum\limits_{i=1}^{n} \omega_i - 1$, $g_3(k_t, \omega_i) = -\omega_i$, $g_4(k_t, \omega_i) = -k_t$. According to the definition of optimistic value, Eq.(14) can be rewritten as

$$\max_{k_t,\omega_i} F\left(k_t,\omega_i\right)=\int_0^T e^{-bt}\frac{\sum\limits_{i=1}^{n}(\omega_i X_t^{1-\alpha})^{\lambda}+\left(k_t X_t^{1-\alpha}\right)^{\lambda}}{\lambda}\mathrm{d}t-\phi P\left(k_t,\omega_i\right), \quad (15)$$

where X_t^{α} satisfies Eq.(11).

From the method of moment [12] and Definition 2.1, if the risky asset S_t satisfies the uncertain differential equation, we have

$$\frac{S_{t_{i+1}}-S_{t_i}-\mu S_{t_i}(t_{i+1}-t_i)}{\sigma S_{t_i}(t_{i+1}-t_i)}\sim\mathcal{N}\left(0,1\right). \quad (16)$$

The sample moments can provide estimates of the corresponding population moments. Therefore,

$$\begin{cases}\mu=\dfrac{1}{n-1}\sum\limits_{i=1}^{n-1}\dfrac{S_{t_{i+1}}-S_{t_i}}{S_{t_i}\left(t_{i+1}-t_i\right)},\\ \sigma=\sqrt{\dfrac{1}{n-1}\sum\limits_{i=1}^{n-1}\left(\dfrac{S_{t_{i+1}}-S_{t_i}}{S_{t_i}\left(t_{i+1}-t_i\right)}\right)^2-\left(\dfrac{1}{n-1}\sum\limits_{i=1}^{n-1}\dfrac{S_{t_{i+1}}-S_{t_i}}{S_{t_i}\left(t_{i+1}-t_i\right)}\right)^2}.\end{cases} \quad (17)$$

Based on Eq.(16), the sample value $z_i=\frac{s_{t_{i+1}}-s_{t_i}-\mu s_{t_i}(t_{i+1}-t_i)}{\sigma s_{t_i}(t_{i+1}-t_i)}$ can be obtained by the observed values $s_{t_1},s_{t_2},\cdots,s_{t_n}$. Then the uncertain hypothesis test [13] is

$$H_0:\mu=\mu_0 \text{ and } \sigma=\sigma_0 \quad \textit{versus} \quad H_1:\mu\neq\mu_0 \text{ or } \sigma\neq\sigma_0.$$

Given a significance level γ, the rejection domain is

$$W=\Big\{(z_1,z_2,\ldots,z_{n-1}):\ \text{there are at least } \alpha \text{ of indexes } i\text{ 's with } 1\le i\le n-1 \\ \text{such that } z_i<\Phi^{-1}\left(\frac{\gamma}{2}\right) \text{ or } z_i>\Phi^{-1}\left(1-\frac{\gamma}{2}\right)\Big\},$$

where $\Phi^{-1}\left(\gamma\right)=\frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}$ is the inverse uncertainty distribution of $\mathcal{N}(0,1)$. If the vector $(z_1,\ z_2,\ \cdots,\ z_{n-1})\notin W$, then the estimated parameters μ and σ pass the uncertain hypothesis test and then are well fitted.

Finally, the sequential least squares programming (SLSQP) method is used to obtain the optimal value of Problem (15).

4 Empirical analysis

This section validates the practicality of the proposed model and the effectiveness of the solution method. Using monthly closing prices from the Shanghai stock market (January 2020 to December 2023), five stocks (Codes: 002783, 603637, 000096, 600513, 002202) are selected, denoted as S_1, S_2, S_3, S_4, S_5 are selected, denoted as S_1, S_2, S_3, S_4, S_5. Parameters μ_i and σ_i are estimated using moment estimation, with results shown in Table 1. All of the parameters

Table 1: Stock parameters μ_i and σ_i.

	S_1	S_2	S_3	S_4	S_5
μ_i	0.2633	0.1391	0.3279	0.1213	0.1197
σ_i	4.5340	4.7109	3.3927	4.7858	4.8785

in the Table 1 pass the uncertain hypothesis test.

Example 4.1. For the model (4), Let $r = 0.02$, $b = 0.05$, $T = 1$, $x_0 = 1$, $\lambda = 0.5$ and $\alpha = 0.8$. Thus the optimal value can be calculated by Eq.(9), with results shown in Table 2.

Table 2: The optimal investment proportions and consumption rate.

	ω_1^*	ω_2^*	ω_3^*	ω_4^*	ω_5^*	k_t^*
model (4)	0.0036	0.0028	0.0109	0.0026	0.0250	0.1587

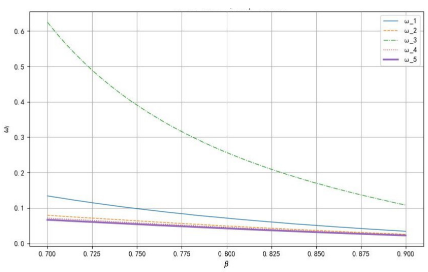
Without portfolio constraint, investors favor high-return assets despite the high risks, driven by their pursuit of returns and macroeconomic factors like low interest rates and inflation, which reduce the appeal of low-risk assets.

Example 4.2. As for the model (15), let $z = 0.2$, $\alpha = 0.8$ and $\phi = 100$. The belief degree β is in the interval $[0.7, 0.9]$. The remaining parameters are the same as in the example 4.1. The optimal value is calculated by SLSQP as shown in Table 3.

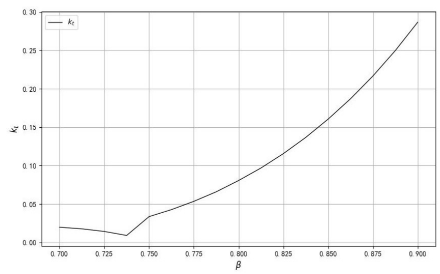
Table 3: The optimal investment proportion and consumption rate for different belief degree.

	ω_1^*	ω_2^*	ω_3^*	ω_4^*	ω_5^*	k_t^*
$\beta = 0.70$	0.1774	0.0798	0.6010	0.0715	0.0701	0.0198
$\beta = 0.75$	0.0958	0.0499	0.5621	0.0443	0.0412	0.0336
$\beta = 0.80$	0.0794	0.0478	0.3639	0.0434	0.0402	0.0806
$\beta = 0.85$	0.0623	0.0410	0.2427	0.0375	0.0349	0.1605
$\beta = 0.90$	0.0454	0.0321	0.1590	0.0296	0.0276	0.2866

Table 3 shows that as k_t increases, ω_i^* decreases while the belief degree β rises. This indicates that investors tend to adopt a more conservative strategy, cutting back on high-risk investments while shifting resources toward consumption. Their focus shifts to ensuring present stability and meeting immediate needs, rather than chasing uncertain returns in the future.



(a) The optimal investment proportions



(b) The optimal consumption rate

Figure 1: The optimal investment proportions and consumption rate for different belief degree

As illustrated in Fig.1, excluding algorithmic fluctuations, the optimal values match our analysis as the risk aversion level varies between $[0.7, 0.9]$. To mitigate bankruptcy risk and enhance potential returns, investors may increase allocations to risky assets, aiming for higher gains to accelerate capital recovery or satisfy constraints.

5 Conclusion

This paper presents continuous-time investment and consumption models under both no portfolio constraint and bankruptcy-constrained environments in an uncertain system, and derives analytical and numerical solutions from the optimistic model. The model's effectiveness was validated through an analysis involving five stocks from the Shanghai Stock Exchange. Results indicate that the proposed model and method are well-suited for portfolio problems with limited data for estimating return distributions. However, due to certain limitations in the code, the model proved overly sensitive to specific values. Future improvements aim to refine the algorithm to enhance its general applicability.

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