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European Option Prices Driven by Standard t-Process

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Abstract

We consider a standard t-process model of a financial market and obtain the solution to the stock price process under the equivalent martingale measure. Using the equivalent martingale measure we derive the European option pricing based on standard t-process. At the end of this paper, the pricing formula by standard t-process is compared with B-S's formula by numerical simulation.

Keywords: European option; standard t-process; equivalent martingale measure

1 Introduction

The classic B-S formula assumes that the price process of risky assets follows a geometric Brownian motion, but this assumption does not match the current reality of the market. To solve this problem, many scholars have improved the B-S option pricing model. Cox [1] and Merton [2] proposed the jump-diffusion model. Hull and White [3] and Heston [4] proposed a random volatility model. Hubalek et al. [5] considered the radial random volatility model with jump. Chan [6] constructed a geometric Lévy process. Schweizer [7] presented a semimartingale model. Borland [8] proposed a non-Gaussian price model driven by Tsallis distribution. Liu and Cui [9] and Wang and Zhang [10] both studied this model. In thie paper,we establish a new process which be composed of standard t-process.

The structure of this paper is as follows. In Section 2, we briefly introduce the market model based on standard t-process and obtian the solution to the stock price process under the equivalent martingale measure. In Section 3, we use price model to deal with the pricing problem of European option based on standard t-process.

2 Price Model Based on Standard t-Process

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete probability space with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the standard assumptions. Given a standard Brownian motion $W(t)$ and a chi-squared random variable Y with degrees of freedom ν, independent of $W(t)$.

Definition 2.1. *A stochastic process $M(t)$ is called a standard $t-$process if it satisfies,*

$$M(t) = \frac{W(t)}{\sqrt{Y/\nu}}. \tag{1}$$

Remark 2.1. The standard $t-$process $M(t)$ in the above definition is a martingale with respect to $\mathcal{F}_t = \sigma\{W(s), Y; s \leq t\}$. In fact, for any $s \leq t$,

$$E\left[M(t)\middle|\mathcal{F}_s\right] = E\left[\frac{W(t)}{\sqrt{Y/\nu}}\middle|\mathcal{F}_s\right] = \frac{W(s)}{\sqrt{Y/\nu}} = M(s).$$

Therefore, the quadratic variation process of $M(t)$ is given by

$$[M]_t = \frac{t\nu}{Y}.$$

Assume there are only bond and stock in the market. The price $S_0(t)$ of the bond with a risk-free return r satisfies.

$$\begin{cases} dS_0(t) = rS_0(t)dt, \\ S_0(0) = 1. \end{cases}$$

The stock price $S(t)(0 \leq t \leq T)$ follows

$$dS(t) = \mu S(t)dt + \sigma S(t)dM(t), \tag{2}$$

where μ and σ are constant, and $M(t)$ is a standard t-process.

Definition 2.2. *Under the equivalent martingale measure, the solution to the stock price process is*

$$S(t) = S(0)\exp[(r - \frac{1}{2}\sigma^2\frac{\nu}{Y})t + \sigma\widetilde{M}(t)], \tag{3}$$

where $\theta = \frac{\mu - r}{\sigma}$, *and* $\widetilde{M}(t) = M(t) + \theta t$ *is a standard* t*-process.*

Proof. Assuming there is no arbitrage opportunity in the market, there is an equivalent martingale measure, such that the process of discounted stock price under the equivalent martingale measure is a martingale. Then the equivalent martingale measure is defined by

$$\frac{dQ}{dP}|\mathcal{F}_t = \exp\{\theta M(t) - \frac{1}{2}\theta^2\frac{t\nu}{Y}\}. \tag{4}$$

We will prove that the discounted stock price process

$$S^*(t) = \frac{S(t)}{S_0(t)} = e^{-rt}S(t)$$

is a martingale under the probability measure Q. In fact, apply Itô's formula to get

$$\begin{aligned} dS^*(t) =& e^{-rt}dS(t) - re^{-rt}S(t)dt \\ =& e^{-rt}S(t)[(\mu - r)dt + \sigma dM(t)] \\ =& S^*(t)\sigma[dM(t) + \frac{\mu - r}{\sigma}dt] \\ =& S^*(t)d\widetilde{M}(t). \end{aligned}$$

Under the equivalent martingale measure Q, the stock price process is

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{M}(t).$$

Apply Itô's formula to get

$$d\ln S(t) = (r - \frac{1}{2}\sigma^2\frac{\nu}{Y})dt + \sigma d\widetilde{M}(t). \tag{5}$$

Integrate both sides of the equation (2.4) to get

$$S(t) = S(0)\exp[(r - \frac{1}{2}\sigma^2\frac{\nu}{Y})t + \sigma\widetilde{M}(t)].$$

3 European Option Pricing Formula

Consider there is a European call option with a strike price of K and a maturity date of T. The payoff on the European call option is

$$C(T) = \max[S(T) - K, 0].$$

Theorem 3.1. Given a European call option with the maturity date T and the strike price K, the call option price is given by

$$C(0) = S(0)M(d_1) - Ke^{-rT}M(d_2), \tag{6}$$

where

$$M(d_1) = \int_0^{+\infty} \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}} \Phi\left(\frac{\ln\frac{S(0)}{K} + (r + \frac{1}{2}\frac{\sigma^2\nu}{y})T}{\sigma\sqrt{T}\sqrt{\frac{\nu}{y}}}\right) dy,$$

$$M(d_2) = \int_0^{+\infty} \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}} \Phi\left(\frac{\ln\frac{S(0)}{K} + (r - \frac{1}{2}\frac{\sigma^2\nu}{y})T}{\sigma\sqrt{T}\sqrt{\frac{\nu}{y}}}\right) dy,$$

and $\Phi(x)$ is the standard normal cumulative distribution function.
Proof. Under equivalent martingale measure Q, the stock price process is

$$\begin{aligned} S(T) &= S(0)\exp\left[(r - \frac{1}{2}\sigma^2\frac{\nu}{Y})T + \sigma\widetilde{M}(T)\right] \\ &= S(0)\exp\left[(r - \frac{1}{2}\sigma^2\frac{\nu}{Y})T + \sigma\frac{\widetilde{W}(T)}{\sqrt{Y/\nu}}\right] \\ &= S(0)\exp\left[(r - \frac{1}{2}\sigma^2\frac{\nu}{Y})T + \sigma\frac{X\sqrt{T}}{\sqrt{Y/\nu}}\right] \end{aligned} \tag{7}$$

where X is a the standard normal random variable, independent of Y.

The price of call option is

$$C(0) = E_Q[e^{-rT}C(T)] = e^{-rT}E_Q\left[(S(T) - K)\, I_{S(T)>K}\right]. \tag{8}$$

From(3.2) and $S(T) > K$, we get

$$X > \left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{Y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{Y}}$$

Thus by the distribution of (X, Y), we obtain that

$$
\begin{aligned}
C(0) &= e^{-rT} \int_0^{+\infty} \int_{-\infty}^{+\infty} (S(0) e^{rT - \frac{1}{2}\frac{\sigma^2 \nu T}{y} + \sigma\sqrt{T}\frac{x}{\sqrt{y/\nu}}} - K)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} dxdy \\
&= S(0) \int_0^{+\infty} \frac{y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} dy \int_{\left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}}^{+\infty} e^{-\frac{1}{2}\frac{\sigma^2 \nu T}{y} + \sigma\sqrt{T}\frac{x}{\sqrt{y/\nu}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&\quad - K e^{-rT} \int_0^{+\infty} \frac{y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} dy \int_{\left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx
\end{aligned} \tag{9}
$$

It can be calculated directly that

$$
\begin{aligned}
&\int_{\left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}}^{+\infty} e^{-\frac{1}{2}\frac{\sigma^2 \nu T}{y} + \sigma\sqrt{T}\frac{x}{\sqrt{y/\nu}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{\left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sigma\sqrt{\frac{T\nu}{y}})^2}{2}} dx \\
&= 1 - \Phi\left(\left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} - \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}\right) \\
&= \Phi\left(\left(\ln\frac{S(0)}{K} + rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}\right) \\
&= \Phi\left(\frac{\ln\frac{S(0)}{K} + (r + \frac{1}{2}\frac{\sigma^2\nu}{y})T}{\sigma\sqrt{T}\sqrt{\frac{\nu}{y}}}\right)
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&\int_{\left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= 1 - \Phi\left(\left(\ln\frac{K}{S(0)} - rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} + \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}\right) \\
&= \Phi\left(\left(\ln\frac{S(0)}{K} + rT\right)\frac{1}{\sigma}\sqrt{\frac{y}{T\nu}} - \frac{1}{2}\sigma\sqrt{\frac{T\nu}{y}}\right) \\
&= \Phi\left(\frac{\ln\frac{S(0)}{K} + (r - \frac{1}{2}\frac{\sigma^2\nu}{y})T}{\sigma\sqrt{T}\sqrt{\frac{\nu}{y}}}\right)
\end{aligned} \tag{11}
$$

From Equations (3.3), (3.4), and (3.5), we see that

$$\begin{aligned} C(0) &= S(0)\int_0^{+\infty} \frac{y^{\frac{\nu}{2}-1}e^{-\frac{y}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\Phi\left(\frac{\ln\frac{S(0)}{K}+(r+\frac{1}{2}\frac{\sigma^2\nu}{y})T}{\sigma\sqrt{T}\sqrt{\frac{\nu}{y}}}\right)dy \\ &- Ke^{-rT}\int_0^{+\infty} \frac{y^{\frac{\nu}{2}-1}e^{-\frac{y}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\Phi\left(\frac{\ln\frac{S(0)}{K}+(r-\frac{1}{2}\frac{\sigma^2\nu}{y})T}{\sigma\sqrt{T}\sqrt{\frac{\nu}{y}}}\right)dy \\ &= S(0)M(d_1) - Ke^{-rT}M(d_2). \end{aligned}$$

This concludes the proof.

4 Simulation Study

The price process for B-S is

$$C(0) = S(0)N(d_1) - Ke^{-rT}N(d_2), \tag{12}$$

where $N(z)$ is standard normal function, and

$$d_1 = \left(\frac{\ln\frac{S(0)}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), d_2 = \left(\frac{\ln\frac{S(0)}{K}+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

Letting $S(0) = 50$, $r = 0.04$, $\sigma = 0.2$, $T = 0.8$ and $v = 5$, we use the price Formulas (3.1) and (4.1) to calculate the option prices.

Table 1 is the European call option prices. As can be seen, with the increase of K, the price of European call option gradually decreases, and the t-process option price is higher than the $B-S$.

Table1 Comparison of numerical results of European options

K	t	$B-S$	K	t	$B-S$
$K = 45$	8.0474	7.4766	$K = 51$	4.5155	3.8532
$K = 46$	7.3617	6.7685	$K = 52$	4.0671	3.3981
$K = 47$	6.7130	6.1005	$K = 53$	3.6579	2.9843
$K = 48$	6.1031	5.4743	$K = 54$	3.2863	2.6100
$K = 49$	5.5332	4.8909	$K = 55$	2.9502	2.2734
$K = 50$	5.0040	4.3503	$K = 56$	2.6476	1.9724

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