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A Characterization of Unicyclic Graphs with the Same Independent Domination Number

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Abstract

A set D of vertices of G is an independent dominating set if no two vertices of D are adjacent and every vertex not in D is adjacent to at lest one vertex in D. The independent domination number of a graph G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set in G. A unicyclic graph is a connected graph containing exactly one cycle. For $k \geq 1$, let $\mathscr{H}(k)$ be the set of unicyclic graphs H satisfying $i(H) = k$. In this paper, we provide a constructive characterization of $\mathscr{H}(k)$ for all $k \geq 1$.

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1 Introduction

One of the famous concepts in graph theory is Domination in graphs. The domination problem is NP-complete for an arbitrary graph [3]. Domination in graphs is now well studied in graph theory. A set D of vertices of G is an independent dominating set (IDS) if no two vertices of D are adjacent and every vertex not in D is adjacent to at lest one vertex in D. The independent domination number of a graph G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set in G. If D is an IDS of G with cardinality

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$i(G)$, then we call D an i-set of G. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [2]. Recently, it was then extensively studied for various classes of graphs in the literature (see [4],[5],[6],[7]).

For $k \geq 1$, let $\mathscr{H}(k)$ be the set of unicyclic graphs H satisfying $i(H) = k$. In this paper, we provide a constructive characterization of $\mathscr{H}(k)$ for all $k \geq 1$.

2 Notations and preliminary results

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. The (open) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *closed neighborhood* $N_G[v]$ is $N_G[v] = N_G(v) \cup \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \bigcup_{v \in A} N_G(v)$ and $N_G[A] = \bigcup_{v \in A} N_G[v]$. The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex x is said to be a *leaf* of G if $\deg_G(x) = 1$. A vertex of G is a *support vertex* if it is adjacent to a leaf in G. We denote by $L(G)$, and $U(G)$ the collections of the leaves and support vertices of G, respectively. For two sets A and B, the *difference of A and B*, denoted by $A - B$, is the set of all the elements of A that are not elements of B. For a subset $A \subseteq V(G)$, the *deletion of A from G* is the graph $G - A$ obtained by removing all vertices in A and all edges incident to these vertices. A *u-v path* $P : u = v_1, v_2, \ldots, v_k = v$ of G is a sequence of k vertices in G such that $v_i v_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k-1$. For any two vertices u and v in G , the *distance between u and v*, denoted by $dist_G(u, v)$, is the minimum length of the u-v paths in G. Denote by P_n a *n-path* with n vertices. The length of P_n is n-1. For other undefined notions, the reader is referred to [1] for graph theory.

The following lemmas are useful.

Lemma 2.1. *For $n \geq 1$, $i(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. It's true for $n = 1, 2$ and 3. For $n \geq 4$, let $k = \lceil \frac{n}{3} \rceil$ and P_n : $v_1, v_2, \ldots, v_n$. Suppose $D = \{v_2, \ldots, v_{3i-1}, \ldots, v_{3k-4}, v_m\}$, where $m = 3k-2$ or $3k-1$ is an IDS of P_n, then $i(P_n) \leq |D| = (k-1) + 1 = k$.

Suppose, by contradiction, $i(P_n) = s \leq k-1$ and $D' = \{v_{i_1}, \ldots, v_{i_s}\}$ is an i-set of P_n, where $i_1 < i_2 < \ldots, i_s$. We can see that $dist(v_j, v_{j+1}) \leq 3$ for $j = 1, \ldots, s-1$. Then $n = |P_n| = |D'| + |P_n - D'| \leq s + [1 + 2(s-1) + 1] = 3s \leq 3(k-1) < n$. This is a contradiction, so $i(P_n) = k = \lceil \frac{n}{3} \rceil$. □

Lemma 2.2. *For $n \geq 3$, $i(C_n) = \lceil \frac{n}{3} \rceil$.*

Proof. It's true for $n = 3$. For $n \geq 4$, let $k = \lceil \frac{n}{3} \rceil$ and $C_n : v_1, v_2, \ldots, v_n, v_1$. Assume D is an i-set of C_n and $v_1 \in D$. Then $v_2 \notin D$ and $v_n \notin D$. Let

$P' = C_n - \{v_1, v_2, v_n\}$ and $D' = D - \{v_1\}$. Then D' is an i-set of P', where $|P'| = n - 3$. By Lemma 2.1, $|D'| = i(P') = \lceil \frac{n-3}{3} \rceil = \lceil \frac{n}{3} \rceil - 1$. Thus $i(C_n) = |D| = |D'| + 1 = (\lceil \frac{n}{3} \rceil - 1) + 1 = \lceil \frac{n}{3} \rceil$. □

Lemma 2.3. *Suppose H is obtained from $H' \in \mathscr{H}(k)$ by adding one vertex v and the edge wv, where $w \in V(H')$, then $k \leq i(H) \leq k+1$. Moreover, the followings hold.*
(i) *The graph $H \in \mathscr{H}(k+1)$ if and only if $w \notin D'$ for every i-set D' of H'.*
(ii) *The graph $H \in \mathscr{H}(k)$ if and only if $w \in D'$ for some i-set D' of H'.*

Proof. We can see that H is unicyclic. If D' is an i-set of H', then D' or $D' \cup \{v\}$ is an IDS of H. So $i(H) \leq |D'| + 1 = k + 1$. The equalities hold if and if $D' \cup \{v\}$ is an i-set of H. Thus we got (i).

If D is an i-set of H, then D, $D - \{v\}$ or $(D - \{v\}) \cup \{w\}$ is an IDS of H'. So $i(H) = |D| \geq i(H') = k$. The equalities hold if and if D_1 is an i-set of H', where $D_1 = D$ or $D_1 = (D - \{v\}) \cup \{w\}$. Note that $w \in D_1$. Thus we got (ii). □

Lemma 2.4. *Suppose H is obtained from $H' \in \mathscr{H}(k)$ by adding a $P_2 : v, v'$ and the edge wv, where $w \in V(H')$, then $H \in \mathscr{H}(k+1)$.*

Proof. We can see that H is unicyclic. Since $v' \notin N_H[V(H')]$, this means that $i(H) \geq i(H') + 1 = k + 1$. Let D' be an i-set of H'. Then $D = D' \cup \{v'\}$ is an ISD of H. So $k + 1 \leq i(H) \leq |D| = |D'| + 1 = k + 1$, thus $H \in \mathscr{H}(k+1)$. □

3 Characterization

In this section, we characterize the set $\mathscr{H}(k)$ for all $k \geq 1$. Suppose H' is a uncyclic graph and H is obtained from H' by one of the following Operations.

Operation O1. Add a new vertex v and the edge wv, where $w \in V(H')$ and $w \notin D'$ for every i-set D' of H'.

Operation O2. Add a new path P_2 and the edge wv, where $w \in V(H')$ and $v \in V(P_2)$.

Operation O3. Add a new vertex v and the edge wv, where $w \in V(H')$ and $w \in D'$ for some i-set D' of H'.

Lemma 3.1. *Let $H' \in \mathscr{H}(k-1)$. Suppose H is obtained from H' by one of the Operation O1 or Operation O2, then $H \in \mathscr{H}(k)$.*

Proof. Suppose H is obtained from some H' by the Operation Oi, where $i = 1, 2$. Then H is a unicyclic graphs.
Case 1. $i = 1$. By Lemma 2.3 (i), then $i(H) = i(H') + 1 = k$ and $H \in \mathscr{H}(k)$.
Case 2. $i = 2$. By Lemma 2.4, $i(H) = i(H') + 1 = k$. Therefore, $H \in \mathscr{H}(k)$.
By Case 1 and Case 2, $H \in \mathscr{H}(k)$. □

Lemma 3.2. *Let $H' \in \mathscr{H}(k)$. Suppose that H is obtained from H' by the Operation O3, then $H \in \mathscr{H}(k)$.*

Proof. We can see that H is unicyclic. By Lemma 2.3(ii), $k \leq i(H) \leq |D'| = i(H') = k$, thus $H \in \mathscr{H}(k)$. □

Let $\mathscr{C}(1) = \{C_3\}$ and $\mathscr{A}(1) = \{C_3\} \cup \mathscr{A}'(1)$, where $\mathscr{A}'(1)$ is the collection of graphs in Figure 1.

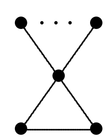

Figure 1: The collection $\mathscr{A}'(1)$ of graphs

For $k \geq 2$, we define the following collections.

(i) $\mathscr{C}(k) = \{C_{3k-2}, C_{3k-1}, C_{3k}\}$.

(ii) $\mathscr{B}(k)$ is the collection of the unicyclic graphs H which is obtained from some $H' \in \mathscr{A}(k-1)$ by one of the Operation O1 or Operation O2.

(iii) $\mathscr{A}'(k)$ is the collection of the unicyclic graphs H which is obtained from a sequence H_1, where $H_1 \in \mathscr{C}(k)$ or $H \in \mathscr{B}(k)$, $H_2, \ldots, H_m = H$ and, if $j = 1, 2, \ldots, m-1$, H_{j+1} is obtained from H_j by the Operation O3.

(iv) $\mathscr{A}(k) = \mathscr{C}(k) \cup \mathscr{B}(k) \cup \mathscr{A}'(k)$

By Lemma 2.2, we have the following lemma.

Lemma 3.3. *For $k \geq 1$, $\mathscr{C}(k) \subset \mathscr{H}(k)$.*

We first prove the following lemma.

Lemma 3.4. *For $k \geq 1$, $\mathscr{A}(k) \subseteq \mathscr{H}(k)$.*

Proof. We prove it by induction on k. It's true for $k = 1$. Assume that it's true for $k-1$, where $k \geq 2$, and $H \in \mathscr{A}(k)$. Then H is unicyclic. We consider three cases.

Case 1. $H \in \mathscr{C}(k)$. By Lemma 3.3, then $H \in \mathscr{H}(k)$.

Case 2. $H \in \mathscr{B}(k)$. Then H is obtained from some $H' \in \mathscr{A}(k-1)$ by one of the Operation O1 or Operation O2. By the hypothesis, $H' \in \mathscr{H}(k-1)$. By Lemma 3.1, $H \in \mathscr{H}(k)$.

Case 3. $H \in \mathscr{A}'(k)$. Then H is obtained from a sequence H_1, where $H_1 \in \mathscr{C}(k)$ or $H \in \mathscr{B}(k)$, $H_2, \ldots, H_m = H$ and, if $j = 1, 2, \ldots, m-1$, H_{j+1} is obtained from H_j by the Operation O3. By Case 1 and Case 2, we have that $H_1 \in \mathscr{H}(k)$. By Lemma 3.2, $i(H) = i(H_m) = i(H_{m-1}) = \cdots = i(H_1) = k$. Thus $H \in \mathscr{H}(k)$.

By Case 1, Case 2 and Case 3, we have that $H \in \mathscr{H}(k)$. □

Theorem 3.5 is the main theorem.

Theorem 3.5. *For $k \geq 1$, $\mathscr{A}(n) = \mathscr{H}(n)$.*

Proof. By Lemma 3.4, we need only prove that $\mathscr{H}(k) \subseteq \mathscr{A}(k)$ for all $k \geq 1$ and it is proved by contradiction. Suppose $H \in \mathscr{H}(k)$ and $H \notin \mathscr{A}(k)$ such that $|H|$ is as small as possible. Let C be the cycle of H. By Lemma 3.3, then $H \neq C$ and $L(H) \neq \emptyset$. Let x be a leaf of H and w be the neighbor of x. Then $H' = H - \{x\}$ is unicyclic. By Lemma 2.3, $k-1 \leq i(H') \leq k$.

Case 1. $i(H') = k$.

Then $H' \in \mathscr{H}(k)$. Since $|H'| < |H|$, by the hypothesis, $H' \in \mathscr{A}(k)$. Since $i(H) = i(H')$, by Lemma 2.3 (ii), $w \in D'$ for some i-set D' of H'. Thus H is obtained from $H' \in \mathscr{A}(k)$ by the Operation O3, it means that $H \in \mathscr{A}(k)$. This is a contradiction.

Case 2. $i(H') = k-1$.

Then $H' \in \mathscr{H}(k-1)$. Since $|H'| < |H|$, by the hypothesis, $H' \in \mathscr{A}(k-1)$. Since $i(H) = i(H') + 1$, by Lemma 2.3 (i), $w \notin D'$ for every i-set D' of H'. Thus H is obtained from $H' \in \mathscr{A}(k-1)$ by the Operation O1, it means that $H \in \mathscr{B}(k)$. So $H \in \mathscr{A}(k)$, this is a contradiction.

By Case 1 and Case 2, $\mathscr{H}(k) \subseteq \mathscr{A}(k)$ for all $k \geq 1$. We complete the proof. □

Hence we provide a constructive characterization $\mathscr{A}(k)$ of $\mathscr{H}(k)$ for all $k \geq 1$.

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