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Articulation Point and Weak Point in a Space and in a Network in Pretopology: the Case of the Connectivity

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Abstract

In this paper, we present the concepts of articulation point and weak point in a space and in a network in Pretopology. We show, in the case of connectivity, some properties allowing to locate them in a network.

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1 Introduction

We have already shown ([8]), in the case of strong connectivity, how to define the concepts of articulation point and weak point in Pretopology. We have also established properties in order to be able to locate them in a network. The aim of this paper is the same but in the case of the connectivity.

2 Different Types of Pretopological Spaces (see [2], [3])

Definition 1. Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping a from P(X) onto P(X) such as :

$$a(\emptyset) = \emptyset$$
$$\forall \ \mathrm{A} \subset \mathrm{X},\ \mathrm{A} \subset a(\mathrm{A})$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1- (X, a) is a V type pretopological space if and only if

$$\forall \ \mathrm{A} \subset \mathrm{X}, \forall \ \mathrm{B} \subset \mathrm{X},\ \mathrm{A} \subset \mathrm{B} \Rightarrow a(\mathrm{A}) \subset a(\mathrm{B}).$$

2- (X, a) is a V_D type pretopological space if and only if

$$\forall \ \mathrm{A} \subset \mathrm{X}, \forall \ \mathrm{B} \subset \mathrm{X},\ a(\mathrm{A} \cup \mathrm{B}) = a(\mathrm{A}) \cup a(\mathrm{B}).$$

3- (X, a) is a V_S type pretopological space if and only if

$$\forall \ \mathrm{A} \subset \mathrm{X},\ a(A) = \bigcup_{x \in A} a(\{x\}).$$

4- (X, a) a V_D type pretopological space, is a topological space if and only if

$$\forall \ \mathrm{A} \subset \mathrm{X},\ a(a(\mathrm{A})) = a(\mathrm{A}).$$

Property 2. *If (X, a) is a V_S space then (X, a) is a V_D space. If (X, a) is a V_D space then (X, a) is a V space.*

Example 3. Let X be a non empty set and R be a binary relationship defined on X.

The pretopology of descendants, noted a_d, is defined by :
$\forall$ A $\subset$ X, a_d(A) = { x $\in$ X/ R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R(x) = { y $\in$ X/ x R y }.

The pretopology of ascendants, noted a_a, is defined by :
$\forall$ A $\subset$ X, a_a(A) = { x $\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R^{-1}(x) = { y $\in$ X/ y R x }.

These pretopologies are V_S ones.

The pretopology of ascendant-descendants, noted a_{ad}, is defined by :
$\forall$ A $\subset$ X, a_{ad}(A) = { x $\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ and R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A.

This pretopology is only V one.

3 Different Pretopological Spaces Defined from a Space (X, a) and Closures (see [2], [3])

Definition 4. Let (X, a) be a V pretopological space. Let A $\subset$ X. A is a closed subset if and only if a (A) = A.

We note $\forall$ A $\subset$ X, a^0(A) = A and $\forall$ n, n$\geq$ 1, a^n(A) = $a(a^{n-1})$(A).

We name closure of A the subset of X, denoted F$_a$(A), which is the smallest closed subset which contains A.

Remark 5. F$_a$(A) is the intersection of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a V space , nor a V_D space, nor a V_S space, nor a topological space), the closure may not exist.

Proposition 6. *Let (X, a) be a V pretopological space. Let A $\subset$ X. If one of the two following conditions is fulfilled :*

- *X is a finite set*
- *a is of V_S type*

then $F_a(A) = \bigcup_{n\geq 0} a^n(A)$.

Remark 7. If a is of V type then a^n and F$_a$ also are of V type. If a is of V_S type then a^n and F$_a$ are also of V_S type.

Definition 8. Let (X, a) be a V pretopological space. Let A $\subset$ X. We define the induced pretopology on A by a, denoted a_A, by : $\forall$ C $\subset$ A, a_A(C) = a(C) $\cap$ A.

(A, a_A) (or more simply A) is said pretopological subspace of (X, a).

We note F$_{aA}$ the closing according to a_A and (F$_a$)$_A$ the closing obtained by restriction of closing F$_a$ on A. (F$_a$)$_A$ is such as $\forall$ C $\subset$ A, (F$_a$)$_A$(C) = F$_a$(C) $\cap$ A.

4 Connectivity in (X, a) (see [2], [3], [7], [9], [10])

Definition 9. Let (X, a) be a V pretopological space.

(X, a) is connected if and only if $\forall$ C $\subset$ X, C $\neq$ ø, F$_a$(C) = X or F$_a$(X - F$_a$(C)) $\cap$ F$_a$(C) $\neq$ ø.

Definition 10. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.

A is a connected subset of (X, a) if and only if A endowed with $(\mathrm{F}_a)_A$ is connected.

A is a connected component of (X, a) if and only if A is a connected subset of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X with A $\neq$ B, B is not a connected subset of (X, a).

A is a connected subspace of (X, a) if and only if (A, a_A), as a pretopological space, is connected.

A is a greatest connected subspace of (X, a) if and only if (A, a_A) is a connected subspace of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X and A $\neq$ B, (B, a_B) is not a connected subspace of (X, a).

Proposition 11 ([2])**.** *Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.*

i- If A is a connected subspace of (X, a) then A is a connected subset of (X, a).

ii- If one of the following three conditions is fulfilled :
** A is a closed subset of X for a*
** complementary of A in X is closed for a and (X, a) is a V_D space*
** a is idempotent*

Then A is a connected subspace of (X, a) $\Leftrightarrow$ A is a connected subset of (X, a).

Remark 12. In general, the converse of i- is not true.

Example 13. Let (X, a) be a pretopological space with X = { a, b, c, d } and a pretopology of descendants defined by the following graph 1 :

x	R(x)
a	{ c }
b	ø
c	{ d }
d	{ b }

Graph 1

Let A = { a, b }. A is a connected subset of (X, a) but A is not a connected subspace of (X, a).

Proposition 14 ([9]). *Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.*

A is a connected component of (X, a) $\Leftrightarrow$ A is a greatest connected subspace of (X, a).

Remark 15 ([10]). Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.

In general, if A is a connected subset of (X, a) then $\forall$ B $\neq \emptyset$, B $\subset$ A, B is not a connected subset of (X, a).

Definition 16 ([3], [5]). Let X a non empty set. Let a_1 and a_2 two pretopologies on X.

a_1 is thinner than a_2 if and only if $\forall$ C $\subset$ X, a_1(C) $\subset$ a_2(C).

Proposition 17. *Let X a non empty set. Let a_1 and a_2 two V type pretopologies on X such as a_1 thinner than a_2. Let $A \subset X$ with A non empty.*

If A is a connected subspace of (X, a_1) then A is a connected subspace of (X, a_2).

Proof.

If A is a connected subspace of (X, a_1)

Then $\forall$ C $\subset$ A, C $\neq \emptyset$, (1) F_{a1A}(C) = A or (2) F_{a1A}(A - F_{a1A}(C)) $\cap$ F_{a1A}(C) $\neq \emptyset$.

Let us show that $\forall$ C $\subset$ A, C $\neq \emptyset$, (1') F_{a2A}(C) = A or (2') F_{a2A}(A - F_{a2A}(C)) $\cap$ F_{a2A}(C) $\neq \emptyset$.

F_{a2A}(C) $\subset$ A so F_{a2A}(C) satisfies (1) or (2)

So we have (1)F_{a1A}(F_{a2A}(C)) = A or (2) F_{a1A}(A - F_{a1A}(F_{a2A}(C))) $\cap$ F_{a1A}(F_{a2A}(C)) $\neq \emptyset$.

But F_{a1A}(F_{a2A}(C)) = F_{a2A}(C) because F_{a2A}(C) $\subset$ F_{a1A}(F_{a2A}(C)) (F_{a1A} is a pretopology) and F_{a1A}(F_{a2A}(C)) $\subset$ F_{a2A}(F_{a2A}(C)) = F_{a2A}(C)) (F_{a1A} is thinner than F_{a2A} [2] and Proposition 37-i of [5], F_{a2A}(C) is closed for F_{a2A})

If (1) is verified then F_{a1A}(F_{a2A}(C)) = A

then F_{a2A}(C)) = A and (1') is verified.

If (2) is verified then F_{a1A}(A - F_{a1A}(F_{a2A}(C))) $\cap$ F_{a1A}(F_{a2A}(C)) $\neq \emptyset$

Then F_{a2A}(A - F_{a2A}(C)) $\cap$ F_{a2A}(C) $\neq \emptyset$ (F_{a1A} is thinner than F_{a2A} [2] and Proposition 37-i of [5])

And (2') is verified.

5 Articulation point and weak point in a pretopological space : the case of the Connectivity

We have shown ([8]) that we cannot claim to decompose a strongly connected component A of a pretocological space (X, a) by finding its articulation points (where a point b of A is said articulation point of A if A - {b} is not a strongly connected subset of (X, a)). Indeed, the set of articulation points of A would be empty.

In the case of connectivity, we have already shown ([10]) that a subset of a connected subset may not be connected, we could therefore decompose the connected components by generalizing the concept of articulation point in the case of a connected subset of A (where a point b of A is said articulation point of A if A - {b} is not a connected subset of (X, a)).

Unlike the case of strong connectivity, the generalization in Pretopology of the notion of articulation point can be done from pretopological subspace or from pretopological subset. But we will seek here to apply this notion only to the greatest connected subspaces. Anyway, according to Proposition 14, the greatest connected subspaces are equivalent to the connected components, so the same contexts will be decomposed.

Remark 18 ([8]). Let (X, a) be a V pretopological space. Let A $\subset$ X. Let B $\subset$ X with A $\subset$ B.

$(a_B)_A = a_A$.

Proposition 19. *Let (X, a) be a V pretopological space. Let A $\subset$ X. Let B $\subset$ X with A $\subset$ B.*

A is a connected subspace of (X, a) $\Leftrightarrow$ A is a connected subspace of (B, a_B).

Proof.

We note $F_{(aB)A}$ the closing according to $(a_B)_A$.

A is a connected subspace of (B, a_B)

$\Leftrightarrow \forall$ C $\subset$ A, C $\neq \emptyset$, $F_{(aB)A}$(C) = A or $F_{(aB)A}$(A - $F_{(aB)A}$(C)) $\cap$ $F_{(aB)A}$(C) $\neq \emptyset$ (by definition)

$\Leftrightarrow \forall$ C $\subset$ A, C $\neq \emptyset$, F_{aA}(C) = A or F_{aA}(A - F_{aA}(C)) $\cap$ F_{aA}(C) $\neq \emptyset$ (Remark 18)

$\Leftrightarrow$ A is a connected subspace of (X, a).

Definition 20. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty and A connected subspace of (X, a). Let b $\in$ A with A $\neq$ { b }.

i- b is an articulation point of A in (X, a) if and only if (A - { b }, $a_{A-\{b\}}$) is not a connected subspace of (X, a).

ii- Let k a natural number with k $\neq$ 0. Let b an articulation point of A in (X, a).

b is a k order articulation point of A in (X, a) if and only if the smallest of the greatest connected subspaces of (A - { b }, $a_{A-\{b\}}$) has k as cardinal.

Remark 21. Let (X, a) be a V pretopological space. Let A $\subset$ X. Let B $\subset$ X with A $\subset$ B. Let k a natural number with k $\neq$ 0.

i- b is an articulation point of A in (X, a) $\Leftrightarrow$ b is an articulation point of A in (B, a_B).

ii- b is a k order articulation point of A in (X, a) $\Leftrightarrow$ b is a k order articulation point of A in (B, a_B).

Proof.

i- and ii- Obvious by Remark 18 and Proposition 19.

Remark 22. Let (X, a) be a V pretopological space. Let A $\subset$ X. Let B $\subset$ X with A $\subset$ B. Let k a natural number with k $\neq$ 0.

We note AP(A) the set of all articulation points of A in (B, a_B) and k-AP(A) the set of all k order articulation points of A in (B, a_B).

We have k-AP(A) $\subset$ AP(A) (by definition).

Definition 23. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty and A connected subspace of (X, a). Let c $\in$ A.

c is a weak point of A in (X, a) if and only if it exists b $\in$ A - { c } with b $\in$ 1-AP(A) in (X, a) and { c } greatest connected subspace of (A - { b }, $a_{A-\{b\ \}}$).

Remark 24. Let (X, a) be a V pretopological space. Let A $\subset$ X. Let B $\subset$ X with A $\subset$ B.

c is a weak point of A in (X, a) $\Leftrightarrow$ c is a weak point of A in (B, a_B).

Proof.

Obvious by Remark 18 and Proposition 19.

Remark 25. Let (X, a) be a V pretopological space. Let A $\subset$ X. Let B $\subset$ X with A $\subset$ B.

We note WP(A) the set of all weak points of A in (B, a_B).

We have 1-AP(A) $\neq$ ø $\Leftrightarrow$ WP(A) $\neq$ ø (by definition).

6 Articulation point and weak point in a network in pretopology : the case of the Connectivity

We will seek here to establish the relation between the set of articulation points (respectively the set of weak points) of a greatest connected subspace of a network studied by the union of the pretopologies (or by the composition of pretopologies) and the set of articulation points (respectively the set of weak points) of the greatest connected subspaces of the different spaces of the network.

But we can show (see Propositions 34 and 40 in [7]) that, in the case where the pretopologies constituting the network are of V type without being of V_S type, establish a relation between the set of articulation points of a greatest connected subspace of the network and of the set of articulation points of the greatest connected subspaces of the separately studied spaces requires to take into account an iterative process of construction of a greatest connected subspace. Indeed, an articulation point can appear in one or the other of the steps. It would therefore be necessary to build as many conditions as there are steps and therefore to make the relationship sought more complex.

This work may seem superfluous in the sense that the results would not appear very pragmatic. Also, we will only develop the case where all the pretopologies forming the network are of V_S type.

Definition 26. Let X a non empty set. Let I a countable family of indices. The family { (X, a_i), $i \in$ I } of pretopological spaces is a network on X.

Definition 27. Let X a non empty set. For any pretopologies a_1 and a_2 defined on X, for any subset A of X, we define the two following mappings :

$(a_1 \cup a_2)$(A) = a_1(A) $\cup$ a_2(A) [union of pretopologies]

$(a_1 \odot a_2)$(A) = $a_1(a_2$(A)) [composition of pretopologies]

More generally, in a network { (X, a_i), $i \in$ I } such as for any $i \in$ I, a_i is of V type, we note $F_\cup$ the closure according to $\bigcup_{i \in I} a_i$.

We define the mapping, denoted $\prod_{i \in I} a_i$, from P(X) onto P(X) by :

$\forall$ A $\subset$ X, $\prod_{i \in I} a_i(A)$= { x $\in$ X/ there exists $n \in$ I such as x $\in a_n(a_{n-1}($ ... (a_1(A))...)) } and we denote $F_{\prod}$ the closure according to $\prod_{i \in I} a_i$.

For any subset A of X, we note $F_{(\cup)A}$ the closure according to $(\bigcup_{i \in I} a_i)_A$, $F_{(\prod)A}$ the closure according to $(\prod_{i \in I} a_i)_A$, $F_{\cup A}$ the closure according to $\bigcup_{i \in I} a_{iA}$.

Proposition 28. *Let { (X, a_i), $i \in I$ } be a network on X such as for any $i \in I$, a_i is of V type. Let $A \subset X$ with A non empty.*

If A is a connected subspace of (X, $\bigcup_{i \in I} a_i$) then A is a connected subspace of (X, $\prod_{i \in I} a_i$).

Proof.

If A is connected subspace of (X, $\bigcup_{i \in I} a_i$)

Then A is connected subspace of (A, $(\bigcup_{i \in I} a_i)_A$) (Proposition 19)

Then A is connected subspace of (A, $\bigcup_{i \in I} a_{iA}$) (Proposition 17-i of [5])

And then A is connected subspace of (A, $\prod_{i\in I} a_{iA}$) (Propositions 33-i and 30-i of [5]).

But $\prod_{i\in I} a_{iA}$ is thinner than $(\prod_{i\in I} a_i)_A$ (Proposition 24-i of [5])

So A is connected subspace of (A, $(\prod_{i\in I} a_i)_A$) (Proposition 17)

And then A is connected subspace of (X, $\prod_{i\in I} a_i$) (Proposition 19).

Remark 29. In general, the converse is not true.

Example 30. Let { (X, a_i), $i \in$ I } be a network on X with X = { a, b, c}, I = { 1, 2 }, a_1 and a_2 prétopologies of descendants defined respectively by the following graph 2 and graph 3 :

x	R(x)
a	ø
b	ø
c	{ a }

Graph 2

x	R(x)
a	ø
b	{ c }
c	ø

Graph 3

Let A = { a, b }. A is a connected subspace of (X, $\prod_{i\in I} a_i$).

Indeed, $(\prod_{i\in I} a_i)_A$({ a }) = $a_2(a_1$({ a })) ∩ A = a_2({ a, c }) ∩ A = X ∩ A = A

So $F_{(\prod)A}$({ a }) = A

And then $(\prod_{i\in I} a_i)_A$({ b }) = $a_2(a_1$({ b })) ∩ A = a_2({ b }) ∩ A = { b }

So $F_{(\prod)A}$({ b }) = { b } with $F_{(\prod)A}$({ a }) ∩ $F_{(\prod)A}$({ b }) = { b } ≠ ø.

But A is not a connected subspace of (X, $\bigcup_{i\in I} a_i$).

Indeed, $(\bigcup_{i\in I} a_i)_A$({ a }) = $(a_1 \cup a_2)$({ a }) ∩ A = { a, c } ∩ A = { a }

So $F_{\cup A}$({ a }) = { a }

And $(\bigcup_{i\in I} a_i)_A$({ b }) = $(a_1 \cup a_2)$({ b }) ∩ A = { b } ∩ A = { b }

So $F_{\cup A}$({ b }) = { b }.

Finally, we have, $F_{\cup A}$({ a }) ≠ A and $F_{\cup A}$(A- $F_{\cup A}$({ a })) ∩ $F_{\cup A}$({ a }) = $F_{\cup A}$({ b }) ∩ { a } = ø.

Proposition 31 ([7]) *Let { (X, a_i), $i \in I$ } a network such as for any $i \in I$, a_i is of V type. Let { S_k, $k \in K$ } a family of subsets non empty of X such as :*

1- $\bigcup_{k\in K} S_k = X$

2- $\forall$ $k \in K$, there exists $\{ A_j, j \in J \}$ a family of subsets non empty of X such as :

2-1- $S_k = \bigcup_{j\in J} A_j$

2-2- $\forall$ $j \in J$, there exists $i \in I$, A_j connected component of (X, a_i)

2-3- $\forall$ $j \in J$, $\forall$ $j' \in J$, there exists a sequence $j_0 \ldots j_r$ of elements of J such as $j_0 = j$, $j_r = j'$ and $\forall$ $l = 0,\ldots,r\text{-}1$, $A_{jl} \cap A_{jl+1} \neq \emptyset$

2-4- $\forall$ $A' \subset X$, $A' \notin \{ A_j, j \in J \}$, if there exists $i \in I$ such as A' connected component of (X, a_i) then $A' \cap S_k = \emptyset$.

We have :

i- $\forall$ $k \in K$, S_k is a connected subset of $(X, \bigcup_{i\in I} a_i)$.

ii- If for any $i \in I$, a_i is of V_S type, $\{ S_k, k \in K \}$ is the family of connected component of $(X, \bigcup_{i\in I} a_i)$.

iii- $\{ S_k, k \in K \}$ is a partition of X.

Proposition 32. *Let $\{ (X, a_i), i \in I \}$ be a network on X such as for any $i \in I$, a_i is of V_S type. Let $\{ S_k, k \in K \}$ a family of subsets non empty of X which satisfied the conditions of the Proposition 31.*

Let $k \in K$, let $b \in S_k$ with $S_k \neq \{ b \}$.

If $b \in AP(S_k)$ in $(X, \bigcup_{i\in I} a_i)$ (respectively in $(X, \prod_{i\in I} a_i)$) then we have

i - it exists $i \in I$, it exists $A \subset S_k$ with A greatest connected subspace of (X, a_i) and $b \in AP(A)$ in (X, a_i)

Or ii - it exists $A \subset S_k$, it exists $A' \subset S_k$, it exists $i \in I$, it exists $i' \in I$ with A greatest connected subspace of (X, a_i) and $b \in A$ and $A \neq \{ b \}$ and A' greatest connected subspace of $(X, a_{i'})$ and $b \in A'$ and $A' \neq \{ b \}$ and $A \cap A' = \{ b \}$.

Proof.

Let us show that if (i) and (ii) are not satisfied then b $\notin$ AP(S_k).

Let b $\in$ S_k.

It exists i $\in$ I, it exists A $\subset$ S_k, A greatest connected subspace of (X, a_i) with b $\in$ A.

Two cases are possible :

1- It exists i $\in$ I, it exists A $\subset$ S_k, A greatest connected subspace of (X, a_i) with b $\in$ A and A $\neq$ { b }

(i) is not satisfied so

$\forall$ i $\in$ I, $\forall$ A $\subset$ S_k, A greatest connected subspace of (X, a_i) with b $\in$ A implies b $\notin$ AP(A) in (X, a_i)

Then A - { b } is a connected subspace of (X, a_i) (by definition of AP(A))

So A - { b } is a connected subspace of (X, $\bigcup_{i\in I} a_i$) (respectively of (X, $\prod_{i\in I} a_i$)) (Proposition 17 and proposition 28).

1-1 If A = S_k

then A - {b } = S_k - {b }

and then S_k - {b } is a connected subspace of (X, $\bigcup_{i\in I} a_i$) (respectively of (X, $\prod_{i\in I} a_i$)).

And then b $\notin$ PA(S_k) in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)).

1-2 If A $\subset$ S_k and A $\neq$ S_k

(ii) is not satisfied so

$\forall$ A $\subset$ S_k, $\forall$ A' $\subset$ S_k, $\forall$ i $\in$ I, $\forall$ i' $\in$ I,

A greatest connected subspace of (X, a_i) with b $\in$ A and A $\neq$ {b } and A' greatest connected subspace of (X, $a_{i'}$) with b $\in$ A' and A' $\neq$ {b } implies A $\cap$ A' $\neq$ {b }.

But we have A - {b } connected subspace of (X, $\bigcup_{i\in I} a_i$) (respectively of (X, $\prod_{i\in I} a_i$)) and A' - {b } connected subspace of (X, $\bigcup_{i\in I} a_i$) (respectively of (X, $\prod_{i\in I} a_i$)).

As A $\cap$ A' $\neq$ {b }, we have (A - {b }) $\cap$ (A' - {b }) $\neq$ ø

so (A - {b }) $\cup$ (A' - {b }) is a connected subspace of (X, $\bigcup_{i\in I} a_i$) (respectively of (X, $\prod_{i\in I} a_i$)) (the union of two connected subspaces which have an intersection non empty is a connected subspace [2]).

then, S_k - {b } is a connected subspace of (X, $\bigcup_{i\in I} a_i$) (respectively of (X, $\prod_{i\in I} a_i$))

And then b $\notin$ PA(S_k) in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)).

2- $\forall$ i $\in$ I, $\forall$ A $\subset$ S_k, A greatest connected subspace of (X, a_i) with b $\in$ A implies card(A) = 1

In this case, S_k = A = {b }

And then b $\notin$ PA(S_k) in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)) (by definition).

Proposition 33. *Let { (X, a_i), i $\in$ I } be a network on X such as for any i $\in$ I, a_i is of V_S type. Let { S_k, k $\in$ K } a family of subsets non empty of X which satisfied the conditions of the Proposition 31.*

Let k $\in$ K, let b $\in$ S_k with S_k $\neq$ { b }.

If b $\in$ 1-AP(S_k) in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)) then we have

1- it exists i $\in$ I, it exists A $\subset$ S_k with A greatest connected subspace of (X, a_i) and b $\in$ 1-AP(A) in (X, a_i)

Or 2- it exists A $\subset$ S_k, it exists A' $\subset$ S_k, it exists i $\in$ I, it exists i' $\in$ I with A greatest connected subspace of (X, a_i) and b $\in$ A and card(A) = 2 and A' greatest connected subspace of (X, $a_{i'}$) and b $\in$ A' and card(A') $\geq$ 2 and A $\cap$ A' = { b }.

Proof.

Let us show that if (1) and (2) are not satisfied then b $\notin$ 1-AP(S_k).

Let b $\in$ S_k.

It exists i $\in$ I, it exists A $\subset$ S_k, A greatest connected subspace of (X, a_i) with b $\in$ A.

Three cases are possible :

1- Card(A) $>$ 2

We have b $\notin$ 1-AP(A) in (X, a_i) ((1) is not satisfied)

So $\forall$ c $\in$ A - { b }, { c } is not greatest connected subspace of (A - { b }, $a_{iA-\{b\ \}}$) (by definition)

Then $\forall$ c $\in$ A - { b }, it exists C $\subset$ A - { b } with { c } $\subset$ C and { c } $\neq$ C and C greatest connected subspace of (A - { b }, $a_{iA-\{b\ \}}$)

Then $\forall$ c $\in$ A - { b }, it exists C $\subset$ A - { b } with { c } $\subset$ C and { c } $\neq$ C and C connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)) (Propositions 19, proposition 17-i of [5] , proposition 28)

So $\forall$ c $\in$ A - { b }, { c } is not greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

Finally, if b $\in$ 1-AP(S_k) in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)) then it exists c $\in$ S_k - A, { c } greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

Then it exists i' $\in$ I, A' $\subset$ S_k and A' greatest connected subspace of (X, $a_{i'}$) with c $\in$ A' and A' $\neq$ { c }.

If b $\notin$ A' then A' is connected subspace of (X, $\bigcup_{i\in I} a_i$) (respectively of (X, $\prod_{i\in I} a_i$)) (Proposition 17, proposition 28)

So A' is a connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)) (Propositions 19, proposition 17-i of [5] , proposition 28)

And { c } is not greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$))

So b $\in$ 1-AP(S_k) implies b $\in$ A'.

If card(A') $>$ 2 then as b $\notin$ 1-AP(A') in (X, $a_{i'}$) ((1) is not satisfied), we have $\forall$ c' $\in$ A' - { b }, {c' } is not greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

So we have card(A') = 2

Then b $\in$ 1-AP(S_k) implies card(A') = 2 and A $\cap$ A' = { b }.

And the result because (2) is not satisfied.

2- Card(A) = 2

2-1 If A = S_k

In this case, b $\notin$ 1-AP(S_k) (by definition).

2-2 - If A $\subset$ S_k and A $\neq$ S_k

2-2-1 If it exists i' $\in$ I, it exists A' $\subset$ S_k with A' greatest connected subspace of (X, $a_{i'}$) and b $\in$ A' and card(A') $>$ 2

Then b $\notin$ 1-AP(S_k) (see the case 1- applied to A').

2-2-2 If $\forall$ A' $\subset$ S$_k$, $\forall$ i' $\in$ I, A' greatest connected subspace of (X, $a_{i'}$) and b $\in$ A' implies card(A') $\leq$ 2

If b $\in$ 1-AP(S$_k$) then is exists c $\in$ S$_k$ - { b }, { c } greatest connected subspace of (S$_k$ - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of (S$_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

2-2-2-1 If c $\in$ S$_k$ - A

Il exists i" $\in$ I, it exists A" $\subset$ S$_k$, A" greatest connected subspace of (X, $a_{i''}$) with c $\in$ A" and A" $\neq$ { c }.

If b $\notin$ A" then { c } is not greatest connected subspace of (S$_k$ - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of (S$_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)) (see the case 1)

So b $\in$ 1-AP(S$_k$) implies card(A") = 2 and A $\cap$ A" = { b }.

And the result because (2) is not satisfied.

2-2-2-2 If c $\in$ A

Then { c } = A - { b }.

If it exists i' $\in$ I, it exists A' $\subset$ S$_k$ with A' $\not\subset$ A and A' greatest connected subspace of (X, $a_{i'}$) and A - { b } $\subset$ A'

Then { c } is not greatest connected subspace of (S$_k$ - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of (S$_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)) (see the case 2-2-2-1 because b $\notin$ A')

So if b $\in$ 1-AP(S$_k$) then

$\forall$ i' $\in$ I, $\forall$ A' $\subset$ S$_k$ with A' $\not\subset$ A, if A' greatest connected subspace of (X, $a_{i'}$) then A - { b } $\not\subset$ A'

Then, as A $\subset$ S$_k$ and A $\neq$ S$_k$, it exists i' $\in$ I, it exists A' $\subset$ S$_k$ with A' $\not\subset$ A and A' greatest connected subspace of (X, $a_{i'}$) and b $\in$ A' and card(A') = 2.

Finally, if b $\in$ 1-AP(S$_k$) then it exists i' $\in$ I, it exists A' $\subset$ S$_k$ with A' greatest connected subspace of (X, $a_{i'}$) and b $\in$ A' and card(A') = 2 and A $\cap$ A' = { b }.

And the result because (2) is not satisfied.

3- Card(A) = 1

In this case, A = { b }.

Two cases are possible :

3-1 If A = S$_k$

Then b $\notin$ 1-AP(S$_k$) by definition.

3-2 If A $\subset$ S$_k$ and A $\neq$ S$_k$

In this case, it exists i'$\in$ I, it exists A' $\subset$ S$_k$ with card(A') $\geq$ 2 and A $\subset$ A' and A' greatest connected subspace of (X, $a_{i'}$)

And the result accordind to the cases 1 and 2 applied to A'.

Proposition 34. *Let { (X, a_i), $i \in I$ } be a network on X such as for any $i \in I$, a_i is of V_S type. Let { S_k, $k \in K$ } a family of subsets non empty of X which satisfied the conditions of the Proposition 31.*

Let $k \in K$, let $c \in S_k$.

If $c \in WP(S_k)$ in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)) then we have

1- it exists $i \in I$, it exists $A \subset S_k$ with A greatest connected subspace of (X, a_i) and $c \in A$ and $c \in WP(A)$ in (X, a_i)

Or 2- it exists $A \subset S_k$ with A = { b, c }, it exists A' $\subset S_k$, it exists $i \in I$, it exists i' $\in I$ with A greatest connected subspace of (X, a_i) and A' greatest connected subspace of (X, $a_{i'}$) and $b \in$ A' and card(A') ≥ 2 and $A \cap$ A' = { b }.

Proof.

Let us show that if (1) and (2) are not satisfied then c $\notin$ WP(S_k).

Let c $\in$ S_k.

It exists i $\in$ I, it exists A $\subset$ S_k, A greatest connected subspace of (X, a_i) with c $\in$ A

Three cases are possible :

1- Card(A) > 2

(1) is not satisfied so c $\notin$ WP(A) in (X, a_i).

So $\forall$ b $\in$ A - { c }, { c } is not greatest connected subspace of (A - { b }, $a_{iA-\{b\ \}}$)

Then $\forall$ b $\in$ A - { c }, it exists C $\subset$ A - { b } with { c } $\subset$ C and { c } $\neq$ C and C greatest connected subspace of (A - { b }, $a_{iA-\{b\ \}}$)

So $\forall$ b $\in$ A - { c }, it exists C $\subset$ A - { b } with { c } $\subset$ C and { c } $\neq$ C and C connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)) (Propositions 19, proposition 17-i of [5] , proposition 28)

Then $\forall$ b $\in$ A - { c }, { c } is not greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

Finally, if c $\in$ WP(S_k) in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)) then it exists b $\in$ S_k - A, { c } greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

But A is greatest connected subspace of (X, a_i) with c $\in$ A and A $\neq$ { c }

So A is connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$))(Propositions 19, proposition 17-i of [5] , proposition 28)

And { c } is not greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

2- Card(A) = 2

2-1 If A = S_k

Then c $\notin$ WP(S_k) by definition.

2-2 - If A $\subset$ S_k and A $\neq$ S_k

2-2-1 If it exists i' $\in$ I, it exists A' $\subset$ S_k with A' greatest connected subspace of (X, $a_{i'}$) and c $\in$ A' and card(A') > 2

Then c $\notin$ WP(S_k) (see the case 1- applied to A').

2-2-2 If $\forall$ A' $\subset$ S_k, $\forall$ i' $\in$ I, A' greatest connected subspace of (X, $a_{i'}$) and c $\in$ A' implies card(A') $\leq$ 2

In this case, let b $\in$ S_k - { c }.

If b $\notin$ A then, as A is greatest connected subspace of (X, a_i), A is connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$))(Propositions 19, proposition 17-i of [5] , proposition 28)

So { c } is not greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

Finally, if c $\in$ WP(S_k) in (X, $\bigcup_{i\in I} a_i$) (respectively in (X, $\prod_{i\in I} a_i$)) then it exists b $\in$ A - { c }, { c } greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$)).

If it exists i' $\in$ I, it exists A' $\subset$ S_k with A' $\not\subset$ A and A' greatest connected subspace of (X, $a_{i'}$) and c $\in$ A'

Then card(A') = 2 and A $\cap$ A' = { c } and b $\notin$ A'

So A' connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$))

Then { c } is not greatest connected subspace of (S_k - { b }, $\bigcup_{i\in I} a_{iSk-\{\ b\ \}}$) (respectively of ($S_k$ - { b }, $(\prod_{i\in I} a_i)_{Sk-\{b\ \}}$))

So if c $\in$ WP(S_k) then

$\forall$ i' $\in$ I, $\forall$ A' $\subset$ S_k with A' $\not\subset$ A, if A' greatest connected subspace of (X, $a_{i'}$) then c $\notin$ A'

Then, as A $\subset$ S_k and A $\neq$ S_k, it exists i' $\in$ I, it exists A' $\subset$ S_k with A' greatest connected subspace of (X, $a_{i'}$) and A $\cap$ A' = { b } and card(A') $\geq$ 2.

And the result because (2) is not satisfied.

3- Card(A) = 1

In this case, A = { c }.

Two cases are possible :

3-1 If A = S_k

Then c $\notin$ WP(S_k) by definition.

3-2 If A $\subset$ S_k and A $\neq$ S_k

In this case, it exists i' $\in$ I, it exists A' $\subset$ S_k with card(A') $\geq$ 2 and A $\subset$ A' and A' greatest connected subspace of (X, $a_{i'}$)

And the result according to the cases 1 and 2 applied to A'.

Remark 35. In general, the converses of the Propositions 32, 33 and 34 are not true.

Example 36. Let { (X, a_i), i $\in$ I } be a network with X = { a, b, c, d }, I = { 1, 2 }, a_1 and a_2 pretopologies of descendants defined respectively by the following graph 4 and graph 5 :

x	R(x)
a	{ b }
b	{a, c }
c	{ b, d }
d	{ c }

Graph 4

x	R(x)
a	{ d}
b	{ c, d }
c	{ b }
d	{ b, a }

Graph 5

X is greatest connected subspace of (X, a_1) and X greatest connected subspace of (X, a_2) with b ∈ 1-AP(X) in (X, a_1) and b ∈ 1-AP(X) in (X, a_2).

So X is greatest connected subspace of (X, $a_1 \cup a_2$) (respectively of (X, $a_1 \odot a_2$)) but b is not articulation point of X in (X, $a_1 \cup a_2$) (respectively of (X, $a1 \odot a2$)). Indeed, { a, c, d } is connected subspace of (X, $a_1 \cup a_2$) (respectively of (X, $a_1 \odot a_2$)).

Then, a ∈ WP(X) in (X, a_1) and a ∈ WP(X) in (X, a_2) but a ∉ WP(X) in (X, $a_1 \cup a_2$) (respectively in (X, $a_1 \odot a_2$)).

7 Conclusion

As for the strong connectivity, the concepts of articulation point and weak point make it possible to propose a decomposition of the greatest strongly connected subspaces of a space or a network in Pretopology ([1], [3], [4], [6], [11]) by progressing from the periphery (the most fragile positions) towards the more central parts (admitting neither weak points nor articulation point) of the space or of the network.

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