Snapback Repellers and Homoclinic Orbits
in a Logistic Map System

Anita Mareno

Penn State Harrisburg, USA

This article is distributed under the Creative Commons by-nc-nd Attribution License.

Copyright © 2024 Hikari Ltd.

Abstract

This paper concerns a two-parameter system of coupled logistic maps. We determine the existence of snapback repellers in this work with respect to an asymmetric and a (nontrivial) symmetric fixed point of the logistic map system. We analyze the stability of the fixed points and employ a perturbation/iterative scheme to construct homoclinic orbits associated with them.

Keywords: discrete dynamical systems, Maratto’s chaos, Snapback repeller, logistic map system, homoclinic orbits

1 Introduction

Chaos in discrete dynamical systems is a phenomenon whose occurrence is linked to continuous changes in a parameter associated with such systems. It is well-known, for instance, that the appearance of a three cycle or a period-doubling bifurcation are routes to chaos. In recent years, several works regarding discrete dynamical systems, specifically logistic map systems, [3],[8],[7],[12],[1], have highlighted how the existence of snapback repellers also leads to chaos.

Marotto, in [9], is the first to coin the phrase ‘snapback repeller’ and proved that their existence leads to chaos. He established sufficient conditions for chaos to occur in an arbitrary n-dimensional (usually) non-linear
system, say, \( x_k = F^k(x_0) \), where for each \( k = 0, 1, ..., n, x_0 \in \mathbb{R}^n \), and where \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous. He explains that snapback repellers exist in dependence upon homoclinic orbits, orbits that are arbitrarily close to an unstable fixed point of a map that are repelled from this point, as \( k \) increases, but *snap back* to meet this point. Homoclinic orbits of a fixed point, orbits that essentially approaching the fixed point in the forward direction and in some backward direction, have been well studied in [5],[6],[4],[2].

Marotto’s original definition of a snapback repeller implied that a repelling point is an expanding fixed point under the usual Euclidean norm which has been shown to be false by way of counterexamples (see for instance [10]). However, the claim is true in a more general sense if the Euclidean norm is replaced by an appropriate vector norm whose existence is guaranteed for eigenvalues whose magnitude exceeds 1. (see [9]). A modified version of his original definition is in [10] which we state below.

In this work we consider a two-parameter system of logistic maps. We numerically show that snapback repellers exist for the nontrivial symmetric and antisymmetric fixed points associated with this system and analyze certain chaotic regimes wherein lie snapback repellers, using the method of critical curves, establish by Mira [11]. This method allows one to determine the boundary of the region, say, of a chaotic attractor, and more generally the boundary of the basin of attraction and any islands of attraction within that region, by iterating ‘critical curves’, the initial set of which, are determined by the gradient of the transformation that represents the discrete system. One can further use this method to confirm the existence of snapback repellers.

### 2 Snapback Repeller Definition

Here we state a modified version of Marotto’s definition of a snapback repeller which appears in [10]:

Suppose \( z \) is a fixed point of \( F \) with all eigenvalues of \( DF(z) \) exceeding 1 in magnitude and suppose there exists a point \( x_0 \neq z \) in a repelling neighborhood of \( z \) such that \( x_M = z \) and \( \det(DF(x_k)) \neq 0 \) for \( 1 \leq k \leq M \) where \( x_k = F^k(x_0) \). Then \( z \) is called a snapback repeller of \( F \).

We note that this definition implies that \( x_k^M \rightarrow \infty \), where \( x_{k+1} = F(x_k) \) for
all $k < M$ satisfies $x_M = z$ and $x_k \to z$ as $k \to -\infty$, is a (non-degenerate) homoclinic orbit. The invertibility of $F$ in a nbd of each $x_k$ for all $k \leq M$ follows as well as $\det(DF(x_k)) \neq 0$ for all $k \in [1, M]$ and all $x_k$ for $k \leq 0$ lie within the local unstable manifold of $F$ at $z$ where $F$ is one-to-one. From this modified definition Marotto proves that if $F$ has a snapback repeller than $F$ is chaotic.

3 The existence of snapback repellers with respect to the asymmetric fixed point

We begin our analysis of the system by solving the equations

\begin{align*}
(1 - \epsilon)rx(1 - x) + \epsilon ry(1 - y) &= x \quad (1) \\
\epsilon rx(1 - x) + (1 - \epsilon)ry(1 - y) &= y \quad (2)
\end{align*}

and obtaining the fixed points of our system:

\begin{align*}
(0, 0), \left(\frac{r}{r - 1}, \frac{r}{r - 1}\right), (x^*, y^*), (y^*, x^*)
\end{align*}

where

\begin{align*}
x^* &= \frac{r(2\epsilon - 1) + 1 - \sqrt{(r(1 - 2\epsilon) - 1)(r(1 - 2\epsilon) + 4\epsilon - 1)}}{2r(2\epsilon - 1)} \quad (4) \\
y^* &= \frac{r(2\epsilon - 1) + 1 + \sqrt{(r(1 - 2\epsilon) - 1)(r(1 - 2\epsilon) + 4\epsilon - 1)}}{2r(2\epsilon - 1)} \quad (5)
\end{align*}

We note that $x^*, y^*$ are real valued if and only if $\Delta = (1 - 4\epsilon)(r - 1)^2 + 4\epsilon^2 r(r - 2) \geq 0$. This occurs when

\begin{align*}
r \in (3, 4) \text{ and } \epsilon \in \left[0, \frac{r - 1}{2r}\right] \text{ or } \epsilon \in \left[\frac{r - 1}{2(r - 2)}, 1\right].
\end{align*}

In addition $x^* = y^*$ if and only if $\Delta = 0$ which occurs when $\epsilon = \frac{r - 1}{2r}$ or $\epsilon = \frac{r - 1}{2(r - 2)}$, and so for these values of $\epsilon$ the fixed point $(x^*, y^*)$ coincides with one of the two symmetric fixed points: $(0, 0)$ or $(\frac{r - 1}{r}, \frac{r - 1}{r})$, respectively.
We now apply Marotto’s definition by using the Jacobian matrix to determine the eigenvalues associated with this system:

\[
DT(x, y) = \begin{pmatrix}
(1 - \epsilon)r(1 - 2x) & \epsilon r(1 - 2y) \\
\epsilon r(1 - 2x) & (1 - \epsilon)(1 - 2y)
\end{pmatrix}
\]  

(7)

Here the characteristic equation is

\[
\lambda^2 + C\lambda + E = 0,
\]  

(8)

where

\[
C = 2(\epsilon - 1)r(1 - y - x)
\]  

(9)

and

\[
E = (1 - \epsilon)^2r^2(1 - 2x)(1 - 2y)
\]  

(10)

The existence of a snapback repeller requires that the discriminant, \(C^2 - 4E < 0\) and that the norm of the eigenvalues exceeds 1. A simple calculation shows that the latter condition requires that \(E > 1\). The region of chaos of interest with respect to the asymmetric fixed point \((x^*, y^*)\), consistent with the two constraints above as well as the requirement that \(\Delta > 0\) is

\[
H = \{(r, \epsilon); r \in (1 + \sqrt{6}, 4), \epsilon \in (k(r), 1)\}
\]  

(11)

where

\[
k(r) = \frac{1}{4} \left[ 3 - 4r + 2r^2 \frac{\sqrt{9 - 16r + 8r^2}}{r(r - 2)} + \frac{\sqrt{9 - 16r + 8r^2}}{r^2(r - 2)^2} \right]
\]  

(12)

It is in this parameter space that we seek snap-back repellers. We now demonstrate how to find a neighborhood of \((x^*, y^*)\) containing a preimage point \((x_{-M}, y_{-M}) \neq (x^*, y^*)\). We consider a point near the fixed point \((x^*, y^*)\) by replacing \(x\) and \(y\) in equations (9) and (10) by \(x^* + \delta_1\) and \(y^* + \delta_2\) respectively. This yields the characteristic equation

\[
\lambda^2 + \tilde{C}\lambda + \tilde{E}
\]  

(13)

where

\[
\tilde{C} = C + 2(1 - \epsilon)r(\delta_1 + \delta_2)
\]  

(14)

and

\[
\tilde{E} = E + (1 - \epsilon)^2r^2[4\delta_1\delta_2 + 4(\delta_1 y^* + \delta_2 x^*) - 2(\delta_1 + 2\delta_2)]
\]  

(15)
We require that the discriminant associated with the modified characteristic equation be negative, i.e. that
\[ \tilde{C}^2 - 4\tilde{E} = C^2 - 4E + 4(1 - \epsilon)^2r^2(\delta_1 + \delta_2)^2 + 4(1 - \epsilon)rA(\delta_1 + \delta_2) - 16\delta_1\delta_2 < 0 \] (16)
\[ + 4(1 - \epsilon)rA(\delta_1 + \delta_2) - 16\delta_1\delta_2 - 16(\delta_1y^* + \delta_2x^*) < 0 \] (17)
We also require that \( \tilde{E} > 1 \), i.e. that,
\[ E + (1 - \epsilon)^2r^2[4\delta_1\delta_2 + 4(\delta_1y^* + \delta_2x^*) - 2(\delta_1 + \delta_2)] \] (18)
It can be shown numerically that \( C^2 - 4E < -0.25 \) for \((r, \epsilon) \in H\). Applying the upper and lower bounds of the two parameters, we see that
\[ \tilde{C}^2 - 4\tilde{E} = C^2 - 4E + 4(1 - \epsilon)^2r^2(\delta_1 + \delta_2)^2 + 4(1 - \epsilon)rA(\delta_1 + \delta_2) - 16\delta_1\delta_2 \]
\[ < -0.25 + 64(1 - \epsilon)^2(\delta_1 + \delta_2)^2 + 0.2(1 - \epsilon)(\delta_1 + \delta_2) - 16(\delta_1\delta_2) - 14\delta_1 - 9\delta_2 \]
\[ < -0.25 + 1.3(\delta_1 + \delta_2)^2 - 14\delta_1 - 9\delta_2 - 16\delta_1\delta_2 \] (21)
By choosing \( \delta_1 \) and \( \delta_2 \) say both less than 0.01, the eigenvalues are guaranteed to be imaginary.

Numerically, restricting ourselves to \( E \in (1.1, 1.2) \) for \( \epsilon \in (k(4), k(1+\sqrt{6})) \), \( r \in (1 + \sqrt{6}, 4) \) yields \( x^* \in (0.35, 0.6), y^* \in (0.8, 0.9) \) and so
\[ E + (1 - \epsilon)^2r^2[4\delta_1\delta_2 + 4(\delta_1y^* + \delta_2x^*) - 2(\delta_1 + \delta_2)] > 1.1 + (1 - \epsilon)^2r^2[4\delta_1\delta_2 + 3.2\delta_1 + 14\delta_2 - 2(\delta_1 + \delta_2)] \] (23)
which obviously holds for the choices of \( \delta_1, \delta_2 \) above.

Using the restrictions stated above we select \( \epsilon = 0.93, r = 3.85 \). For these parameter values, \( x^* = 0.409969, y^* = 0.892054 \). We show that the fixed point \((0.409969, 0.892054)\) is associated with a non-degenerate homoclinic orbit by constructing the chaotic area around this fixed point, using the method of critical curves. We begin by computing the determinant of the Jacobian matrix DT:
\[ [(1 - \epsilon)^2 - \epsilon^2]r^2(1 - 2x)(1 - 2y) \] (25)
We note that the determinant vanishes when \( x = \frac{1}{2} \) or \( y = \frac{1}{2} \). The union of these two curves constitute the critical line
\[ LC_{-1} = \left\{ (x, y) : y = \frac{1}{2} \right\} \cup \left\{ (x, y) : x = \frac{1}{2} \right\} := LC_{-1}^a \cup LC_{-1}^b \]
Anita Mareno

Figure 1: The graphs of $LC$ and $LC_{-1}$ and the $Z_0 - Z_4$ zones.

where we define

$$LC_{-1}^b = \{(x, y) : x = \frac{1}{2}\}$$

and

$$LC_{-1}^a = \{(x, y) : y = \frac{1}{2}\}.$$

The image of $LC_{-1}$ under $T$ is called the the critical line $LC$, where $LC$ has two component parts:

$$LC_a := \{(x, y) : y = \frac{r(2\epsilon - 1) + 4x(1 - \epsilon)}{4\epsilon}\}$$

and

$$LC_b := \{(x, y) : y = \frac{r(2\epsilon - 1) + 4x\epsilon}{4(\epsilon - 1)}\}.$$

So,

$$LC = LC_a \cup LC_b.$$

The curves $LC_a$ and $LC_b$ separate the $Z_0$ and $Z_4$ regions. We note here that our logistic map system is invariant with respect to the line $y = x$. As a result the two curves $LC_a, LC_b$ share this symmetry as well.

To assert the existence of a homoclinic orbit that reaches the fixed point in the backwards direction, we find all the preimages of our mapping $T$:

$$T_{i-1}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{r(2\epsilon - 1)\pm\sqrt{(2\epsilon - 1)r[(r(2\epsilon - 1))] + 4x(1 - \epsilon) - 4\epsilon y}}{2r(2\epsilon - 1)} \\ \frac{r(2\epsilon - 1)\pm\sqrt{(2\epsilon - 1)r[(r(2\epsilon - 1))] + 4y(1 - \epsilon) - 4\epsilon x}}{2r(2\epsilon - 1)} \end{pmatrix}$$

(26)

Here we relabel these four preimages according to the pairs of signs in between the terms:

$$T_1^{-1} = T_{-,-}, T_2^{-1} = T_{-,+}, T_3^{-1} = T_{+,+}, T_4^{-1} = T_{+,+}$$

(27)
Iterate | x-coordinate | y-coordinate
---|---|---
x_{-1} | 0.590031 | 0.892054
x_{-2} | 0.390856 | 0.821139
x_{-3} | 0.333807 | 0.896958
x_{-4} | 0.428461 | 0.918572
x_{-35} | 0.409966 | 0.892052

Table 1: Partial sequence of iterates converging to $x_0 = (x^*, y^*)$

Now, for points on $LC_a$, $T_1^{-1}$ and $T_3^{-1}$ coincide and $T_2^{-1}$ and $T_4^{-1}$ coincide. Similarly on $LC_b$, we have two pairs of coinciding preimages, and hence only two distinct preimages, $T_1^{-1} = T_3^{-1}$ and $T_2^{-1} = T_4^{-1}$.

In this table we generate a sequence of preimages of the fixed point. We note that $x_{-1}$ was obtained by applying the mapping $T_{-1}^{-1}$. Thereafter we applied the mapping $T_{+1}^{-1}$. The remaining inverse maps applied to our fixed point converge to either the trivial fixed point or the nontrivial symmetric fixed point. We note that the critical curves generate the boundary of the chaotic region to which our repelling fixed point $x_0 = (x^*, y^*)$ belongs.

In figure 2, we re-label the fixed point $x_0 = (x^*, y^*)$ as $B$ and its preimage $x_{-1}$ as $B_{-1}$. In 2(a) we see that the preimage point lies outside the chaotic attractor, resulting from a saddle-node bifurcation. Subsequently no sequence of additional preimages can converge to $B$ and so there is no homoclinic orbit associated with the fixed point. The phase portrait in (b), shows a chaotic attractor resulting from a crisis occurring at $\epsilon = 0.9249$. Both the fixed point $B$ and its preimage $B_{-1}$ lie in this region. The rank-1 preimage of the fixed point, $x_{-1}$, lies on the critical curve $LC_3$ for $\epsilon = 0.93$. This implies there exist a point $c \in LC_{-1}$ such that $T^4(c) = B_{-1}$ and so $T^5(c) = B$. Thus our fixed point is a snapback repeller. A critical homoclinic orbit of $B$, containing our iterates from above exists. We also note that in 2(d) $B$ intersects the critical curves $LC_k, k = 4, 5, 6$; this is a typical characteristic of a homoclinic orbit of a snapback repeller (see [6]).

We now consider the symmetric fixed $(\frac{r-1}{r}, \frac{r-1}{r})$. The eigenvalues associated
Figure 2: (a) An attracting set at $r = 3.85$, $\epsilon = 0.921$; (b) An attracting set at $r = 3.85$, $\epsilon = 0.93$. (c) Boundary of the attracting set in (b), along with the attracting set itself. (d) Boundary of the attracting set in (b), with additional critical curves, along with the attracting set itself.
Iterate | x-coordinate | y-coordinate |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{-1}$</td>
<td>0.746193</td>
<td>0.253807</td>
</tr>
<tr>
<td>$x_{-2}$</td>
<td>0.948188</td>
<td>0.712885</td>
</tr>
<tr>
<td>$x_{-3}$</td>
<td>0.776355</td>
<td>0.54512</td>
</tr>
<tr>
<td>$x_{-4}$</td>
<td>0.844711</td>
<td>0.71395</td>
</tr>
<tr>
<td>$x_{-25}$</td>
<td>0.746199</td>
<td>0.746187</td>
</tr>
</tbody>
</table>

Table 2: Partial sequence of iterates converging to $(x^*, x^*)$

with this fixed point,

$$\lambda = 2 - r, \quad \lambda = (2\epsilon - 1)(r - 2)$$

are both real. For, say, values of $r \in (3, 4)$, the eigenvalues becomes unstable for

$$\epsilon \in \left(\frac{r - 1}{2(r - 2)}, 1\right] \cup \left[0, \frac{r - 3}{2(r - 2)}\right).$$

Here $C = 2(\epsilon - 1)(2 - r), E = (1 - 2\epsilon)(2 - r)^2$ (recall the eigenvalues are both real so $C^2 - 4E > 0$). Without repeating the analysis for the asymmetric fixed point, we note the norms of the eigenvalues exceed 1 for

$$\epsilon \in \left(\frac{r - 1}{2(r - 2)}, \frac{r - 3}{2(r - 2)}\right)$$

and $\epsilon \in (0.67, 0.75)$ Here we select $r = 3.94$ and $\epsilon = 0.9017$. These choices for our parameters yield the fixed point $(0.746193, 0.746193)$.

Here we note that $x_{-1}$ was obtained by applying the map $T_{-1}^{-1}$. Thereafter we applied the map $T_{+, +}^{-1}$ and so the fixed point $(0.746193, 0.746193)$ is a snapback repeller.

In figure 3(a), both the symmetric fixed point and its preimage lie outside the annular chaotic attractor so no homoclinic orbit to $A$ exists. In figure 3(b) we see the phase portrait of the system after a crisis has occurred. Here both $A$ and $A_{-1}$ are within the chaotic attractor. In figure 3(c) we note that $A_{-1}$ does not yet lie on of the three critical curves in the plot, $LC_1, LC$ or $LC_i$, for $i = 1, 2, 3$, so $A$ is not yet a snapback repeller. In figure 3(d), $A_{-1}$ lies on $LC_4$ and $A$ intersects $LC_5, LC_6$, indicating that a homoclinic orbit to $A$ exists and so $A$ is a snapback repeller.
Figure 3: (a) An attracting set at $r = 3.94$, $\epsilon = 0.889$; (b) Two attracting sets at $r = 3.94$, $\epsilon = 0.9017$. (c) Boundary of the attracting sets in (b), along with the attracting sets. (d) Boundary of the attracting sets in (b), with additional critical curves, along with the attracting sets.
4 Conclusion

In this work we have demonstrated the existence of snapback repellers in a coupled system of logistic maps. Specifically we have shown that non-degenerate homoclinic orbits exists with respect to asymmetric and non-trivial symmetric fixed points. Herein, we observed that the transition from a repelling fixed point to a snapback repeller coincides with the appearance of chaotic regimes. In future work, we will explore the existence of snapback repellers for n-cycles with $n \geq 2$, the associated homoclinic orbits and chaotic regions.

References


Received: May 5, 2024; Published: June 19, 2024