Comparison of Approximate and Exact Solution of Some Fractional Differential Equations

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Abstract

Three numerical methods for nonlinear fractional differential equations (FDE) have been presented, Fractional Adams-Bashforth Method (FAB), Fractional Adams-Bashforth-Moulton Method (FABM) and Fractional Multistep Differential Transformation Method (FMDTM) and used for very specific type of problems. Numerous problems in many applications are modelled mathematically by FDE (systems). For solving the FDE with fractional derivatives described in the Caputo sense, this work presents an approach solutions using the constructed methods, with aim to compare them with the exact ones obtained by Laplace Transform (LT). The methods are illustrated by two examples and solutions are obtained with comparisons made between FMDTM, FAB, FABM and the exact solutions at each integration point, given, both, graphically and tabularly, for arbitrary fractional order, $0 < \alpha \leq 1$, and the routine available in the Mathematica software. According to the investigated fractional order $\alpha$, we plot the characteristic deviation pronounced by $R(t)$, to show the behaviour of the methods in comparison with the exact solution. The superiority of each numerical method over the others depends on the taken fractional problem.

Mathematics Subject Classification: 34C28, 34A08, 74H15
Keywords: Fractional Differential Equations, Fractional Adams-Bashforth and Adams-Bashforth-Moulton Method, Fractional Multistep Differential Transformation Method, Laplace Transform, Exact Solution

1 Introduction

Fractional calculus is not a new topic, it can be dated back to the Leibnizs letter to LHopital, dated 30 September 1695, in which the meaning of the one-half order derivative was first discussed with some remarks about its possibility. Numerous problems in many applications are modelled mathematically by FDE (systems) [8, 9]. In recent years considerable interest in fractional calculus has been stimulated by the applications that it finds in different fields of science, including numerical analysis, economics and finance, engineering, physics, biology, etc. FDE have been successfully modelled for many physical and engineering phenomena [1, 2]. In this paper we test three numerical algorithms comparing with exact solution of initial value problem of the form:

\[ D^\alpha y(t) = f(y(t), t), \quad y_0 = y(0), \quad 0 < \alpha \leq 1 \]  

(1)

where \( D^\alpha \) denotes the Caputo fractional operator [2].

2 Basic Definitions

**Definition 2.1** Fractional integral of order \( \alpha \) for function \( f(t) \) can be expressed as follows:

\[ aI_t^\alpha f(t) = D_t^{-\alpha} = \frac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} f(\tau) d\tau \]  

(2)

**Definition 2.2** For \( 0 < \alpha < 1 \), Caputo definition of fractional derivative of order \( \alpha \) is:

\[ D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \]  

(3)

**Remark:** The Caputo’s definition is used, like an modification of the Riemann-Liouville definition, which has an advantage of dealing properly with the initial value problem since the initial condition is given in terms of the field variables and their integer order. This case is widely used in physical applications [9].

**Definition 2.3** [2] The Gamma function \( \Gamma : (0, \infty) \rightarrow R \) and Beta function \( B(x, y) \) are defined by (4) and (5):

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \]  

(4)
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\[ B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \quad x > 0, \ y > 0 \quad (5) \]

**Definition 2.4** [2, 9] The one and two-parameter Mittag-Leffler functions are defined by (6) and (7):

\[ E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma[\alpha k + 1]}, \quad \alpha > 0 \quad (6) \]

\[ E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma[\alpha k + \beta]}, \quad \alpha > 0, \ \beta > 0 \quad (7) \]

**Definition 2.5** [7, 9] Laplace transform or image of the original function \( f(t) \), is called the function

\[ F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (8) \]

with corresponding inverse Laplace transform of the form:

\[ f(t) = \frac{1}{2\pi i} \lim_{t \to \infty} \int_{\gamma-it}^{\gamma+it} F(s) e^{st} dt = L^{-1}[F(s)] \quad (9) \]

where \( \gamma \in \mathbb{R} \) and contour path of the integration is contained in the convergence region.

**Laplace Transform of Caputo Operator** [8]:

\[ L[D^\alpha f(t)] = L\left[ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-1} f^{(n)}(\tau) d\tau \right] = L[I^{n-\alpha} f^{(n)}(t)] \]

\[ L[D^\alpha f(t)] = s^n F(s) - \sum_{k=0}^{n-1} s^{n-\alpha - 1} [I^{n-\alpha} f^{(k)}(t)]_{t=0} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-\alpha - 1} f^{(k)}(0) \quad (10) \]

3 Numerical Methods

3.1 Fractional Multistep Differential Transformation Method

The DTM deals with the approximated solutions to integer-order differential equations and is based on polynomial approximations [4, 10]. The differential transform of the \( k \)-th derivative of function \( f(t) \) is defined as: \( F(k) = \frac{1}{k!} \left[ \frac{d^k f(t)}{dt^k} \right]_{t=t_0} \) with \( f(t) \) the original function

\[ f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^k \quad (11) \]
From \( F(k) \) and (11), we get \( f(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \frac{d^k f(t)}{dt^k} \), which implies that the concept of DTM, where the (1) can be expressed by the finite series \( y(t) = \sum_{n=0}^{N} a_n t^n \), \( t \in [0,T] \). It is simply formed on the idea of dividing the time interval \([0,T]\) into \( M \) subintervals \([t_{m-1}, t_m]\), \( m = 1, 2, ..., M \) of equal step size \( h = \frac{T}{M} \) by \( t_m = mh \). We apply the DTM to (1) over the interval \([0,t_1]\), using the initial conditions \( y_1(t_0) = c_k \). For \( m \geq 2 \) and at each subinterval \([t_{m-1}, t_m]\) using the initial conditions \( y_m(t_{m-1}) = y_{m-1}(t_{m-1}) \) and applying DTM to (1) over the interval \([t_{m-1}, t_m]\), where \( t_0 \) in (11) is replaced by \( t_{m-1} \). The process is repeated and generates a sequence of approximate solutions \( y_m(t), m = 1, 2, ..., M \), for the solution \( y(t) \), \( y_m(t) = \sum_{n=0}^{K} a_{mn} (t - t_{m-1})^n \), \( t \in [t_m, t_{m+1}] \), where \( N = KM \) [1, 4]. In fact the FMDTM assumes the following solution:

\[
y(t) = \begin{cases} 
y_1(t), & t \in [0, t_1] 
y_2(t), & t \in [t_1, t_2] 
\vdots 
y_M(t), & t \in [t_{M-1}, t_M] 
\end{cases}
\]  
(12)

The FMDTM implements a step-by-step procedure which is not adequate to discretize non-local operators such as fractional derivatives.

### 3.2 Adams-Bashforth and Adams-Bashforth-Moulton Method

The FAB method is a natural candidate for a predictor with Adams-Moulton method (FAM) like an candidate for corrector in process of constructing the Predictor-Corrector method or FABM [1, 6]. To construct them we shall always assume that a solution of (1) is sought on some interval \([0,T]\) to construct approximate solution values \( y_j \approx y(t_j) \) at the grid points \( t_j = hj \), \( (j = 0, 1, 2, ..., N) \) for \( h = \frac{T}{N} \). The initial value problem (1) is equivalent to the Volterra integral equation, which means that we shall approximate it in front of (1). The main part of the algorithm for FAB:

\[
y[j] = \sum_{k=0}^{[\alpha]-1} \frac{(jh)^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{j-1} b[j-k](f(kh, y[k])) \tag{13}
\]

where \( b[j - k] \) is the weight depended only by difference \((j - k)\) because of the convolution structure of \( b_{k,j} \) [3]. Using FAB and FAM, the main part of the FABM or "Predictor and corrector" [3, 8] of the algorithm:

\[
p[j] = \sum_{k=0}^{[\alpha]-1} \frac{(jh)^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{j-1} b[j-k](f(kh, y[k]))
\]
\[
y[j] = \sum_{k=0}^{[^{\alpha-1]} \frac{jh}{k!}} (jh)^k y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha + 2)} f(jh, p) + ((j - 1)^{\alpha+1} - (j - \alpha - 1)j^\alpha) f(0, y[0]) \\
+ \sum_{k=0}^{j-1} a[j - k](f(kh, y[k]))
\] (14)

The corrector part of (21) is denoted by \( y[j] \) with weight \( a[j - k] \) [3], where \( p \) represent FABM, which in this case act like a predictor.

The deviation of the approximation curves from the exact one can hardly be perceived from the figure. To characterize this deviation, and hence the quality of particular approximation method, we introduce the relative difference parameter \( R_y(t) \) defined as the absolute difference between the exact value \( y_{\text{exact}} \) of the solution curve \( y(t) \) at the fixed time \( t \) and the approximate value \( y_{\text{approx}} \) at the same instant of time divided by the maximum absolute value of these two numbers (expressed in percent by multiplying with 100):

\[
R_y(t) = \frac{|y_{\text{exact}}(t) - y_{\text{approx}}(t)|}{\max\{|y_{\text{exact}}(t)|, |y_{\text{approx}}(t)|\}} \times 100
\] (15)

4 Numerical Examples


\[
D^\alpha y(t) = -y^2(t) + 1, \quad 0 < \alpha \leq 1, \quad y(0) = 0
\] (16)

Solution: Exact solution of Eq. (16) for \( 0 < \alpha \leq 1, \ y(0) = 0 \), is:

\[
y(t) = \frac{e^{\frac{2}{\alpha} e^{\frac{1}{\alpha}} - 1}}{e^{\frac{2}{\alpha} e^{\frac{1}{\alpha}}} + 1}
\] (17)

It is taken by using transformed function by FDTM:

\[
Y[k + 1] = \frac{1}{\alpha(k + 1)}[\delta[k] - \sum_{l=0}^{k} Y[l]Y[k - l]]
\] (18)

For \( k = 0, 1, 2, ..., n - 1 \), the solution by means of FDTM is found as:

\[
y(t) = \frac{t^\alpha}{\alpha} - \frac{t^{3\alpha}}{3\alpha} + \frac{2t^{5\alpha}}{15\alpha} - \frac{17t^{7\alpha}}{315\alpha} + \frac{62t^{9\alpha}}{2835\alpha} - \frac{1382t^{11\alpha}}{155925\alpha} + \frac{21844t^{13\alpha}}{6081075\alpha}...
\] (19)

The main part of algorithms for FAB (20) and FABM (20, 21):

\[
p[j] = y_0 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{j-1} b[j - k](-y[k]y[k] + 1)
\] (20)
\[ y[j] = y_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)}((-p[j]p[j] + 1)((j - 1)^{\alpha+1} - (j - \alpha - 1)j^{\alpha})(-y[0]y[0] + 1)
+ \sum_{k=1}^{j-1} a[j - k](-y[k]y[k] + 1)) \]  

(21)

The obtained solution (19) is the fractional power series expansion of the exact solution (17) for the first 14 terms. We show the results of exact and approximated solutions for \( \alpha = 0.9 \) and \( \alpha = 0.5 \), step size \( h = 0.01 \) and time interval \( t \in [0, 20] \).

In Fig. 1 (a), we depict the plots of \( y(t) \) versus the time \( t \) of the initial value problem (16) consisting of the exact integration curve, as well as the approximation time-series obtained from the three different numerical methods. The exact curve is depicted with a black (solid) line, and the dotted (red) line, the dashed (green) line and the dash-dotted (blue) line are the approximation time-series obtained with FMDTM, FAB, FABM respectively.

In Fig.1 (b) we plot the characteristic deviation pronounced by \( R_y(t) \). For the three methods the curves of the deviation have a large increasing behavior in a small time interval. Then they become decreasing as the integration time becomes larger. This observation suggests that after the time interval \( t \in [0, 2] \) where everything seems different, the methods diverge from the exact solution, then all of them gradually converge to the exact solution with increasing integration time. Although all three methods approximate the exact solution in a practical range of relative deviations, in this case FMDTM give us better approximations for a large time interval. The same conclusions can also be extracted by looking at the tabular values of the exact time-series \( y_{\text{exact}} \) and the approximation time series \( y_{\text{FMDTM}}, y_{\text{FAB}} \) and \( y_{\text{FABM}} \), along with the corresponding relative difference \( R_{\text{FMDTM}}, R_{\text{FAB}} \) and \( R_{\text{FABM}} \). Table 1. Let see
Comparison of approximate and exact solution of some FDE

Table 1: Partial data values of the time-series $y(t)$ for increasing $t$ of the exact solution of system Eq.(16), numerical approximation values for FMDTM, FAB, and FABM and corresponding values for the relative difference $R_y(t)$ for each method.

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The behavior of the three methods for the time interval $t \in [0, 20]$, with the same initial condition $y(0) = 0$, fractional order $\alpha = 0.5$ and step size $h = 0.01$, see Fig. 2. In Fig. 2 (b) we can see approximately the same behavior of the curve $R_y(t)$ as in Fig. 1, for $\alpha = 0.5$, FMDTM is in a better approximation than FAB and FABM. The curve for three methods, firstly, rapidly increases till the time interval $t \in [0, 1]$, then they begin slowly decreasing till the best approximations.

Example 2. [5] Using LT, FMDTM, FAB and FABM, let solve the FDE (22):

$$D^\alpha y(t) = 2y(t) + t^2, \quad 0 < \alpha \leq 1, \quad y(0) = 1$$

(22)
Solution: Exact solution of (22) for arbitrary $\alpha$ is taken by starting from the general form of the equation

$$D^\alpha y(t) - ay(t) = h(t), \ n \in N, y^{(k)}(0) = b_k, \ k = 0, 1, ..., n - 1$$  \hspace{1cm} (23)

Using the LT to equation (23):

$$Y(s)(s^\alpha - a) = H(s) + \sum_{k=0}^{n-1} s^{\alpha-k-1}b_k \Rightarrow Y(s) = \sum_{k=1}^{n-1} b_k \frac{s^{\alpha-k-1}}{(s^\alpha - a)} + \frac{H(s)}{(s^\alpha - a)}$$  \hspace{1cm} (24)

Applying the inverse LT to (24) and using the equation

$$L^{-1}(\frac{s^{\alpha-\beta}}{s^\alpha + a}) = x^{\beta-1}E_{\alpha,\beta}(-at^\alpha),$$

we have the final exact solution of (23):

$$y(t) = \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(at^\alpha) + \int_0^t ((t - \tau)^{\alpha-1}E_{\alpha,\alpha}(a(t - \tau))h(t)dt$$  \hspace{1cm} (25)

For $\alpha = 0.9$, (25) takes the form $y(t) = E_{0.9,1}(2t^{0.9}) + \Gamma[3]t^{2.9}E_{0.9,3.9}(2t^{0.9})$.

Transformed function using FMDTM is:

$$Y(k + \alpha\beta) = \frac{\Gamma[1 + \frac{k}{\beta}]}{\Gamma[1 + \alpha + \frac{k}{\beta}]}[2Y[k] + \delta[k - 2\beta]], \hspace{0.5cm} Y[k] = 0, k = 0, 1, ..., \alpha\beta - 1$$  \hspace{1cm} (26)

The main part of algorithms for FABM (27) and FABMM (27, 28):

$$p[j] = y_0 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{j-1} b[j - k](2y[k] + (kh)^2)$$  \hspace{1cm} (27)

$$y[j] = y_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)}(2p[j] + t^2)((j - 1)^{\alpha+1} - (j - \alpha - 1)(y[0])$$

$$+ \sum_{k=0}^{j-1} a[j - k](2y[k] + (kh)^2))$$  \hspace{1cm} (28)

In Fig. 3 (a) we depict the plots of $y(t)$ versus the time $t$ of the initial value problem (22) consisting of the exact integration curve, as well as the approximation time-series obtained from the three different numerical methods. The exact curve is depicted with a black (solid) line, and the dotted (red) line, the dashed (green) line and the dash-dotted (blue) line are the approximation time-series obtained with FMDTM, FAB, FABM respectively. In all numerical simulations, we take the integration step-size $h = 0.01$, and the integration
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Figure 3: (a) Time-series $y(t)$ and (b) Relative difference $R_y(t)$ of Eq. (22). The exact integration curve (black/solid line) and the approximation curves for FMDTM (dotted/red line), FAB (dashed/green line), and FABM (dot-dashed/blue line); $\alpha = 0.9$, $y(0) = 1$, $h = 0.01$ and $t \in [0, 20]$

Table 2: Partial data values of the time-series $y(t)$ for increasing $t$ of the exact solution of system Eq.22, numerical approximation values for FMDTM, FAB, and FABM, and corresponding values for the relative difference $R_y(t)$ for each method.

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time-span is $t \in [0, 20]$. It is seen that all the curves are characterized with a typical exponential-like increase as $t \to \infty$.

In Fig. 3 (b) we plot the characteristic deviation pronounced by $R_y(t)$. The curve $R_y(t)$ of FMDTM shows an increasing behavior at time interval, approximately $[0, 4]$ then it becomes constant as the integration time becomes larger. The numerical method gradually diverges from the exact solution in the interval $[0, 4]$ as the time rises. For FAB and FABM we have an constant behaviour of $R_y(t)$ and in this case they are better approximations than FMTDM at small and large time values. The same conclusions can also be extracted by looking at the tabular values of the exact time-series $y_{\text{exact}}$ and the approximation time series $y_{\text{FMDTM}}, y_{\text{FAB}}$ and $y_{\text{FABM}}$, along with the corresponding relative difference parameters $R_{\text{FMDTM}}, R_{\text{FAB}}$ and $R_{\text{FABM}}$. See Table 2.
5 Conclusion

Considerable attention is paid to FDE because they appear to be more effective for modelling dynamical processes in natural and applied sciences. For this reason, finding and successfully applying numerical methods for solving these equations is and will be one of the essential concerns for mathematical and applied scientists. In this paper, FMDTM, FAB, FABM are extended to solve FDE. The numerical techniques were tested on linear (Example 2) and nonlinear (Example 1) fractional problems. The study emphasized our belief that the methods are in a reliable technique to handle FDE with advantages in terms of their straightforward applicability, their computational effectiveness and their accuracy. FMDTM shows good agreements in Example 1 and constant behaviour in Example 2. The superiority of each numerical method over the other methods depends on the particular equation under investigation: in the first case (Example 1), FMDTM is shown to be a better approximation than FAB and FABM, according to the taken fractional orders $\alpha = 0.9$ and $\alpha = 0.5$. In the second case (Example 2), FAB and FABM are in great agreement with the exact solution, and FMDTM which diverges for a small time interval and then shows constant behaviour as the time rises. Not all numerical methods gradually diverge from the exact solution with increasing integration time. The recent appearance of FDE as models in some fields in applied mathematics makes it necessary to investigate methods of solutions for such equations, even they are analytical or numerical methods, and we hope that this work is a step on this direction. Successfully, FMDTM, FAB and FABM were applied to solve the FDE of order $0 < \alpha \leq 1$. All ideas were illustrated to be efficient in applying the proposed technique to several linear and nonlinear FDE of that order.

References


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