Calculation of \( n \)-th Partial Sums \( S_n \) of Power Series and Its Relation with the Calculation of Bernoulli Numbers

Carlos Oscar Rodriguez Leal

Fraccionamiento Los Manzanos, Jalisco
45200 Tesistan, Jal., Mexico

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Abstract

In this work, the general formula of the \( n \)-th partial sums \( S_n \) of sums of powers of the form \( 1^n + 2^n + \cdots + m^n \) is obtained by an algebraic method, and said formula is applied to obtaining the Bernoulli numbers by a new simple method.

Keywords: partial sums of sequences – power series – Bernoulli numbers

1 Introduction

In this article, the general formula of the \( n \)th partial sums \( S_n \) of power sums of the form \( 1^n + 2^n + \cdots + m^n \) is obtained, by a very simple novel algebraic method and it is applies to obtaining Bernoulli numbers in a very simple way, through recurrence relations or through determinants of triangular matrices.

2 Deduction of the \( n \)-th partial sums of power series

To deduce the \( n \)-th partial sums of power series \( S_n: \sum_{m=1}^{p} m^n \), we first observe the graphical behavior of the area under the step curve of the partial sums and their similarity and bounding with the definite integrals of the respective polynomials lower (with area less than the sum) \( 0^n + 1^n + \ldots + p^n \) and upper
(with area greater than the sum) $1^n + 2^n + \ldots + (p + 1)^n$ (see figure 1). With which we establish the following conjecture:

$$\Sigma_{m=1}^{p} m^n = A_0^n + A_1^n p + \ldots + A_n^n p^n + A_{n+1}^n p^{n+1}, \quad (1)$$

where $A_j^n$ is the $j$-th coefficient of the respective polynomial of degree $n + 1$.

But from [1,2,3], we see that this conjecture is correct and that $A_0^n$ is zero.

Figure 1: Power Sum Conjecture

That is, there is no constant term.

Now, if we take the sum with one more last term, i.e. $\Sigma_{m=1}^{p+1} m^n$, we will get

$$\Sigma_{m=1}^{p+1} m^n = A_1^n (p + 1) + A_2^n (p + 1)^2 + \ldots + A_n^n (p + 1)^n + A_{n+1}^n (p + 1)^{n+1} = A_1^n p + A_2^n p^2 + \ldots + A_n^n p^n + (p + 1)^n, \quad (2)$$

from which, when developing the polynomials, it follows that,

$$A_1^n (p + 1) + A_2^n (p^2 + 2p + 1) + \ldots + A_n^n (C_0^n p^0 + C_1^n p + \ldots + C_{n-1}^n p^{n-1}) + A_{n+1}^n (C_0^{n+1} p^0 + C_1^{n+1} p^1 + \ldots + C_{n+1}^{n+1} p^{n+1}) = A_1^n p + A_2^n p^2 + \ldots + A_n^n p^n + A_{n+1}^n p^{n+1} + C_0^n p^0 + C_1^n p + \ldots + C_{n}^n p^n, \quad (3)$$

whence, eliminating the equal terms from both sides of the equation,

$$A_1^n + A_2^n (2p + 1) + \ldots + A_n^n (C_0^n p^0 + C_1^n p^1 + \ldots + C_{n-1}^n p^{n-1}) = A_{n+1}^n (C_0^{n+1} p^0 + C_1^{n+1} p^1 + \ldots + C_n^{n+1} p^n) \quad (4)$$
and thus, equating coefficients of like terms, we arrive at the system of equations

\[ C^n_0 = A^n_1 C^n_0 + A^n_2 C^n_0 + \ldots + A^n_n C^n_0 + A^n_{n+1} C^{n+1}_0 \]
\[ C^n_1 = A^n_2 C^n_1 + \ldots + A^n_n C^n_1 + A^n_{n+1} C^{n+1}_1 \]
\[ C^n_{n-1} = A^n_n C^n_{n-1} + A^n_{n+1} C^{n+1}_{n-1} \]
\[ C^n_n = A^n_{n+1} C^{n+1}_n \]  

(5)

2.1 Solution of the system of equations

The above system can be solved recursively with the following formulas:

\[ A^n_{n+1} = C^n_n / C^n_{n+1} = 1/(n+1) \]

(6)

\[ A^n_{i+1} = \frac{C^n_i - \sum^{n+1-i}_{k=2} A^n_{i+k} C^i_{i+k}}{C^i_{i+1}}, \]

(7)

with \( i = 1, \ldots, n - 1 \).

On the other hand, expressing the system matrixly, we will obtain, for some fixed degree \( n \), another method of solving the system (matrix method)

\[ C^n_m = A^n_i C^i_m \]  

(8)

where we have used a sum convention like Einstein’s in tensor analysis, with \( i = m + 1, \ldots, n + 1 \), and \( m = 0, \ldots, n \), so, solving the system matrix with the inverse of a triangular matrix (which greatly reduces the order of complexity of the calculation), we obtain the values of the vector of coefficients

\[ 1^4 + 2^4 + \ldots + m^4 = \frac{m(m+1)(2m+1)(3m^2 + 3m + 1)}{30}. \]

(9)

In this example \( n = 4 \), so, solving the system for the variables \( A^4_m \), first by the recursive method, we will have

\[ A^4_5 = 1/5, A^4_4 = 1/2, A^4_3 = 1/3, A^4_2 = 0, A^4_1 = -1/30, \]

(10)

these coefficients coinciding with those given in [1,2].

And now, finding the solution of the matrix equation using the octave program (version 8.2.0), we will obtain the same values of the polynomial coefficients \( A^n_m \) (see the appendix for the code in octave).
3 Obtaining Bernoulli numbers

In this section we find a simple alternative method for calculating Bernoulli numbers. According to [1], the nth Bernoulli number corresponds to the coefficient of the degree 1 term $A_n$. Therefore, using formulas (6) and (7) or (9) (depending on whether the recursive or matrix method is used, respectively), we can obtain, by this alternative method, $B_0 = 1, B_n = A_n^1, n > 0$.

4 Appendix

This appendix shows the code in Octave to find the polynomial coefficients $A_n^m$.

```
# Programa de n-ésimas sumas parciales de series de potencias
# Carlos Oscar Rodriguez Leal.
# 03/09/2023

grado = 5; # Indicamos el grado de nuestra sumatoria
A = zeros(grado+1,1); # Vector de coeficientes polinomiales
Cm = zeros(grado+1,grado+1); # Matriz de combinaciones

# Cargamos el vector de combinaciones
for m = 1:grado-1
    Cm(m+1,1:m) = factorial(m)/factorial(m-1)*factorial(1:m+1);
endfor

# Cargamos la matriz de combinaciones
for m = 1:grado-1
    for i = m:grado+1
        Cm(i,m) = factorial(i)/factorial(m-1)*factorial(1:m-1);
    endfor
endfor

Cn = Cm;

# Despejamos el vector de coeficientes polinomiales
An = (Cn'*(-1))'
```

Figure 2: Code in Octave
References

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