An Improvement of Massera’s Theorem
for the Liénard Equation

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Abstract

We improve the Massera’s theorem for a generalized Liénard equation. The method is easy and applied to effective examples.

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1 Introduction and Result

In 1954, the effective tool for the existence and the uniqueness of the limit cycle for a classical Liénard equation:

\[ \ddot{x} + f(x)\dot{x} + x = 0 \]  \hspace{1cm} (1)

was given by J. L. Massera [9]. It consists of a good idea and is proved from the facts that the limit cycle is star-like (for the definition see [9] or [10]) and stable. Moreover, it is a powerful and significant tool for equation (1) as is seen in the famous ”Van der Pol equation”. But, it can’t be applied to the generalized equation of (1):

\[ \ddot{x} + f(x)\dot{x} + g(x) = 0. \]  \hspace{1cm} (2)
Though the result of [1] is a generalization of Massera’s theorem, it is the case of \( g(x) = x \). Our purpose is to give for equation (2) the generalization of Proposition 1.1 below and to show the power for several examples.

Massera’s result is the following

**Proposition 1.1** Assume that the conditions

(C1) \( f(x) \) is a continuous function,
(C2) there exist \( a \) and \( b \) \((a < 0 < b)\) such that \( f(x) < 0 \) \((a < x < b)\) and \( f(x) > 0 \) \((x \leq a, x \geq b)\),
(C3) \( f(x) \) is nondecreasing as \(|x|\) increase

are satisfied. Then equation (1) has a unique limit cycle.

To generalize the above result, we assume the conditions

(C4) \( f(x) \) and \( g(x) \) are locally Lipschitz continuous functions and \( g(x)/x > 0 \),
(C5) \( g(x) = xh(x) \) where \( h(x) > 0 \) \((x \neq 0)\).

It is well-known under the above conditions (C2), (C4) and (C5) that the uniqueness of solutions of equation (2) for initial value problems is guaranteed and the only equilibrium point \((0,0)\) is unstable. For instance see [6] or [10].

Our main result is the following

**Theorem 1.2** Assume that the conditions (C2), (C4), (C5) and besides

(C6) \( f(x) \) and \( h(x) \) are nondecreasing as \(|x|\) increase,
(C7) \( \limsup_{x \to \pm\infty} [G(x) \pm F(x)] = +\infty \)

are satisfied, where \( F(x) = \int_0^x f(\xi)d\xi \) and \( G(x) = \int_0^x g(\xi)d\xi \). Then equation (2) has a unique limit cycle. Moreover, it is stable and hyperbolic.

Note that the system

\[ \dot{x} = y, \quad \dot{y} = -f(x)y - g(x) \]  

or

\[ \dot{x} = z - F(x), \quad \dot{z} = -g(x) \]

is equivalent to equation (2). This is a system called the Liénard system. It is important that the existence and the non-existence of limit cycles of system (3) or (4) coincide with those of equation (2).

Assume that there exists \( \alpha \) and \( \beta \) such that \( F(\alpha) = F(\beta) = F(0) = 0 \) and \( \alpha < 0 < \beta \) from the condition (C2). Remark that if there don’t exist \( \alpha \) and \( \beta \), then system (4) has no limit cycles.
When \( G(\alpha) \geq G(\beta) \) without loss of generality, let \( \Omega \) be the domain surrounded by the closed plane curve \((1/2)z^2 + G(x) = G(\beta)\). From the fact (see [4]) that the limit cycle of system (4) can’t exist in \( \Omega \) by the conditions (C2) and (C4), we have the following

**Corollary 1.3** Under the conditions in Theorem 1.2, a limit cycle of system (4) must exist outside \( \Omega \) and intersect both the lines \( x = \beta^* \) and \( x = \beta \), where \( \beta^* \) is a negative number such that \( G(\beta) = G(\beta^*) \).

2 Proof

We prove Theorem 1.2 on the analogy of the discussion in [9]. The outline of the proof consists of three steps.

- **Existence of closed orbit \( \Gamma \).**
  
  It is known under the conditions (C2), (C4) and (C7) that system (4) has at least one closed orbit. For instance see [2] or [3]. Thus, a closed orbit \( \Gamma \) of system (3) exists under theses conditions.

- **\( \Gamma \) is star-like.**
  
  Changing system (3) to the polar coordinate \( x = \rho \cos \theta \) and \( y = \rho \sin \theta \), the system is transformed to the differential system with respect to \( \rho \) and \( \theta \):

  \[
  \begin{aligned}
  \dot{\rho} &= \rho \cos \theta \sin \theta \{1 - h(\rho \cos \theta)\} - \rho \sin^2 \theta f(\rho \cos \theta) \\
  \dot{\theta} &= -\cos^2 \theta h(\rho \cos \theta) - \sin^2 \theta - \sin \theta \cos \theta f(\rho \cos \theta).
  \end{aligned}
  \]

  Assume that \( \Gamma \) is not star-like. Then there exists a half-ray \( \theta = \theta_0 \) which intersects \( \Gamma \) at three points \( P_1, P_2 \) and \( P_3 \) on the \( xy \)-plane. Thus, \( \theta \) must change sign twice at these points as \( \rho \) increases. By the monotonicity of \( f(x) \) and \( h(x) \), this is impossible. Therefore \( \Gamma \) is star-like.

- **\( \Gamma \) is unique.**
  
  In system (3) we consider a closed orbit \( \Gamma \) (this is star-like) and make the transformation \((x, y) \rightarrow (kx, ky)\) for \( k \in \mathbb{R}^+ \). Letting \( \Gamma(kx, ky) = \Gamma_k(x, y) \), we obtain a family of simple closed curves \( \Gamma_k \) on the plane which \( \Gamma_k \supset \Gamma \) for \( k > 1 \), \( \Gamma_k \subset \Gamma \) for \( k < 1 \) on both sides of \( \Gamma \) and \( \Gamma_1 = \Gamma \).

  Take a point \( P(x, y) \) on \( \Gamma \) and consider its tangent vector \( T(x, y) \). This is given by the slope, which in the case is

  \[
  T(x, y) = -f(x) - \frac{g(x)}{y} = -f(x) - \frac{xh(x)}{y}.
  \]

  From the similarity transformation, the tangent vector \( \tilde{T}(x, y) \) at the point \( P_k(kx, ky) \) on the closed curve \( \Gamma_k \) is parallel to the one at the point \( P \). The slope of system (3) in this point is

  \[
  T(kx, ky) = -f(kx) - \frac{g(kx)}{ky} = -f(kx) - \frac{xh(kx)}{y}.
  \]
From the monotonicity assumptions on \( f(x) \) and \( h(x) \), the slope at the point \( P_k \) is

\[
\tilde{T}(P_k) = -f(kx) - \frac{xh(kx)}{y} \leq -f(x) - \frac{xh(x)}{y} = T(P_k) \text{ if } k > 1.
\]

Similarly, we have

\[
\tilde{T}(P_k) = -f(kx) - \frac{xh(kx)}{y} \geq -f(x) - \frac{xh(x)}{y} = T(P_k) \text{ if } k < 1.
\]

This implies that the orbit \( \Gamma^+(P_k) \) of system (3) can’t move from the inside (resp. outside) of \( \Gamma_k \) to the outside (resp. inside) when \( k > 1 \) (resp. \( k < 1 \)). Thus, system (3) can’t have any closed orbit other than \( \Gamma \).

Moreover, it is seen from the mentioned facts above that the only one limit cycle \( \Gamma \) is stable and hyperbolic. □

**Remark 2.1** The case of which is not satisfied the condition (C5) or (C6) has been discussed in [5].

### 3 Applications

**Example 3.1** Our result can be applied to system (4) with

\[
F(x) = \frac{1}{42}x^7 + \frac{m + n}{15}x^6 + \frac{m^2 + 4mn + n^2}{20}x^5 + \frac{m^2n + mn^2}{6}x^4 + \frac{m^2n^2}{6}x^3 + rx
\]

\[
g(x) = px^3 + qx,
\]

where \( m < 0 < n, r < 0, p > 0 \) and \( q > 0 \).

This is equivalent to the Liénard equation \( \ddot{x} + F'(x)\dot{x} + g(x) = 0 \) in the form (2). Since \( f'(x) = F''(x) = x(x + m)^2(x + n)^2 \), we see that \( f(x) \) and \( h(x) = px^2 + q \) are nondecreasing as \( |x| \) increase. Thus, we conclude from Theorem 1.2 that the system has a unique limit cycle.

To give the same result above by another methods, for instance [5] or [10] etc., isn’t easy. This method is easy and excellent for this system.

**Example 3.2** We consider the following system called a continuous piecewise linear Liénard system (CPWL):

\[
F(x) = \begin{cases}
T_R(x - u_R) + T_C u_R & (x \geq u_R) \\
T_C x & (u_L \leq x \leq u_R) \\
T_L(x - u_L) + T_C u_L & (x \leq u_L),
\end{cases}
\]
A Massera’s theorem for the Liénard equation

\[ g(x) = \begin{cases} 
  l_R(x - v_R) + l_C v_R & (x \geq v_R) \\
  l_C x & (v_L \leq x \leq v_R) \\
  l_L(x - v_L) + l_C v_L & (x \leq v_L),
\end{cases} \]

where \( T_R > 0, T_C < 0, T_L > 0, l_R > 0, l_C > 0, l_L > 0, u_L < 0 < u_R \) and \( v_L < 0 < v_R \).

Llibre-Ordóñez-Ponce [7] gave the unique existence of the limit cycle of this system in the form \(-u_L = u_R\) and \(-v_L = v_R\). Then the solution orbits have a special character (see Remark in [7] or [8]). Our result is an improvement of it. Using Theorem 1, we have the following

**Corollary 3.3** System (CPWL) has a unique limit cycle if \( l_R \geq l_C > 0 \) and \( l_L \geq l_C > 0 \).

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**References**


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