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New Coincidence Point and Fixed Point Theorems with $\mathcal{MT}$-Functions

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Abstract

In this paper, we establish new coincidence point theorems which generalize Mizoguchi-Takahashi's fixed point theorem and extend Chen-Du's fixed point theorem from single-valued mappings to multivalued mappings.

Mathematics Subject Classification: 47H10, 54H25

Keywords: $\mathcal{MT}$-function (or $\mathcal{R}$-function), fixed point, coincidence point, Chen-Du's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem

1. Introduction and preliminaries

Let (X, d) be a metric space. Denote by $\mathcal{N}(X)$ the family of all nonempty subsets of X and $\mathcal{CB}(X)$ the class of all nonempty closed and bounded subsets of X. For $x \in X$ and a subset A of X, define $d(x, A) = \inf_{y \in A} d(x, y)$. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max\left\{\sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B)\right\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X, where Let $g : X \to X$ be a self-mapping and $T : X \to \mathcal{N}(X)$ be a multivalued mapping. A point v in X is said to be a *coincidence point* (see, for instance, [1-4]) of g and T if $gv \in Tv$. The set of coincidence points of g and T is denoted by $\mathcal{COP}(g,T)$. If $v \in Tv$, then v is called a fixed point of T. The set of fixed points of T is denoted by $\mathcal{F}(T)$.

Recall that a function $\varphi : [0,+\infty) \to [0,1)$ is said to be an $\mathcal{MT}$*-function* (or $\mathcal{R}$*-function*) [2-4] if $1 > \limsup_{s\to t^+} \varphi(s) := \inf_{\epsilon>0} \sup_{t<s<t+\epsilon} \varphi(s)$ for all $t \in [0,+\infty)$. Clearly, if $\varphi : [0,+\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is a $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

In 2012, Du [4] presented some new characterizations of $\mathcal{MT}$-functions as follows.

Theorem 1.1 (Du [4, Theorem 2.1]). *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*

(b) *For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.*

(c) *For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.*

(d) *For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.*

(e) *For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.*

(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

(g) *φ is a function of contractive factor; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

The famous Mizoguchi-Takahashi's fixed point theorem [6, 8] is a real generalization of Nadler's fixed point theorem [5, 7] which extends Banach contraction principle [1, 5, 8] from single-valued mappings to multivalued mappings.

Theorem 1.2 (Mizoguchi and Takahashi [6]). *Let (X, d) be a complete metric space, $\varphi : [0, +\infty) \to [0, 1)$ be an $\mathcal{MT}$-function and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. Assume that*

$$\mathcal{H}(Tx, Ty) \le \varphi(d(x, y))d(x, y) \quad \text{for all } x, y \in X.$$

Then $\mathcal{F}(T) \ne \emptyset$.

In [2], Chen and Du studied convergence theorems, and applied them to obtain some interesting fixed point theorems. One of these fixed point theorems established in [2] is as follows:

Theorem 1.3 (Chen and Du [2, Corollary 2.3]). *Let (X, d) be a complete metric space, $G : X \to X$ be a selfmapping and $\kappa : [0, +\infty) \to (0, 1)$ be an $\mathcal{MT}$-function. Assume that*

$$d(Gx, Gy) \le \kappa(d(x, y)) \max\left\{d(x, y), \frac{1}{4}(d(x, Gx) + d(y, Gy) + d(y, Gx))\right\}$$

for all $x, y \in X$. Then G admits a unique fixed point in X.

In this work, we will establish new coincidence point theorems which not only extends Theorem 1.3 from single-valued mappings to multivalued mappings, but also generalizes Theorem 1.3 from fixed point results to coincidence point results.

2. Main results

The following lemma is very important for proving our main theorems.

Lemma 2.1. *Let (X, d) be a metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued mapping and $g : X \to X$ be a self-mapping. Suppose that $T(X) = \bigcup_{x \in X} T(x) \subseteq g(X)$. Then the following statements hold:*

(a) For any $a \in X$, there exists $b \in X$ such that $gb \in Ta$.

(b) Fixed $x \in X$. If $z \in Tx$, then there exists $y \in X$ such that $gy = z \in Tx$.

Proof. First, we prove (a). For any $a \in X$, since $T(X) = \bigcup_{x \in X} T(x) \subseteq g(X)$, we have $Ta \subseteq g(X)$. So for each $v \in Ta$, $v \in g(X)$. Hence there exists $b \in X$ such that $gb = v \in Ta$. Following a similar argument as in the proof of (a), one can verify (b). $\square$

In this section, we first establish a new coincidence point theorem which not only extends Theorem 1.3 from single-valued mappings to multivalued mappings, but also generalizes Theorem 1.3 from fixed point results to coincidence point results.

Theorem 2.1. *Let (X,d) be a metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued mapping and $g : X \to X$ be a self-mapping. Suppose that*

(A1) $T(X) = \bigcup_{x\in X} T(x) \subseteq g(X)$,

(A2) *$g(X)$ is a complete subspace of X,*

(A3) *there exists an $\mathcal{MT}$-function $\varphi : [0,+\infty) \to [0,1)$ such that*

$$\mathcal{H}(Tx,Ty) \le \varphi(d(gx,gy)) \max\left\{d(gx,gy), \frac{1}{4}(d(gx,Tx)+d(gy,Ty)+d(gy,Tx))\right\}$$

for all $x,y \in X$. Then the following hold:

(a) *There exists a sequences $\{x_n\}$ in X such that $gx_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$, $\{gx_n\}$ is a Cauchy sequence in $g(X)$ and $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \inf_{n\in\mathbb{N}} d(gx_n, gx_{n+1}) = 0$.*

(b) $\mathcal{COP}(g,T) \ne \emptyset$.

Proof. Let $x_1 \in X$. Then, by Lemma 2.1, there exists $x_2 \in X$ such that $gx_2 \in Tx_1$. If $gx_1 = gx_2$, then $gx_1 \in Tx_1$ and $x_1 \in \mathcal{COP}(g,T) \ne \emptyset$. Otherwise, if $gx_2 \ne gx_1$, $d(gx_1, gx_2) > 0$. Define $\kappa : [0,+\infty) \to (0,1)$ by

$$\kappa(t) = \frac{\varphi(t)+1}{2} \quad \text{for all } t \in [0,+\infty).$$

Then $0 \le \varphi(t) < \kappa(t) < 1$ for all $t \in [0,+\infty)$. By (A3), since $d(gx_2, Tx_1) = 0$, we have

$$\begin{aligned} d(gx_2, Tx_2) &\le \mathcal{H}(Tx_1, Tx_2) \\ &< \kappa(d(gx_1,gx_2)) \max\left\{d(gx_1,gx_2), \tfrac{1}{4}(d(gx_1,Tx_1)+d(gx_2,Tx_2))\right\}. \end{aligned} \tag{2.1}$$

Then, by (2.1), there exists $z \in Tx_2$ such that

$$d(gx_2, z) < \kappa(d(gx_1,gx_2)) \max\left\{d(gx_1,gx_2), \frac{1}{4}(d(gx_1,Tx_1)+d(gx_2,Tx_2))\right\}. \tag{2.2}$$

By (A1), we have $z \in Tx_2 \subseteq g(X)$. So there exists $x_3 \in X$ such that $gx_3 = z \in Tx_2$. Hence, using (2.2), we get

$$d(gx_2, gx_3) < \kappa(d(gx_1, gx_2)) \max\left\{d(gx_1, gx_2), \frac{1}{4}(d(gx_1, gx_2) + d(gx_2, gx_3))\right\}. \tag{2.3}$$

Assume that $d(gx_1, gx_2) < d(gx_2, gx_3)$. Then (2.3) yields

$$d(gx_2, gx_3) < \kappa(d(gx_1, gx_2))d(gx_2, gx_3) < d(gx_2, gx_3),$$

a contradiction. Hence it must be $d(gx_2, gx_3) \leq d(gx_1, gx_2)$ and (2.3) deduces

$$d(gx_2, gx_3) < \kappa(d(gx_1, gx_2))d(gx_1, gx_2).$$

If $gx_2 = gx_3 \in Tx_2$, then $x_2 \in \mathcal{COP}(g, T) \neq \emptyset$. Otherwise, if $gx_3 \neq gx_2$, then $d(gx_2, gx_3) > 0$. By (A3), we have

$$d(gx_3, Tx_3) < \kappa(d(gx_2, gx_3)) \max\left\{d(gx_2, gx_3), \frac{1}{4}(d(gx_2, Tx_2) + d(gx_3, Tx_3))\right\}. \tag{2.4}$$

So, from (2.4), there exists $x_4 \in X$ such that $gx_4 \in Tx_3$ satisfying

$$d(gx_3, gx_4) < \kappa(d(gx_2, gx_3)) \max\left\{d(gx_2, gx_3), \frac{1}{4}(d(gx_2, gx_3) + d(gx_3, gx_4))\right\}. \tag{2.5}$$

Suppose that $d(gx_2, gx_3) < d(gx_3, gx_4)$. Thus, by (2.5), we get

$$d(gx_3, gx_4) < \kappa(d(gx_2, gx_3))d(gx_3, gx_4) < d(gx_3, gx_4),$$

which leads to a contradiction. So it must be $d(gx_3, gx_4) \leq d(gx_2, gx_3)$ and (2.5) implies

$$d(gx_3, gx_4) < \kappa(d(gx_2, gx_3))d(gx_2, gx_3).$$

By induction, we can obtain a sequences $\{x_n\}$ in X satisfying for each $n \in \mathbb{N}$,

(i) $gx_{n+1} \in Tx_n$,

(ii) $d(gx_{n+1}, gx_{n+2}) < \kappa(d(gx_n, gx_{n+1}))d(gx_n, gx_{n+1})$.

Since $\kappa(t) < 1$ for all $t \in [0, +\infty)$, (ii) implies that the sequence $\{d(gx_n, gx_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0, +\infty)$. Hence we obtain

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \inf_{n\in\mathbb{N}} d(gx_n, gx_{n+1}) \geq 0 \quad \text{exists.} \tag{2.6}$$

Since φ is an $\mathcal{MT}$-function, by (g) of Theorem 1.1, we have

$$0 \le \sup_{n\in\mathbb{N}} \varphi\left(d(gx_n, gx_{n+1})\right) < 1.$$

So,

$$0 < \xi := \sup_{n\in\mathbb{N}} \kappa\left(d(gx_n, gx_{n+1})\right) = \frac{1}{2}\left[1 + \sup_{n\in\mathbb{N}} \varphi\left(d(gx_n, gx_{n+1})\right)\right] < 1.$$

By (ii) again, we get

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}) &< \kappa(d(gx_n, gx_{n+1}))d(gx_n, gx_{n+1}) \\ &\le \xi d(gx_n, gx_{n+1}) \\ &\le \cdots \\ &\le \xi^n d(gx_1, gx_2) \qquad \text{for each } n \in \mathbb{N}. \end{aligned} \tag{2.7}$$

Since $\lim_{n\to\infty} \xi^n = 0$, by (2.7), we have

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{2.8}$$

Combining (2.6) with (2.8), we obtain

$$\inf_{n\in\mathbb{N}} d(gx_n, gx_{n+1}) = \lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0.$$

Next, we verify that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Let $\beta_n := gx_n$ for all $n \in \mathbb{N}$. Then $d(\beta_1, \beta_2) = d(gx_1, gx_2) > 0$. For $m, n \in \mathbb{N}$ with $m > n$, by (2.7), we obtain

$$d(\beta_n, \beta_m) \le \sum_{j=n}^{m-1} d(\beta_j, \beta_{j+1}) < \frac{\xi^{n-1}}{1-\xi} d(\beta_1, \beta_2). \tag{2.9}$$

Since $\xi \in (0, 1)$, (2.9) yields

$$\lim_{n\to\infty} \sup\{d(\beta_n, \beta_m) : m > n\} = 0,$$

which shows $\{gx_n\} \equiv \{\beta_n\}$ is a Cauchy sequence in $g(X)$. Therefore, the conclusion (a) is proved.

Finally, we will verify the conclusion (b). By the completeness of $g(X)$, there exists $v \in X$ such that $gx_n \to gv$ as $n \to \infty$. For any $n \in \mathbb{N}$, using condition (A3), we obtain

$$\begin{aligned} &d(gx_{n+1}, Tv) \le \mathcal{H}(Tx_n, Tv) \\ &\qquad < \max\left\{d(gx_n, gv), \frac{1}{4}(d(gx_n, gx_{n+1}) + d(gv, Tv) + d(gv, gx_{n+1}))\right\}. \end{aligned}$$

By taking the limit as $n \to \infty$ on both sides of the last inequality yields $d(gv, Tv) \leq \frac{d(gv,Tv)}{4}$, which deduces $d(gv, Tv) = 0$. By the closedness of Tv, we obtain $gv \in Tv$ or $v \in \mathcal{COP}(g, T)$. The proof is completed. $\square$

The following new coincidence point theorems are immediate from Theorem 2.1.

Corollary 2.1. *Let (X, d) be a metric space, $T : X \to CB(X)$ be a multivalued mapping and $g : X \to X$ be a self-mapping. Suppose that*

(A1) $T(X) = \bigcup_{x \in X} T(x) \subseteq g(X)$,

(A2) *$g(X)$ is a complete subspace of X,*

(A4) *there exists an $\mathcal{MT}$-function $\varphi : [0, +\infty) \to [0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \leq \frac{1}{4}\varphi(d(gx, gy))(d(gx, Tx) + d(gy, Ty) + d(gy, Tx))$$

for all $x, y \in X$. Then the following hold:

(a) *There exists a sequences $\{x_n\}$ in X such that $gx_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$, $\{gx_n\}$ is a Cauchy sequence in $g(X)$ and $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \inf\limits_{n\in\mathbb{N}} d(gx_n, gx_{n+1})$.*

(b) $\mathcal{COP}(g, T) \neq \emptyset$.

Corollary 2.2. *Let (X, d) be a metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued mapping and $g : X \to X$ be a self-mapping. Suppose that*

(A1) $T(X) = \bigcup_{x \in X} T(x) \subseteq g(X)$,

(A2) *$g(X)$ is a complete subspace of X,*

(A5) *there exists an $\mathcal{MT}$-function $\varphi : [0, +\infty) \to [0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \leq \varphi(d(gx, gy))d(gx, gy) \qquad \textit{for all } x, y \in X.$$

Then the following hold:

(a) *There exists a sequences* $\{x_n\}$ *in* X *such that* $gx_{n+1} \in Tx_n$ *for each* $n \in \mathbb{N}$, $\{gx_n\}$ *is a Cauchy sequence in* $g(X)$ *and* $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \inf\limits_{n\in\mathbb{N}} d(gx_n, gx_{n+1})$.

(b) $\mathcal{COP}(g,T) \neq \emptyset$.

If we take $g(x) = x$ for all $x \in X$ in Theorem 2.1, then we obtain the following fixed point theorem for multivalued mappings.

Theorem 2.2. *Let* (X,d) *be a complete metric space and* $T : X \to \mathcal{CB}(X)$ *be a multivalued mapping. Suppose thatthere exists an* $\mathcal{MT}$*-function* $\varphi : [0,+\infty) \to [0,1)$ *such that*

$$\mathcal{H}(Tx,Ty) \leq \varphi(d(x,y)) \max\left\{d(x,y), \frac{1}{4}(d(x,Tx)+d(y,Ty)+d(y,Tx))\right\}$$

for all $x,y \in X$. *Then the following hold:*

(a) *There exists a Cauchy sequences* $\{x_n\}$ *in* X *such that* $x_{n+1} \in Tx_n$ *for each* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf\limits_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(b) $\mathcal{F}(T) \neq \emptyset$.

Corollary 2.3. *Let* (X,d) *be a complete metric space and* $T : X \to \mathcal{CB}(X)$ *be a multivalued mapping. Suppose thatthere exists an* $\mathcal{MT}$*-function* $\varphi : [0,+\infty) \to [0,1)$ *such that*

$$\mathcal{H}(Tx,Ty) \leq \frac{1}{4}\varphi(d(x,y))(d(x,Tx)+d(y,Ty)+d(y,Tx))$$

for all $x,y \in X$. *Then the following hold:*

(a) *There exists a Cauchy sequences* $\{x_n\}$ *in* X *such that* $x_{n+1} \in Tx_n$ *for each* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf\limits_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(b) $\mathcal{F}(T) \neq \emptyset$.

Remark 2.1. Mizoguchi-Takahashi's fixed point theorem and Theorem 1.3 are special cases of Theorems 2.1 and 2.2.

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