Applied Mathematical Sciences, Vol. 18, 2024, no. 7, 299 - 305
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2024.919146

Uncertain Demand and the Gathering of Information: A Discrete Dynamic Perspective

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Abstract

I study a dynamic production problem with uncertainty about demand in a monopolistic market regime. The monopolist's objective is to find a production plan that maximizes the expected profit by balancing the incentive to increase revenues and the cost of information gathering. A closed solution is given under the assumption of a uniform probability distribution of demand. The optimal production in each period is shown to be an adaptive learning process.

Keywords: discrete difference equation, adaptive learning process, infinite horizon

1 Introduction

In the presence of uncertainty, there are several ways for firms to gather information. Among them, the supply of goods is a source of continuous flows of information. This paper aims at analyzing the learning process of a monopolist that maximizes the flow of profits given an uncertain demand by assuming an infinite horizon. As Zeira (1987) [6] has pointed out, if a firm offers a marginal unit of product, it will be able to learn something about the production process. If so, the firm has an incentive to exploit this learning effect when maximizing intertemporal profits. Note that the learning effect is internalised if the firm operates in an imperfect market regime. In fact, in a perfect competition model [5], firms do not internalize the learning effect, and consequently, the equilibrium is Pareto suboptimal. Unlike [6], who assumes uncertainty regarding the production process, in this paper we assume that uncertainty is related to demand. By producing and offering goods in the market, some new demand-related information can be

discovered. However, there remains some similarity with [6] in that the information gathered over a period has a cost (if any) in case more is produced than the market demands. However, such information, even if it is costly, may be useful in subsequent periods. In other papers, for example [4] and the references cited therein, production decisions are expressed in terms of an excess capacity that can be interpreted as a form of insurance against uncertain demand. However, their objective is different since they are interested in studying dynamic pricing strategies in terms of temporal price discrimination. For an in-depth review of the literature on dynamic pricing with learning, see [2].
The role of information differs depending on the market regime in which the enterprise operates. In this paper, I study the effect of learning in the case of a monopolist. The analysis is completed in [1] by assuming a strategic environment where two monopolists share the same information set about demand uncertainty.

2 Assumptions

Consider a model in which a monopolist must decide how much to produce to satisfy an uncertain demand. The monopoly power of the firm does not derive from a technology with increasing returns to scale (in fact, we will assume that the production function is at constant returns) but from some other external reason, such as the case of a patent for the production of a certain good exclusively. The monopolist is also an information monopolist. He estimates an a priori probability of demand by means of a probability function. The information set can be updated by producing larger and larger quantities of goods and then attempting to sell them. The monopolist maximises inter-temporal profit, taking into account the benefit that the information collected in one period produces on the profit of the next period. In this sense, there is an incentive to produce more in order to know more.
First, I assume that the monopolist does not know the exact level of demand; he can obtain information by trying to offer a certain amount of goods on the market. If production remains below the (unknown) level of demand, the firm will be able to sell its entire output at a price $p = 1$. If production is higher, the monopolist will be able to sell the quantity demanded by consumers, which we denote by x, at a price equal to the marginal cost, $p = c$. The remaining quantity remains unsold, resulting in a cost that reduces net profits. However, it can be seen that it is convenient for the monopolist to offer a quantity slightly less than x, which is $x - dx$, to get a price $p = 1$.

Assumption 1: The monopolist sells the quantity y if $y < x$ at the price $p = 1$, and the quantity $x - dx$ if $y \geq x$ at the price $p = 1$.

From now on, I will refer to an offer equal to x, but implicitly, I refer to the quantity $(x - dx)$. Second, I assume that the probability density function of demand is a uniform distribution.

Assumption 2: The probability density function of the random variable x is defined on the interval [L, H], where $H > L > 0$. The distribution function is assumed to be uniform, e.g., $pr(x = z) = f(z) = 1/(H - L)$ and the cumulative probability that x is less than z is $pr(x \leq z) = F(z) = (z - L)/(H - L)$.

Third, the technology is characterized by constant returns to scale.

Assumption 3: The production function is $y = k$, where y is the output and k is the capital employed in production.

Fourth, production requires an operating cost c per unit of good actually sold, e.g., a distribution cost, and a unit cost g that is incurred before demand is known, so it is related to output and can be interpreted as a fixed cost relative to actual sales. If the quantity produced exceeds demand, the cost of unsold goods cannot be recovered. The production of large quantities is a way of obtaining useful information to investigate demand. However, this can prove very costly if production is excessive. In this sense, collecting information is costly.

Assumption 4: The cost function is $C(y) = (g + c)y$ if $y < x$ and $C(y) = gy + cx$ if $y \geq x$.

The optimal level of output is obtained by bringing the marginal expected profit to zero. Marginal profit is composed of two parts. The marginal profit that is obtained if a marginal unit of output does not allow the level of demand x to be discovered, and the marginal profit if x is discovered with the same additional unit. These two components must be corrected for the change in probability due to the output change. In other words, if the monopolist decides to produce more, it increases the probability of discovering x and, correspondingly, decreases the probability of not discovering x. As a first step, the static model is studied as a benchmark.

3 Static equilibrium

In the static model, the following optimal production (and input) is obtained:

Proposition 1: The optimal output (and capital input) in a static framework is

$$y^* = k^* = (1 - a)H + aL \tag{1}$$

where $a = g/(1 - c)$.

Proof: Given the probability distribution function for demand is defined over the compact interval [L, H], the maximization problem of the monopolist's expected profit is the following:

$$\max_{L\leq y\leq H} E\Pi = (1-c)\int_L^y zf(z)dz + (1-c)\int_y^H xzf(z)dz + gy$$

The first-order condition is $dE\Pi/dy = 0$ or

$$(1-c)\left[y\int_y^H zf(z)dz - yf(y) + yf(y)\right] - g = 0$$

The second and third terms into the square brackets are identical because of the monopolist's incentive to sell a quantity slightly less than x to maintain a price above marginal cost, $p = 1$. Therefore, it is found

$$(1-c)y\int_y^H f(z)dz - g = 0$$

that, given the uniform probability distribution function, is

$$(1-c)\frac{H-y}{H-L} - g = 0$$

So that the optimal production level y^* is

$$y^* = H - \tfrac{g}{1-c}(H-L).$$

By imposing $a \equiv \frac{g}{1-c}$, I can write it as

$$y^* = (1-a)H + aL$$

Given the production function $y = k$, it is also true that $y^* = k^*$.

4 A discrete dynamic model with an infinite horizon

Consider a monopolist's decision problem in an infinite-horizon framework. The monopolist's intertemporal profit function is

$$\Pi(x) = \sum_{t=0}^{\infty}\frac{1}{(1+r)^t}\{(1-c-g)min(y_t,x) - max(g(y_{t+1}-y_t),0)\}$$

Given that the demand level x is uncertain, the monopolist maximizes the expected profit that takes into account the probability that x is discovered with y_{t+1}

$$\mathrm{E}\Pi(x) = \sum_{t=0}^{\infty} pr(y_t < x \leq y_{t+1})E[\Pi(x)|y_t < x \leq y_{t+1}]$$

where

$$E[\Pi(x)|y_t < x \leq y_{t+1}] = \sum_{s=0}^{t}\frac{1}{(1+r)^s}\{(1-c-g)y_s - g(y_{s+1}-y_s)\} +$$
$$+E\left[\frac{(1-c-g)x}{(1+r)^{t+1}} + \frac{(1-c-g)x}{(1+r)^{t+2}} + \cdots \mid y_t < x \leq y_{t+1}\right]$$

Therefore, the *unconditional* expected profit to be maximized is

$$\max_{y_{t+1}} \mathrm{E}\Pi = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [(1-c)y_t - g y_{t+1}] \int_{y_t}^{H} z f(z) dz +$$

$$+ \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \frac{(1-c-g)}{r} \int_{y_t}^{y_{t+1}} z f(z) dz \tag{2}$$

As stated in Proposition 2, the optimal production plan that maximizes (2) is an adaptive learning scheme, as in [3].

Proposition 2: The optimal output (and capital input) in a discrete dynamic model with an infinite horizon is

$$y_t = (1-\lambda_1)H + \lambda_1 y_{t-1} \tag{3}$$

where $\lambda_1 = \frac{(a+1+r)-\sqrt{(a+1+r)^2-4a^2(1+r)}}{2a}$ is the eigenvalue between 0 and 1 associated with the difference equation

$$y_{t+2} - \frac{a+1+r}{a} y_{t+1} + (1+r) y_t + \frac{(1+r)(1-a)}{a} H = 0 \tag{4}$$

where $a \equiv \frac{g}{1-c}$.

Proof: The optimal production plan can be solved by the gradient of (2) along the control variable y_{t+1}. Hence, the optimal production plan satisfies

$$-\int_{y_t}^{H} f(z) dz + \frac{1}{1+r} f(y_{t+1})(y_{t+2} - y_{t+1}) + \frac{1-c}{g(1+r)} \int_{y_{t+1}}^{H} f(z) dz = 0$$

Since the probability distribution is uniform, e.g. $f(z) = 1/(H-L)$, the equation can be written as the second-order linear difference equation (4). The solution to this difference equation is

$$y_t = -\frac{(1+r)(1-a)}{a(1-\lambda_1)(1-\lambda_2)} H + c_1 \lambda_1^t + + c_2 \lambda_2^t$$

where λ_1 and λ_2 are the eigenvalues of the difference equation (4). Moreover, two conditions to obtain a solution are as follows: i) there is a positive initial value of production $y_0 > 0$; ii) the numerical sequence $\{y_t\}, t \geq 1$ is non-decreasing due to the monotonic property of the production adjustment over time. The two conditions imply that

$$\lim_{t\to\infty} y_t = H$$

$$\lim_{t\to\infty} \left[-\frac{(1+r)(1-a)}{a(1-\lambda_1)(1-\lambda_2)} H + c_1 \lambda_1^t + + c_2 \lambda_2^t \right] = H$$

$$-\frac{(1+r)(1-a)}{a(1-\lambda_1)(1-\lambda_2)}H = H$$
$$(1-\lambda_1)(1-\lambda_2) = -\frac{(1+r)(1-a)}{a}$$

The eigenvalues are

$$\lambda_1 = \frac{(a+1+r) - \sqrt{(a+1+r)^2 - 4a^2(1+r)}}{2a}$$

$$\lambda_2 = \frac{(a+1+r) + \sqrt{(a+1+r)^2 - 4a^2(1+r)}}{2a}$$

λ_1 is between 0 and 1 if $(1+r) < [(g+1-c)/(2g)]^2$, while $\lambda_2 > 1$ everywhere. The optimal adjustment program requires that the eigenvalue be between 0 and 1. Given the initial value of production y_0, the solution to the difference equation is an adaptive learning process, with a rate of adjustment λ_1, that is reported in (3).

Corollary 1: The monopolist produces more output over an infinite time horizon than it would produce in a static model.

Proof: the output at time t, y_t, is a weighted average of output at time $t-1$, y_{t-1}, and the maximum demand level H, with weights a and $(1-a)$ in the static model; λ_1 and $(1-\lambda_1)$ over an infinite time horizon. By comparing (1) and (3), the optimal output at time t of the infinite horizon model is greater than the corresponding optimal output of the static model, given that $\lambda_1 < a$.

5 Concluding remarks

In this paper, I describe in terms of a discrete dynamic model the optimal production level of a monopolist who is faced with the trade-off between producing to maximize revenues and demand learning. When the demand curve is unknown, the optimal production plan is an adaptive learning scheme. A closed solution is obtained when assuming a uniform probability distribution for demand. When the monopolist’s horizon is infinite, the exploration force is stronger than a static horizon since the monopolist takes into account the benefit that the information gathered in one period produces on future profits.

As a further development, the analysis of the problem is completed in [1], where I consider two independent monopolists. Information is a key tool through which the firm seeks to strategically interact with other firms.

Acknowledgement. This work is supported by the Department of Economics and Management, University of Ferrara, Italy.

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Received: June 5, 2024; Published: July 5, 2024