Nonlinear Parabolic Equations Involving Measure Data in Musielak-Orlicz-Sobolev Spaces

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Abstract

We prove the existence of solutions of nonlinear parabolic problems with measure data in Musielak-Orlicz-Sobolev spaces.

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1 Introduction

Let $\Omega$ a bounded open subset of $\mathbb{R}^n$ and let $Q$ be the cylinder $\Omega \times (0, T)$ with some given $T > 0$.
We consider the following nonlinear parabolic problem:

$$
\begin{align*}
\frac{\partial u}{\partial t} + A(u) &= \mu \text{ in } Q \\
u(x, t) &= 0 \text{ on } \partial\Omega \times (0, T) \\
u(x, 0) &= 0 \text{ in } \Omega
\end{align*}
$$

(1)
where $A = - \text{div} (a(x,t,u,\nabla u))$ is an operator of Leray-Lions defined on $D(A) \subset W^{1,x}_0 L^\varphi(\Omega)$, $\varphi$ is an appropriate Musielak-Orlicz function related to the growth of $a(x,t,u,\nabla u)$, and $\mu$ is a given Radon measure.

Solution to problem (1) has been provided firstly by Boccardo-Gallout, [19] (see also [18, 20]) in the setting of classical spaces $L^p(0,T;W^{1,p})$. Meskine, in [10] prove the existence of solution for problem (1) in the setting of inhomogeneous Orlicz-Sobolev space $W^{1,x}_0 L_B$ for any $B \in \mathcal{P}_M$, where $\mathcal{P}_M$ is a special class of N-functions and $M$ the N-function. Our purpose in this paper is to prove existence solutions for the problem (1) in the setting of inhomogeneous Musielak-Orlicz-Sobolev spaces $W^{1,x}_0 L^\varphi(Q)$ for any $\varphi \in \mathcal{P}_\varphi$, where $\mathcal{P}_\varphi$ is a special class of Musielak-Orlicz functions and $\varphi$ the Musielak-Orlicz function.

Let us point out that our result can be applied in the particular case when $\varphi(x,t) = t^p(x)$, in this case we use the notations $L^{p(x)}(\Omega) = L^\varphi(\Omega)$, and $W^{m,p(x)}(\Omega) = W^m L^\varphi(\Omega)$. These spaces are called Variable exponent $L^p$ and Sobolev spaces.

For some classical and recent results on elliptic and parabolic problems in Orlicz-Sobolev spaces and a Musielak-Orlicz-Sobolev spaces, we refer to [16, 15, 14, 22, 6, 7, 8, 9, 10, 17].

2 Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [11]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later.

Musielak-Orlicz-Sobolev spaces: Let $\Omega$ be an open subset of $\mathbb{R}^n$.
A Musielak-Orlicz function $\varphi$ is a real-valued function defined in $\Omega \times \mathbb{R}^+$ such that:

a) $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t \to 0} \sup_{x \in \Omega} \frac{\varphi(x,t)}{t} = 0$$

$$\lim_{t \to \infty} \inf_{x \in \Omega} \frac{\varphi(x,t)}{t} = 0.$$

b) $\varphi(.,t)$ is a Lebesgue measurable function

Now, let $\varphi_x(t) = \varphi(x,t)$ and let $\varphi_x^{-1}$ be the non-negative reciprocal function with respect to $t$, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\frac{1}{\varphi_x}) = t.$$

For any two Musielak-Orlicz functions $\varphi$ and $\gamma$ we introduce the following ordering:
c) if there exists two positives constants \( c \) and \( T \) such that for almost everywhere \( x \in \Omega \) :

\[
\varphi(x, t) \leq \gamma(x, ct) \text{ for } t \geq T
\]

we write \( \varphi \prec \gamma \) and we say that \( \gamma \) dominates \( \varphi \) globally if \( T = 0 \) and near infinity if \( T > 0 \).

d) if for every positive constant \( c \) and almost everywhere \( x \in \Omega \) we have

\[
\lim_{t \to 0} \left( \sup_{x \in \Omega} \frac{\varphi(x, ct)}{\gamma(x, t)} \right) = 0 \quad \text{or} \quad \lim_{t \to \infty} \left( \sup_{x \in \varphi} \frac{\varphi(x, ct)}{\gamma(x, t)} \right) = 0
\]

we write \( \varphi \ll \gamma \) at 0 or near \( \infty \) respectively, and we say that \( \varphi \) increases essentially more slowly than \( \gamma \) at 0 or near infinity respectively.

In the sequel the measurability of a function \( u : \Omega \mapsto R \) means the Lebesgue measurability.

We define the functional

\[
\varrho_{\varphi, \Omega}(u) = \int_{\Omega} \varphi(x, |u(x)|) dx
\]

where \( u : \Omega \mapsto \mathbb{R} \) is a measurable function.

The set

\[
K_{\varphi}(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable } | \varrho_{\varphi, \Omega}(u) < +\infty \}
\]

is called the Musielak-Orlicz class (the generalized Orlicz class).

The Musielak-Orlicz space (the generalized Orlicz spaces) \( L_{\varphi}(\Omega) \) is the vector space generated by \( K_{\varphi}(\Omega) \), that is, \( L_{\varphi}(\Omega) \) is the smallest linear space containing the set \( K_{\varphi}(\Omega) \).

Equivalently:

\[
L_{\varphi}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } | \varrho_{\varphi, \Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0 \right\}
\]

Let

\[
\psi(x, s) = \sup_{t \geq 0} \{ st - \varphi(x, t) \},
\]

\( \psi \) is the Musielak-Orlicz function complementary to ( or conjugate of ) \( \varphi(x, t) \) in the sense of Young with respect to the variable \( s \).

On the space \( L_{\varphi}(\Omega) \) we define the Luxemburg norm:

\[
|||u|||_{\varphi, \Omega} = \sup_{||v||_{\psi} \leq 1} \int_{\Omega} |u(x)v(x)| dx.
\]

and the so-called Orlicz norm:

\[
||u||_{\varphi, \Omega} = \inf_{\lambda > 0} \lambda \left\{ \int_{\Omega} \varphi(x, \frac{|u(x)|}{\lambda}) dx, \leq 1 \right\}.
\]
where \( \psi \) is the Musielak-Orlicz function complementary to \( \varphi \). These two norms are equivalent [11].

The closure in \( L_\psi(\Omega) \) of the set of bounded measurable functions with compact support in \( \Omega \) is denoted by \( E_\varphi(\Omega) \). It is a separable space and \( E_\varphi(\Omega)^* = L_\psi(\Omega) \) [11].

The following conditions are equivalent:

\[
e) \quad E_\varphi(\Omega) = K_\varphi(\Omega)
\]

\[
f) \quad K_\varphi(\Omega) = L_\varphi(\Omega)
\]

\[
g) \quad \varphi \text{ has the } \Delta_2 \text{ property.}
\]

We recall that \( \varphi \) has the \( \Delta_2 \) property if there exists \( k > 0 \) independent of \( x \in \Omega \) and a nonnegative function \( h \), integrable in \( \Omega \) such that \( \varphi(x, 2t) \leq k \varphi(x, t) + h(x) \) for large values of \( t \), or for all values of \( t \), according to whether \( \Omega \) has finite measure or not.

Let us define the modular convergence: we say that a sequence of functions \( u_n \in L_\varphi(\Omega) \) is modular convergent to \( u \in L_\varphi(\Omega) \) if there exists a constant \( k > 0 \) such that

\[
\lim_{n \to \infty} \varrho_{\varphi, \Omega}(u_n - u) = 0.
\]

For any fixed nonnegative integer \( m \) we define

\[
W^m L_\varphi(\Omega) = \{ u \in L_\varphi(\Omega) : \forall |\alpha| \leq m \quad D^\alpha u \in L_\varphi(\Omega) \}
\]

where \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \) with nonnegative integers \( \alpha_i \); \( |\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n| \) and \( D^\alpha u \) denote the distributional derivatives.

The space \( W^m L_\varphi(\Omega) \) is called the Musielak-Orlicz-Sobolev space.

Now, the functional

\[
\overline{\varrho}_{\varphi, \Omega}(u) = \sum_{|\alpha| \leq m} \varrho_{\varphi, \Omega}(D^\alpha u),
\]

for \( u \in W^m L_\varphi(\Omega) \) is a convex modular. and

\[
||u||^m_{\varphi, \Omega} = \inf \{ \lambda > 0 : \overline{\varrho}_{\varphi, \Omega}(\frac{u}{\lambda}) \leq 1 \}
\]

is a norm on \( W^m L_\varphi(\Omega) \).

The pair \( (W^m L_\varphi(\Omega), ||u||^m_{\varphi, \Omega}) \) is a Banach space if \( \varphi \) satisfies the following condition:

\[
\text{there exist a constant } c > 0 \text{ such that } \inf_{x \in \Omega} \varphi(x, 1) \geq c,
\]

as in [11].
The space \( W^m L_\varphi(\Omega) \) will always be identified to a \( \sigma(\Pi L_\varphi, \Pi E_\psi) \) closed subspace of the product \( \prod_{\|\alpha\| \leq m} L_\varphi(\Omega) = \prod L_\varphi. \)

Let \( W^m_0 \varphi(\Omega) \) be the \( \sigma(\Pi L_\varphi, \Pi E_\psi) \) closure of \( D(\Omega) \) in \( W^m \varphi(\Omega). \)

The following spaces of distributions will also be used:

\[
W^{-m} L_\psi(\Omega) = \{ f \in D'(\Omega); f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \} 
\]

\[
W^{-m} E_\psi(\Omega) = \{ f \in D'(\Omega); f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \} 
\]

As we did for \( L_\varphi(\Omega) \), we say that a sequence of functions \( u_n \in W^m L_\varphi(\Omega) \) is modular convergent to \( u \in W^m L_\varphi(\Omega) \) if there exists a constant \( k > 0 \) such that

\[
\lim_{n \to \infty} \varrho_{\varphi,\Omega}(u_n - u) = 0. 
\]

From \([\Pi]\), for two complementary Musielak-Orlicz functions \( \varphi \) and \( \psi \) the following inequalities hold:

\[ h) \] the young inequality:

\[ t.s \leq \varphi(x,t) + \psi(x,s) \text{ for } t, s \geq 0, \ x \in \Omega \]

\[ i) \] the Hölder inequality:

\[
\left| \int_{_\Omega} u(x)v(x) \ dx \right| \leq ||u||_{\varphi,\Omega}||v||_{\psi,\Omega}. 
\]

for all \( u \in L_\varphi(\Omega) \) and \( v \in L_\psi(\Omega). \)

**Inhomogeneous Musielak-Orlicz-Sobolev spaces:**

Let \( \Omega \) an bounded open subset of \( \mathbb{R}^n \) and let \( Q = \Omega \times ]0,T[ \) with some given \( T > 0. \) Let \( \varphi \) be a Musielak function. For each \( \alpha \in \mathbb{N}^n, \) denote by \( D^\alpha_x \) the distributional derivative on \( Q \) of order \( \alpha \) with respect to the variable \( x \in \mathbb{R}^n. \) The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

\[
W^{1,x} L_\varphi(Q) = \{ u \in L_\varphi(Q) : \forall |\alpha| \leq 1 \ D^\alpha_x u \in L_\varphi(Q) \} 
\]
and
\[ W^{1,x}E_\varphi(Q) = \{ u \in E_\varphi(Q) : \forall |\alpha| \leq 1 \ D_x^\alpha u \in E_\varphi(Q) \} \]

The last space is a subspace of the first one, and both are Banach spaces under the norm
\[ \| u \| = \sum_{|\alpha| \leq m} \| D_x^\alpha u \|_{\varphi,Q}. \]

We can easily show that they form a complementary system when \( \Omega \) is a Lipschitz domain [3]. These spaces are considered as subspaces of the product space \( \Pi L_\varphi(Q) \) which has \((N+1)\) copies. We shall also consider the weak topologies \( \sigma(\Pi L_\varphi, \Pi E_\psi) \) and \( \sigma(\Pi L_\varphi, \Pi L_\psi) \).

If \( u \in W^{1,x}L_\varphi(Q) \) then the function : \( t \mapsto u(t) = u(t,.) \) is defined on \((0,T)\) with values in \( W^1L_\varphi(\Omega) \). If, further, \( u \in W^{1,x}E_\varphi(Q) \) then this function is a \( W^1E_\varphi(\Omega) \)-valued and is strongly measurable. Furthermore the following imbedding holds: \( W^{1,x}E_\varphi(Q) \subset L^1(0,T; W^1E_\varphi(\Omega)) \). The space \( W^{1,x}L_\varphi(Q) \) is not in general separable, if \( u \in W^{1,x}L_\varphi(Q) \), we cannot conclude that the function \( u(t) \) is measurable on \((0,T)\). However, the scalar function \( t \mapsto \| u(t) \|_{\varphi,\Omega} \) is in \( L^1(0,T) \). The space \( W^{1,x}_0E_\varphi(Q) \) is defined as the (norm) closure in \( W^{1,x}E_\varphi(Q) \) of \( D(Q) \).

We can easily show as in [3] that when \( \Omega \) a Lipschitz domain then each element \( u \) of the closure of \( D(Q) \) with respect of the weak * topology \( \sigma(\Pi L_\varphi, \Pi E_\psi) \) is limit, in \( W^{1,x}L_\varphi(Q) \), of some subsequence \( (u_i) \subset D(Q) \) for the modular convergence; i.e., there exists \( \lambda > 0 \) such that for all \( |\alpha| \leq 1 \),

\[ \int_Q \varphi(x, (\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda})) dx dt \to 0 \text{ as } i \to \infty, \]

this implies that \( (u_i) \) converges to \( u \) in \( W^{1,x}L_\varphi(Q) \) for the weak topology \( \sigma(\Pi L_M, \Pi L_\psi) \).

Consequently
\[ D(Q)^\sigma(\Pi L_\varphi, \Pi E_\psi) = \overline{D(Q)^\sigma(\Pi L_\varphi, \Pi L_\psi)}, \]

this space will be denoted by \( W^{1,x}_0L_\psi(Q) \). Furthermore, \( W^{1,x}_0E_\varphi(Q) = W^{1,x}_0L_\varphi(Q) \cap \Pi E_\psi \).

We have the following complementary system
\[ \begin{pmatrix} W^{1,x}_0L_\varphi(Q) \\ W^{1,x}_0E_\varphi(Q) \end{pmatrix} \bigg/ F, \]

\( F \) being the dual space of \( W^{1,x}_0E_\varphi(Q) \). It is also, except for an isomorphism, the quotient of \( \Pi L_\psi \) by the polar set \( W^{1,x}_0E_\varphi(Q)^{\perp} \), and will be denoted by \( F = W^{-1,x}L_\psi(Q) \) and it is shown that
\[ W^{-1,x}L_\psi(Q) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \right\}. \]

This space will be equipped with the usual quotient norm
\[ \| f \| = \inf \sum_{|\alpha| \leq 1} \| f_\alpha \|_{\psi,Q}. \]
where the inf is taken on all possible decompositions

\[ f = \sum_{|\alpha| \leq 1} D^\alpha_x f_\alpha, \quad f_\alpha \in L_\psi(Q). \]

The space \( F_0 \) is then given by

\[ F_0 = \left\{ f = \sum_{|\alpha| \leq 1} D^\alpha_x f_\alpha : f_\alpha \in E_\psi(Q) \right\} \]

and is denoted by \( F_0 = W^{-1,a}E_\psi(Q) \).

In order to deal with the time derivative, we introduce a time mollification of a function \( u \in L_\phi(Q) \).

Thus we define, for all \( \mu > 0 \) and all \( (x, t) \in Q \)

\[ u_\mu(x, t) = \mu \int_{-\infty}^t \tilde{u}(x, s) \exp(\mu(s - t)) ds, \]

where \( \tilde{u}(x, s) = u(x, s)\chi_{(0,T)}(s) \) is the zero extension of \( u \).

**Proposition 1.** If \( u \in L_\phi(Q) \) then \( u_\mu \) is measurable in \( Q \) and \( \frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu) \) and if \( u \in L_\psi(Q) \) then

\[ \int_Q \varphi(x, u_\mu) dx dt \leq \int_Q \varphi(x, u) dx dt. \]

**Proof.** Since \( (x, t, s) \mapsto u(x, s)\exp(\mu(s - t)) \) is measurable in \( \Omega \times [0, T] \times [0, T] \), we deduce that \( u_\mu \) is measurable by Fubini’s theorem. By Jensen’s integral inequality we have, since \( \int_{-\infty}^0 \exp(\mu s) ds = 1 \),

\[ \varphi(x, \int_{-\infty}^t \mu \tilde{u}(x, s) \exp(\mu(s - t)) ds) = \varphi(x, \int_{-\infty}^0 \mu \exp(\mu s) \tilde{u}(x, s + t) ds) \leq \int_{-\infty}^0 \mu \exp(\mu s) \varphi(x, \tilde{u}(x, s + t)) ds \]

which implies

\[ \int_Q \varphi(x, u_\mu(x, t)) dx dt \leq \int_{\Omega \times \mathbb{R}} \left( \int_{-\infty}^0 \mu \exp(\mu s) \varphi(x, \tilde{u}(x, s + t)) ds \right) dx dt \]

\[ \leq \int_{-\infty}^0 \mu \exp(\mu s) \left( \int_{\Omega \times \mathbb{R}} \varphi(x, \tilde{u}(x, s + t)) dx dt \right) ds \]

\[ \leq \int_{-\infty}^0 \mu \exp(\mu s) \left( \int_Q \varphi(x, u(x, t)) dx dt \right) ds \]

\[ = \int_Q \varphi(x, u) dx dt. \]
Furthermore
\[ \frac{\partial u}{\partial t} = \lim_{\delta \to 0} \frac{1}{\delta} (\exp(-\mu \delta) - 1) u_\mu(x, t) + \lim_{\delta \to 0} \frac{1}{\delta} \int_t^{t+\delta} u(x, s) \exp(\mu(s-(t+\delta))) ds = -\mu u_\mu + \mu u. \]

**Proposition 2.** Assume that \((u_n)_n\) is a bounded sequence in \(W^{1,x}_0(Q)\) such that \(\frac{\partial u_n}{\partial t}\) is bounded in \(W^{-1,x}_0(Q) + L^1(Q)\), then \(u_n\) is relatively compact in \(L^1(Q)\).

**Proof.** It is easily by using Corollary 1 of [14].

### 3 The Main Result

Let \(\mathcal{P}_\varphi\) be a subset of Musielak-Orlicz functions defined by:

\[ \mathcal{P}_\varphi = \left\{ \phi : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ is a Musielak-Orlicz function, such that } \phi \ll \varphi \right\} \]

where \(H(x, r) = \varphi(x, r)/r\).

we assume that

\[ \mathcal{P}_\varphi \neq \emptyset \quad (2) \]

Let \(A : D(A) \subset W^{1,x}_0(Q) \rightarrow W^{-1,x}_0(Q)\) be a mapping given by \(A(u) = -\text{div} a(x, t, u, \nabla u)\) where \(a : Q \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) be Caratheodory function satisfying for a.e \((x, t) \in \Omega\) and all \(s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^n\) with \(\xi \neq \eta\):

\[
|a(x, t, s, \xi)| \leq \beta \varphi(x, |\xi|)/|\xi| \quad (3)
\]

\[
(a(x, t, s, \xi) - a(x, s, \eta))(\xi - \eta) > 0 \quad (4)
\]

\[
a(x, t, s, \xi) \xi \geq \alpha \varphi(x, |\xi|) \quad (5)
\]

where \(\alpha, \beta > 0\). Furthermore, assume that there exists \(D \in \mathcal{P}_\varphi\) such that

\[ D \circ H^{-1} \text{ is a Musielak-Orlicz function}. \quad (6) \]

Set \(T_k(s) = (-k, \min(k, s)), \forall s \in \mathbb{R}\), for all \(k \geq 0\).

Denote by \(\mathcal{M}_b(Q)\) the set of all bounded Radon measure defined on \(Q\) and by \(T_0^{1,\varphi}(Q)\) as the set of measurable functions \(Q \rightarrow \mathbb{R}\) such that \(T_k(u) \in W^{1,\varphi}_0(Q) \cap D(A)\). Assume that \(f \in \mathcal{M}_b(\Omega)\), and consider the following nonlinear parabolic problem with Dirichlet boundary

\[ \frac{\partial u}{\partial t} + A(u) = f \text{ in } Q. \quad (7) \]

**Theorem 1.** Assume that (2)-(6) hold and \(f \in \mathcal{M}_b(Q)\). Then there exists at least one weak solution of the problem

\[
\begin{cases}
    u \in T^{1,\varphi}_0(Q) \cap W^{1,x}_0(Q), \forall \phi \in \mathcal{P}_\varphi \\
    - \int_Q u \frac{\partial \phi}{\partial t} + \int_\Omega a(x, t, u, \nabla u) \nabla v dx = \langle f, v \rangle, \forall v \in D(Q).
\end{cases}
\]
Proof. The proof will be given in two steps.

step 1. A priori estimates.
Consider now the following approximate equations:

\[
\begin{align*}
\left\{
\begin{array}{l}
  u_n \in W_0^{1,x}(Q), u_n(x, 0) = 0, \\
  \frac{\partial u_n}{\partial t} - \text{div}(a(x, t, u_n, \nabla u_n)) = f_n
  \end{array}
\right.
\]

(8)

where \( f_n \) is a smooth function which converges to \( f \) in the distributional sense and
\( \|f_n\|_{L^1(Q)} \leq \|f\|_{\mathcal{M}_b(Q)}. \) By Theorem 2 of [15], there exists at least one solution \( u_n \)
of (8).

For \( k > 0 \), by taking \( T_k(u_n) \) as test function in (8), one has

\[
\int_{\Omega} a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq Ck.
\]

In view of (5), we get

\[
\int_{\Omega} \phi(x, |\nabla T_k(u_n)|) dx \leq Ck.
\]

Take a \( C^2(\mathbb{R}) \), and nondecreasing function \( \beta_k \) such that \( \beta_k(s) = s \) for \( |s| \leq \frac{k}{2} \) and \( \beta_k(s) = k \text{ sign } s \) if \( |k| > s \).

Multiplying the approximate equation (8) by \( \beta_k'(u_n) \), we get

\[
\frac{\partial \beta_k(u_n)}{\partial t} - \text{div}(a(x, t, u_n, \nabla u_n)\beta_k'(u_n)) + a(x, t, u_n, \nabla u_n)\nabla u_n \beta_k''(u_n) = f_n \beta_k'(u_n) \text{ in } D'(Q).
\]

Which implies easily that \( \frac{\partial \beta_k(u_n)}{\partial t} \) is bounded in \( W^{-1,x}(\Phi, L^1(Q) + L^1(Q). \) Thanks to Proposition 2, we deduce that \( \beta_k(u_n) \) is compact in \( L^1(\Omega) \).

Then as in [21] and by the proof of Theorem 3 of [13], we deduce that there exists \( u \in L^\infty(0, T; L^1(\Omega)) \) such that: \( u_n \to u \) almost everywhere in \( Q \) and

\[
T_k(u_n) \rightharpoonup T_u \text{ weakly in } W_0^{1,x}(\Phi, L^1(Q) \text{ for } \sigma(\Pi L_\Phi, \Pi E_\psi).
\]

(9)

Now, let \( \phi \in \mathcal{P}_\Phi \). By a slight adaptation of the context of Lemma 2.1. of [16], it follows that

\[
\int_Q \phi(x, |\nabla(u_n)|) dx \leq C, \forall n.
\]

(10)

We shall show that \( a(x, t, (u_n), \nabla(u_n)) \nabla(u_n) \) is bounded in \( (\Phi(Q)) \).

Let \( \omega \in (E_\Phi(Q))^n \) with \( \|\omega\|_{\Phi} = 1 \). By (5) and Young inequality, one has

\[
\int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \omega dx
\]

\[
\leq \beta \int_Q \psi(x, \frac{\phi(x, |\nabla T_k(u_n)|)}{|\nabla T_k(u_n)|}) dx + \beta \int_Q \phi(x, |\omega|) dx
\]

\[
\leq \beta \int_Q \phi(x, |\nabla T_k(u_n)|) dx + \beta
\]
This last inequality is deduced from the fact that $\psi(x, \varphi(x, u)/u) \leq \varphi(x, u)$, for all $u > 0$, and 
\[ \int_Q \varphi(x, |\omega|)dx \leq 1. \] In view of (10),
\[ \int_{\Omega} a(x, t, T_k(u_n), \nabla T_k(u_n))\omega dx \leq Ck + \beta, \]
which implies that $(a(x, t, T_k(u_n), \nabla T_k(u_n)) \omega)$ is a bounded sequence in $(L_\psi(Q))^n$.

**step 2**. Almost everywhere convergence of the gradient and passage to the limit. Since $T_k(u) \in W_0^{1,x} L_c(Q)$, then there exists a sequence $(\alpha_j^k) \subseteq D(Q)$ such that $(\alpha_j^k) \rightarrow T_k(u)$ for the modular convergence in $W_0^{1,x} L_c(Q)$. For the remaining of this article, $\chi_s$ and $\chi_{j,s}$ will denoted respectively the characteristic functions of the sets $Q_s = \{(x, t) \in Q; |\nabla T_k(u(x, t))| \leq s\}$ and $Q_{j,s} = \{(x, t) \in Q; |\nabla T_k(v_j(x, t))| \leq s\}$.

For the sake of simplicity, we will write only $\varepsilon(n, j, \mu, s)$ to mean all quantities (possibly different) such that
\[ \lim_{n \rightarrow \infty} \lim_{j \rightarrow \infty} \lim_{\mu \rightarrow \infty} \lim_{s \rightarrow \infty} \varepsilon(n, j, \mu, s) = 0. \]

For every $\mu > 0$, we define
\[ w_\mu(x, t) = \mu \int_{-\infty}^{t} \exp(\mu(s - t))w(x, t)\chi[0,T](s)ds, \]
the time regularized of any function $w \in W_0^{1,x} L_c(Q)$.

Taking now $\nabla_T(u_n - T_k(\alpha_j^k) \mu)$ as test function in (8), we obtain
\[ \langle \partial u_n \partial t, T_\mu(u_n - T_k(\alpha_j^k), \mu) \rangle + \int_Q a(x, t, u_n, \nabla(u_n))\nabla T_\mu(u_n - T_k(u))dx \leq C_\eta \]

The first term of the left hand side of the last equality reads as
\[ \langle \partial u_n \partial t, T_\mu(u_n - T_k(\alpha_j^k), \mu) \rangle = \langle \partial u_n \partial t, T_k(\alpha_j^k) \mu \rangle + \langle T_k(\alpha_j^k) \mu, T_\mu(u_n - T_k(\alpha_j^k), \mu) \rangle. \]

The second term of the last equality can be easily to see that is positive and the third term can be written as
\[ \langle \partial T_k(\alpha_j^k) \mu \partial t, T_\mu(u_n - T_k(\alpha_j^k), \mu) \rangle = \mu \int_Q (T_k(\alpha_j^k) - T_k(\alpha_j^k), \mu)(T_\mu(u_n - T_k(\alpha_j^k), \mu))dxdt, \]
thus by letting $n, j \rightarrow \infty$ and since $(\alpha_j^k) \rightarrow T_k(u)$ a.e. in $Q$ and by using Lebesgue Theorem,
\[ \int_Q (T_k(\alpha_j^k) - T_k(\alpha_j^k), \mu)(T_\mu(u_n - T_k(\alpha_j^k), \mu))dxdt = \int_Q (T_k(u) - T_k(u), \mu)(T_\mu(u - T_k(u), \mu))dxdt + \varepsilon(n, j). \]
Consequently
\[
\langle \frac{\partial u_n}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \rangle \geq \varepsilon(n, j).
\]

On the other hand,
\[
\int_Q a(x, t, u_n, \nabla(u_n)) \cdot \nabla T_\eta(u_n - T_k(\alpha_j^k)_\mu) \, dx \, dt
= \int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}) \, dx \, dt
+ \int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \, dx \, dt
- \int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_j^k)_\mu \chi(\nabla T_k(\alpha_j^k)_\mu > s) \, dx \, dt,
\]
which implies, by using the fact that
\[
\int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \, dx \, dt \geq 0,
\]
\[
\int_{\{T_k(u_n) - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}) \, dx \, dt \leq C \eta
\]
\[
+ \int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_j^k)_\mu \chi(\nabla T_k(\alpha_j^k)_\mu > s) \, dx \, dt.
\]
Since \(a(x, t, T_k(\eta)(u_n), \nabla T_k(\eta)(u_n))\) is bounded in \((L^\psi(\Omega))^n\), there exists some \(h_{k+\eta} \in (L^\psi(\Omega))^n\) such that
\[
a(x, t, T_k(\eta)(u_n), \nabla T_k(\eta)(u_n)) \to h_{k+\eta}
\]
weakly in \((L^\psi(\Omega))^n\) for \(\sigma(\Pi L^\psi, \Pi E_\varphi)\).
Consequently
\[
\int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_j^k)_\mu \chi(\nabla T_k(\alpha_j^k)_\mu > s) \, dx \, dt
= \int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} h_{k+\eta} \nabla T_k(\alpha_j^k)_\mu \chi(\nabla T_k(\alpha_j^k)_\mu > s) \, dx \, dt + \varepsilon(n),
\]
where we have used the fact that \(\nabla T_k(\alpha_j^k)_\mu \chi_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}}\) tends strongly to \(\nabla T_k(\alpha_j^k)_\mu \chi_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}}\) in \((E_\varphi(Q))^n\). Letting \(j \to \infty\), we obtain
\[
\int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_j^k)_\mu \chi(\nabla T_k(\alpha_j^k)_\mu > s) \, dx \, dt
= \int_{\{k < |u_n| \cap \{u_n - T_k(\alpha_j^k)_\mu \mid \eta\}} h_{k+\eta} \nabla T_k(u)_\mu \chi(\nabla T_k(u)>s) \, dx \, dt + \varepsilon(n, j).
\]
Thanks to Proposition 1, one easily has
\[
\int_{\{k<\mu\} \cap \{|u-T_k(u)\_\mu|<\eta\}} h_{k+\eta} \nabla T_k(u) \mu \chi_{\{|\nabla T_k(u)|>s\}} \, dx dt \\
= \int_{\{k<\mu\} \cap \{|u-T_k(u)\_\mu|<\eta\}} h_{k+\eta} \nabla T_k(u) \chi_{\{|\nabla T_k(u)|>s\}} \, dx dt + \varepsilon(\mu).
\]

Hence
\[
\int_{\{|T_k(u_n) - T_k(\alpha_j^k)\_\mu|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \mu \chi_{j,s}) \, dx dt + C \eta + \varepsilon(n, j, \mu, s).
\]

On the other hand, remark that
\[
\int_{\{|T_k(u_n) - T_k(\alpha_j^k)\_\mu|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s}) \, dx dt \\
= \int_{\{|T_k(u_n) - T_k(\alpha_j^k)\_\mu|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s}) \, dx dt \\
+ \int_{\{|T_k(u_n) - T_k(\alpha_j^k)\_\mu|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k) \mu \chi_{j,s}) \, dx dt.
\]

The latest integral tends to 0 as \(n\) and \(j\) go to \(\infty\). Indeed, we have
\[
\int_{\{|T_k(u_n) - T_k(\alpha_j^k)\_\mu|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k) \mu \chi_{j,s}) \, dx dt
\]
tends to
\[
\int_{\{|T_k(u) - T_k(\alpha_j^k)\_\mu|<\eta\}} h_k [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k) \mu \chi_{j,s}] \, dx dt
\]
as \(n \to \infty\), since
\[
a(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L^p(\Omega))^n \text{ for } \sigma(\Pi L^p, \Pi E^p),
\]
while \(\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k) \mu \chi_{j,s} \in (E^p(Q))^n\). It’s obvious that
\[
\int_{\{|T_k(u) - T_k(\alpha_j^k)\_\mu|<\eta\}} h_k [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k) \mu \chi_{j,s}] \, dx dt
\]
goes to 0 as \(j, \mu \to \infty\) by using Lebesgue theorem. We deduce then that
\[
\int_{\{|T_k(u_n) - T_k(\alpha_j^k)\_\mu|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s}) \, dx dt + C \eta + \varepsilon(n, j, \mu, s).
\]

Let \(0 < \delta < 1\). We have
\[
\int_{Q^r} \left[ a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)) \right] \times [\nabla T_k(u_n) - \nabla T_k(u)]^\delta \, dx dt \leq C \text{ mea } \{|T_k(u_n) - T_k(\alpha_j^k)\_\mu| < \eta\}^\delta
\]
\[
+ C [\int_{\{|T_k(u_n) - T_k(\alpha_j^k)\_\mu|<\eta\} \cap Q^r} [a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)]^\delta \, dx dt].
\]
On the other hand, we have for every $s \geq r$

\[
\int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta \cap Q_r\}} \left[ a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)) \right] \\
\times [\nabla T_k(u_n) - \nabla T_k(u)] dxdt \\
\leq \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} \left[ a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u) \chi_{s}) \right] \\
\times [\nabla T_k(u_n) - \nabla T_k(u) \chi_{s}] dxdt \\
\leq \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} \left[ a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \right] \\
\times [\nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s}] dxdt \\
+ \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_{s}] dxdt \\
+ \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} [a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \chi_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}}] \\
- a(x, t, T_k(u_n), \nabla T_k(u) \chi_{s})] \nabla T_k(u_n) dxdt \\
- \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(\alpha_j^k) \chi_{j,s} dxdt \\
+ \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u) \chi_{s}) \nabla T_k(u) \chi_{s} dxdt \\
\leq I_1(n, j, \mu, s) + I_2(n, j, \mu, s) + I_3(n, j, \mu, s) + I_4(n, j, \mu, s) + I_5(n, j, \mu, s). \tag{12}
\]

We shall go to limit as $n, j, \mu$ and $s \to \infty$ in the last fifth integrals of the last side. Starting with $I_1$, we have

\[
I_1(n, j, \mu, s) \leq C \eta + \varepsilon(n, j, \mu, s) - \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(\alpha_j^k) \chi_{j,s} dxdt
\]

since

\[
a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \chi_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} \to a(x, t, T_k(u), \nabla T_k(\alpha_j^k) \chi_{j,s}) \chi_{\{T_k(u) - T_k(\alpha_j^k)_{\mu} < \eta\}} \text{ in } (E_\psi(Q))^n
\]

while

\[
\nabla T_k(u_n) \to \nabla T_k(u) \text{ weakly}
\]

we deduce then that

\[
\int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) [\nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s}] dxdt
\]

\[
= \int_{\{T_k(u_n) - T_k(\alpha_j^k)_{\mu} < \eta\}} a(x, t, T_k(u), \nabla T_k(\alpha_j^k) \chi_{j,s}) [\nabla T_k(u) - \nabla T_k(\alpha_j^k) \chi_{j,s}] dxdt + \varepsilon(n)
\]
which gives by letting \( j \to \infty \) and using the modular convergence of \( \nabla T_k(\alpha_j^k) \)

\[
\int\int_{\{T_k(u_n) - T_k(\alpha_j^k) < \eta\}} a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s})[\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s}] \, dx \, dt + \varepsilon(n)
\]

\[
= \int\int_{Q} a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s})[\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s}] \, dx \, dt + \varepsilon(j) = \varepsilon(j).
\]

Finally

\[
I_1(n, j, \mu, s) \leq C \eta + \varepsilon(n, j, \mu, s) + \varepsilon(n, j) = \varepsilon(n, j, \mu, s, \eta).
\]

For what concerns \( I_2 \), by letting \( n \to \infty \), we have

\[
I_2(n, j, \mu, s) = \int\int_{\{T_k(u_n) - T_k(\alpha_j^k) < \eta\}} h_k[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(u)\chi_{j,s}] \, dx \, dt + \varepsilon(n)
\]

since

\[
a(x, t, T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s})\chi_{\{T_k(u_n) - T_k(\alpha_j^k) < \eta\}} \to h_k \text{ for } \sigma(\Pi L \psi, E_\varphi)
\]

while

\[
\chi_{\{T_k(u_n) - T_k(\alpha_j^k) < \eta\}}[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(u)\chi_{j,s}] \to \\
\chi_{\{T_k(u) - T_k(\alpha_j^k) < \eta\}}[\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(u)\chi_{j,s}] \text{ strongly in } (E_\varphi(Q))^n.
\]

By letting now \( j \to \infty \), and using Lebesgue theorem, we deduce then that

\[
I_2(n, j, \mu, s) = \varepsilon(n, j).
\]

Similar tools as above, give

\[
I_3(n, j, \mu, s) = \varepsilon(n, j)
\]

\[
I_4(n, j, \mu, s) = -\int\int_{Q} a(x, t, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u) + \varepsilon(n, j, \mu, s)
\]

\[
I_5(n, j, \mu, s) = \int\int_{Q} a(x, t, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u) + \varepsilon(n, j, \mu, s).
\]

Combining (11), (12), (13), (14) and (15), we have

\[
\int\int_{Q} [(a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u_n))) \times (\nabla T_k(u_n) - \nabla T_k(u))]^\delta \, dx \, dt
\]

\[
\leq C(\text{meas}\{T_k(u_n) - T_k(\alpha_j^k) < \eta\})^\delta + C(\varepsilon(n, j, \mu, s, \eta))^\delta
\]
and by passing to the limit sup over \(n, j, \mu, s\) and \(\eta\)
\[
\lim_{n \to \infty} \int_{Q_r} \left[ [a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)] \right] \delta \, dx \, dt = 0
\]
and thus, there exists subsequence also denote by \((u_n)\) such that
\[
\nabla u_n \to \nabla u \text{ a.e. in } Q_r,
\]
and since \(r\) is arbitrary, we have
\[
\nabla u_n \to \nabla u \text{ a.e in } Q.
\]

On the other hand, thanks to (3), (6) and (10), we deduce that
\[
\int_Q D \circ H^{-1}(s, \frac{|a(x, t, u_n, \nabla u_n)|}{\beta}) \, dx \, dt \leq \int_\Omega D(x, |\nabla u_n|) \, dx \, dt \leq C
\]
Hence
\[
a(x, t, u_n, \nabla u_n) \rightharpoonup a(x, t, u, \nabla u)
\]
weakly for \(\sigma(\Pi_{D_0H^{-1}}, \Pi_{E_{D_0H^{-1}}})\)

Going back to approximate equations (8), and using \(v \in D(Q)\) as the test function, one has
\[
- \int_Q u_n \frac{\partial v}{\partial t} \, dx \, dt + \int_Q a(x, t, u_n, \nabla u_n) \nabla v \, dx \, dt = \langle f_n, v \rangle
\]
in which we can pass to the limit since we have
\[
u_n \to u \text{ strongly in } (E_\gamma(Q))^n \text{ for every } \gamma \ll \phi \in \mathcal{P}_\phi
\]

This completes the proof of Theorem 1.

References


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