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On Dynamics in a Solow-Swan Type Model

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Abstract

This paper examines a Solow-Swan type model with delay. The local stability and Hopf bifurcation of the system are investigated from a theoretical perspective.

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1 Introduction

Dohtani et al. [1] reconsider the modified version of the Solow-Swan model [5, 6] introduced in [2]. Their consideration of expected permanent income is based on Friedman [3], and the EPCP income of the representative household is thus determined by the following distributed lag of y,

$$y_p(t) = \int_{-\infty}^{t} \beta e^{\beta(\tau - t)} y(\tau) d\tau, \tag{1}$$

where y is income per capita and y_p is EPCP income per capita. Their resulting model happens to be described by

$$\begin{aligned} \dot{k} &= f(k) - (\delta + n)k - c, \\ \dot{c} &= \beta \left[\alpha f(k) - c \right], \end{aligned} \tag{2}$$

where δ, n, α and β are positive parameters. Friedman [3] actually proposed an estimate of the permanent component at time t as

$$y_p(t) = \int_{-\infty}^{t} W(\tau - t) y(\tau) d\tau,$$

where $W(\tau - t) = \beta e^{\beta(\tau-t)} = \beta e^{-\beta(t-\tau)}$, namely he considered the gamma distribution function

$$W(u,\beta,l) = \beta^l \frac{u^{l-1}e^{-\beta u}}{(l-1)!},$$

with $l = 1$, i.e. the weak kernel case. In this paper, for simplicity, we assume $f(k) = k^m$, $m \in (0,1)$, and examine the cases where $W(\tau - t)$ is the Dirac delta function or the gamma distribution function with $l = 2$, i.e. the strong kernel case. Conditions required for the presence of recurring cycles around the model's equilibrium point and their stability are determined.

2 Case Dirac delta function

If $W(\tau - t)$ is the Dirac delta function, then $y_p(t) = y(t - \tau)$. Since the consumption decision is assumed to depend on the EPCP income, i.e. $c = \alpha y_p$, system (2) is changed into the following model

$$\dot{k} = Ak^m - (\delta + n)k - \alpha Ak^m(t - \tau). \tag{3}$$

Eq. (3) has a unique positive equilibrium k_* if $\alpha < 1$, where $(1-\alpha)Ak_*^{m-1} = \delta + n$. To investigate the stability of the delayed system, we first make a coordinate transformation such that a new system is centered at the origin, and then linearize the resultant system at the origin to derive its characteristic equation. Then, we obtain

$$\lambda - \frac{(\delta+n)[m-(1-\alpha)]}{1-\alpha} + \frac{(\delta+n)\alpha m}{1-\alpha}e^{-\lambda\tau} = 0. \tag{4}$$

When there is no delay, i.e. $\tau = 0$ in (3), the characteristic equation (4) becomes $\lambda = -(\delta+n)(1-\alpha) < 0$. Thus, k_* is locally asymptotically stable. Suppose $\tau > 0$ in (3). Let $\lambda = i\omega$, $\omega > 0$, be a root of the characteristic equation (4), and rewrite (4) in terms of its real and imaginary parts as

$$\omega = \frac{(\delta+n)\alpha m}{1-\alpha}\sin\omega\tau, \qquad \frac{(\delta+n)[m-(1-\alpha)]}{1-\alpha} = \frac{(\delta+n)\alpha m}{1-\alpha}\cos\omega\tau.$$

It follows by taking the sum of squares that

$$\omega^2 = \frac{(\delta+n)^2}{1-\alpha}\left[-(1+\alpha)m^2 + 2m - (1-\alpha)\right].$$

It is immediate to see that if the condition $-(1+\alpha)m^2 + 2m - (1-\alpha) > 0$ holds, i.e. $(1-\alpha)/(1+\alpha) < m$, then there exists $\omega > 0$. As a result, Eq. (4) has a pair of purely imaginary roots $\lambda = \pm i\omega_0$ at the critical values τ_j, where

$$\omega_0 = (\delta+n)\sqrt{\frac{-(1+\alpha)m^2 + 2m - (1-\alpha)}{1-\alpha}}, \tag{5}$$

and

$$\tau_j = \frac{1}{\omega_0}\cos^{-1}\left\{\frac{m-(1-\alpha)}{\alpha m}\right\} + \frac{2j\pi}{\omega_0}, \quad j = 0, 1, 2, ... \tag{6}$$

Denote by $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ the root of Eq. (4) satisfying $\nu(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$. This step is to verify that the following transversality condition hold. To see this, differentiating (4) with respect to τ, we have

$$\left[1 - \frac{(\delta+n)\alpha m}{1-\alpha}\tau e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \frac{(\delta+n)\alpha m}{1-\alpha}\lambda e^{-\lambda\tau}, \tag{7}$$

which implies

$$sign\left\{\left.\frac{d\,(Re\lambda)}{d\tau}\right|_{\tau=\tau_j}\right\} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_j}\right\} = sign\left\{\left[\frac{1-\alpha}{(\delta+n)\alpha m}\right]^2\right\} > 0.$$

Consequently, the root of characteristic equation (4) near τ_j crosses the imaginary axis from the left to the right as τ continuously varies from a number less than τ_j to one greater than τ_j. It remains to show that $\lambda = i\omega_0$ is a simple purely imaginary root of the characteristic equation (4). If $\lambda = i\omega_0$ is a repeated root, then (7) yields that $i\omega_0 e^{-i\omega_0\tau}\tau_j = 0$, which is clearly a contradiction. Summing up all works above lead us to state the following results.

Theorem 2.1. *Let ω_0 and τ_0 be defined as in* (5) *and* (6)*, respectively. If $\alpha < 1$ and $1-\alpha < (1+\alpha)m$, then the positive equilibrium k_* of* (3) *is locally asymptotically stable for $\tau \in [0, \tau_0)$, and unstable for $\tau > \tau_0$. Furthermore,* (3) *undergoes a Hopf bifurcation at k_* when $\tau = \tau_0$.*

3 Case gamma distribution with strong kernel

Let $W(\tau - t)$ be the gamma distribution function with $l = 2$, i.e. the strong kernel case. Setting

$$x = \int_{-\infty}^{t} \beta e^{-\beta(t-\tau)} y(\tau) d\tau,$$

by the linear chain trick technique (MacDonald [4]), we have $\dot{y}_p = \beta(x - y_p)$ and $\dot{x} = \beta(y - x)$. From $c = \alpha y_p$, we deduce

$$\dot{c} = \alpha\beta(x - y_p) = \alpha\beta x - \beta c.$$

Therefore, the model to study is

$$\begin{cases} \dot{k} &= Ak^m - (\delta+n)k - c, \\ \dot{c} &= \alpha\beta x - \beta c, \\ \dot{x} &= \beta A k^m - \beta x. \end{cases} \tag{8}$$

Proceeding as done before we obtain the following characteristic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{9}$$

where

$$a_1 = -\frac{(\delta+n)[m-(1-\alpha)]}{1-\alpha} + 2\beta, \quad a_2 = \left\{-\frac{2(\delta+n)[m-(1-\alpha)]}{1-\alpha} + \beta\right\}\beta,$$

$$a_3 = (\delta+n)(1-m)\beta^2 > 0.$$

According to the Routh-Hurwitz criterion, the stability conditions are validated if $a_1 > 0, a_3 > 0$ and $a_1a_2 > a_3$.

Lemma 3.1. *$a_1 > 0$ if and only if one of the following conditions holds.*

1) $\alpha \leq 1-m$;

2) $\alpha > 1-m$ *and* $\beta > (\delta+n)[m-(1-\alpha)]/[2(1-\alpha)]$.

Proof. The statement follows from the definition of a_1. □

Lemma 3.2. *$a_1a_2 > a_3$ if and only if one of the following conditions holds.*

1) $\alpha < 8(1-m)/(8+m)$;

2) $\alpha = 8(1-m)/(8+m)$ *and* $\beta \neq (\delta+n)(1-m)/(9m)$;

3) $8(1-m)/(8+m) < \alpha < 1$, $\beta < \beta_1$ *and* $\beta > \beta_2$, *where*

$$\beta_1 = \frac{(\delta+n)\left\{\alpha m + 4\left[m-(1-\alpha)\right] - \sqrt{\alpha m + 8\left[m-(1-\alpha)\right]}\right\}}{4(1-\alpha)} \tag{10}$$

and

$$\beta_2 = \frac{(\delta+n)\left\{\alpha m + 4\left[m-(1-\alpha)\right] + \sqrt{\alpha m + 8\left[m-(1-\alpha)\right]}\right\}}{4(1-\alpha)}. \tag{11}$$

Proof. Expliciting the terms in the inequality $a_1a_2 > a_3$ yields the following second order inequality in the term β,

$$\begin{aligned} 2(1-\alpha)^2\beta^2 + (\delta+n)(1-\alpha)\left\{-(1-m)(1-\alpha) - 5[m-(1-\alpha)]\right\}\beta \\ + 2(\delta+n)^2[m-(1-\alpha)]^2 > 0. \end{aligned} \tag{12}$$

Let $-(1-m)(1-\alpha) - 5[m-(1-\alpha)] = -\alpha m - 4\left[m-(1-\alpha)\right] \geq 0$, i.e. $\alpha \leq 4(1-m)/(4+m)$. Then, (12) is always verified. Let $-(1-m)(1-\alpha) -$

$5[m-(1-\alpha)] = -\alpha m - 4\,[m-(1-\alpha)] < 0$, i.e. $4(1-m)/(4+m) < \alpha < 1$. Then, we calculate the discriminant Δ of the polynomial in (12), and find

$$\Delta = (\delta+n)^2(1-\alpha)^2\,(\alpha m + 8\alpha + 8m - 8)\,\alpha m.$$

If $\alpha m + 8\,[m-(1-\alpha)] < 0$, i.e. $4(1-m)/(4+m) < \alpha < 8(1-m)/(8+m)$, then $\Delta < 0$, and (12) holds true since the coefficient of β^2 is positive. If $\alpha m + 8\,[m-(1-\alpha)] = 0$, i.e. $\alpha = 8(1-m)/(8+m)$, then the polynomial in (12) becomes

$$2\left(1-\frac{9m}{8+m}\right)^2\left[\beta - \frac{(\delta+n)(1-m)}{9m}\right]^2.$$

If $\alpha m + 8\,[m-(1-\alpha)] > 0$, i.e. $8(1-m)/(8+m) < \alpha < 1$, then $\Delta > 0$, and (12) is solved by $\beta < \beta_1$ and $\beta > \beta_2$. □

We have the following results on the stability of system (8).

Proposition 3.3. *Let β_1, β_2 be defined as in* (10), (11)*, respectively. The equilibrium point of* (8) *is locally asymptotically stable if and only if*

1) $\alpha < 8(1-m)/(8+m)$;

2) $\alpha = 8(1-m)/(8+m)$ *and* $\beta \neq (\delta+n)(1-m)/(9m)$;

3) $8(1-m)/(8+m) < \alpha < 1$, $\beta < \beta_1$ *and* $\beta > \beta_2$.

Proof. The statement follows from the previous two lemmas, noticing that $8(1-m)/(8+m) < 1-m$ as well as, when $\alpha > 1-m$ and $\beta > (\delta+n)[m-(1-\alpha)]/[2(1-\alpha)]$, that one has $(\delta+n)[m-(1-\alpha)]/[2(1-\alpha)] < \beta_1$. Notice that this last inequality is equivalent to

$$3(1-m)^2(1-\alpha)^2 + 27[m-(1-\alpha)]^2 + 14(1-m)(1-\alpha)[m-(1-\alpha)] > 0,$$

and this holds true since we are in the case $\alpha > 1-m$. □

When $8(1-m)/(8+m) < \alpha < 1$, one can establish the existence of a cyclic solution at $\beta = \beta_*$, with $\beta_* = \beta_1, \beta_2$ if the characteristic equation (9) has a pair of purely imaginary roots and the real parts of these roots change signs. It is clear that $a_1^* a_2^* = a_3^*$ if $\beta = \beta_*$, where a_j^* means a_j $(j = 1,2,3)$ evaluated at $\beta = \beta_*$. Consequently, Eq. (9) rewrites as $(\lambda + a_1^*)(\lambda^2 + a_2^*) = 0$. Hence, there exist a pair of purely imaginary roots $\lambda_{1,2} = \pm i\omega_*$, with $\omega_* = \sqrt{a_2^*}$, and a real root $\lambda_3 = -a_1^*$. In order to show that the equilibrium point of system (8) may undergo a Hopf bifurcation for $\beta = \beta_*$ we need to prove that $\lambda_{1,2}$ are simple roots of (9) and satisfy the transversality condition. Differentiating Eq. (9) with respect to β, we obtain

$$(3\lambda^2 + 2a_1\lambda + a_2)\frac{d\lambda}{d\beta} = -\left(a_1'\lambda^2 + a_2'\lambda + a_3'\right), \tag{13}$$

where

$$a_1' = 2, \quad a_2' = -\frac{2(\delta+n)[m-(1-\alpha)]}{1-\alpha} + 2\beta_*, \quad a_3' = 2(\delta+n)(1-m)\beta_*.$$

To see that $\lambda = i\omega_*$ is a simple root of (9), we proceed assuming it is a repeated root. Then, (13) yields $-a_1'\omega_*^2 + ia_2'\omega_* + a_3' = 0$, and so the contradiction $\omega_* = 0$. Using $\omega_*^2 = a_2^*$, we can derive from (13) that

$$Re\left[\frac{d\lambda}{d\beta}\right]_{\lambda=i\omega_*} = -\frac{a_1^*\left(a_2'^*\right) + a_1'^* a_2^* - a_3'^*}{2\left(a_2^* + a_1^{*2}\right)}.$$

Then, we conclude that

$$sign\left\{\left.\frac{d\left(Re\lambda\right)}{d\beta}\right|_{\lambda=i\omega_*}\right\} = sign\left\{-a_1^*a_2'^* - a_1'^*a_2^* + a_3'^*\right\}. \tag{14}$$

If the sign in (14) is positive, then only crossing of the imaginary axis from left to right is possible as β increases. On the other hand, if the sign in (14) is negative, then only crossing from left to right with increasing β occurs. In order to investigate the sign in (14), we observe that

$$a_2'^* = \frac{a_2^*}{\beta_*} + \beta_*, \quad a_3'^* = \frac{2}{\beta_*}a_3^* = \frac{2}{\beta_*}a_1^*a_2^*.$$

In this way, a direct calculation allows us to shows that

$$sign\left\{-a_1^*a_2'^* - a_1'^*a_2^* + a_3'^*\right\} = sign\left\{\frac{(\delta+n)^2[m-(1-\alpha)]^2}{(1-\alpha)^2} - \beta_*^2\right\}. \tag{15}$$

Let us consider the following three cases: $m-(1-\alpha)=0$, $m-(1-\alpha)>0$ and $m-(1-\alpha)<0$. If $m-(1-\alpha)=0$, i.e. $\alpha = 1-m$, then

$$sign\left\{-a_1^*a_2'^* - a_1'^*a_2^* + a_3'^*\right\} = sign\left\{-\beta_*^2\right\} < 0.$$

If $m-(1-\alpha)>0$, i.e. $1-m<\alpha<1$, then

$$\begin{aligned} -a_1^*a_2'^* - a_1'^*a_2^* + a_3'^* \quad &> \quad 0 \Leftrightarrow \frac{(\delta+n)[m-(1-\alpha)]}{(1-\alpha)} > \beta_* \\ &\Leftrightarrow \quad \alpha m \pm \sqrt{\alpha m + 8\left[m-(1-\alpha)\right]} < 0. \end{aligned}$$

If $\beta_* = \beta_2$, then $sign\left\{-a_1^*a_2'^* - a_1'^*a_2^* + a_3'^*\right\} < 0$. If $\beta_* = \beta_1$, then $\alpha m - \sqrt{\alpha m + 8\left[m-(1-\alpha)\right]} < 0$ holds true. In fact, $\sqrt{\alpha m + 8\left[m-(1-\alpha)\right]} >$

$\sqrt{\alpha m} > \alpha m$ because $m - (1-\alpha)$ and $\alpha m < 1$. Finally, if $m - (1-\alpha) < 0$, i.e. $8(1-m)/(8+m) < \alpha < 1-m$, then

$$\begin{aligned} -a_1^* a_2'^* - a_1'^* a_2^* + a_3'^* &> 0 \Leftrightarrow -\frac{(\delta+n)^2[m-(1-\alpha)]^2}{(1-\alpha)^2} > \beta_* \\ &\Leftrightarrow \alpha m + 8\left[m-(1-\alpha)\right] \pm \sqrt{\alpha m + 8\left[m-(1-\alpha)\right]} < 0. \end{aligned}$$

If $\beta_* = \beta_2$, then $sign\left\{-a_1^* a_2'^* - a_1'^* a_2^* + a_3'^*\right\} < 0$ because $\alpha m + 8\left[m-(1-\alpha)\right] > 0$. If $\beta_* = \beta_1$, then $\alpha m + 8\left[m-(1-\alpha)\right] - \sqrt{\alpha m + 8\left[m-(1-\alpha)\right]} < 0$ when $\alpha m + 8\left[m-(1-\alpha)\right] < 1$, i.e. if $\alpha < (9-8m)/(8+m)$. Noticing that $1 - m < (9-8m)/(8+m)$, we can conclude that this inequality to holds true. Hence, we have $sign\left\{-a_1^* a_2'^* - a_1'^* a_2^* + a_3'^*\right\} > 0$. According to the Hopf bifurcation theorem, the previous analysis leads us to the following conclusions.

Theorem 3.4. *Let* $8(1-m)/(8+m) < \alpha < 1$.

1) *If* $m - (1-\alpha) \neq 0$, *i.e.* $\alpha \neq 1 - m$, *then system* (8) *undergoes a Hopf bifurcation at the equilibrium point when* $\beta_* = \beta_1$ *and* $\beta_* = \beta_2$.

2) *If* $m - (1-\alpha) = 0$, *i.e.* $\alpha = 1 - m$, *then system* (8) *undergoes a Hopf bifurcation at the equilibrium point when* $\beta_* = \beta_2$.

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