

Articulation Point and Weak Point in a Space and in a Network in Pretopology: the Case of the Strong Connectivity

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Abstract

In this paper, we present the concepts of articulation point and weak point in a space and in a network in Pretopology. We show, in the case of strong connectivity, some properties allowing to locate them in a network.

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1 Introduction

In Pretopology (see [2], [4]), some authors have developed a way of structuring the pretopological space using a particular method based on research known as minimal closed subsets. We have already shown ([9]) that this method is very strongly linked to the search of the strongly connected components of the pretopological space.

But sometimes, it is more interesting to break down each context (components or greatest subspaces) in order to give a finer structure ([4], [1], [12], [8]).

This is why we generalized in Pretopology the notion of articulation point (resulting from the graph Theory) and that we defined the notion of weak point. We show here, in the case of strong connectivity, how these concepts are defined in a space and then we indicate properties allowing to locate them in a network in Pretopology.

2 Different Types of Pretopological Spaces (see [2], [4], [6])

Definition 1. Let X be a non empty set. $P(X)$ denotes the family of subsets of X . We call pseudoclosure on X any mapping a from $P(X)$ onto $P(X)$ such as :

$$\begin{aligned} a(\emptyset) &= \emptyset \\ \forall A \subset X, A &\subset a(A) \end{aligned}$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1- (X, a) is a V type pretopological space if and only if

$$\forall A \subset X, \forall B \subset X, A \subset B \Rightarrow a(A) \subset a(B).$$

2- (X, a) is a V_D type pretopological space if and only if

$$\forall A \subset X, \forall B \subset X, a(A \cup B) = a(A) \cup a(B).$$

3- (X, a) is a V_S type pretopological space if and only if

$$\forall A \subset X, a(A) = \bigcup_{x \in A} a(\{x\}).$$

4- (X, a) a V_D type pretopological space, is a topological space if and only if

$$\forall A \subset X, a(a(A)) = a(A).$$

Property 2. *If (X, a) is a V_S space then (X, a) is a V_D space. If (X, a) is a V_D space then (X, a) is a V space.*

Example 3. Let X be a non empty set and R be a binary relationship defined on X .

The pretopology of descendants, noted a_d , is defined by :
 $\forall A \subset X, a_d(A) = \{ x \in X / R(x) \cap A \neq \emptyset \} \cup A$ with $R(x) = \{ y \in X / x R y \}$.

The pretopology of ascendants, noted a_a , is defined by :

$\forall A \subset X, a_a(A) = \{ x \in X / R^{-1}(x) \cap A \neq \emptyset \} \cup A$ with $R^{-1}(x) = \{ y \in X / y R x \}$.

These pretopologies are V_S ones.

The pretopology of ascendant-descendants, noted a_{ad} , is defined by :

$\forall A \subset X, a_{ad}(A) = \{ x \in X / R^{-1}(x) \cap A \neq \emptyset \text{ and } R(x) \cap A \neq \emptyset \} \cup A$.

This pretopology is only V one.

3 Different Pretopological Spaces Defined from a Space (X, a) and Closures (see [2], [4])

Definition 4. Let (X, a) be a V pretopological space. Let $A \subset X$. A is a closed subset if and only if $a(A) = A$.

We note $\forall A \subset X, a^0(A) = A$ and $\forall n, n \geq 1, a^n(A) = a(a^{n-1})(A)$.

We name closure of A the subset of X , denoted $F_a(A)$, which is the smallest closed subset which contains A .

Remark 5. $F_a(A)$ is the intersection of all closed subsets which contain A . In the case where (X, a) is a "general" pretopological space (i.e. is not a V space, nor a V_D space, nor a V_S space, nor a topological space), the closure may not exist.

Proposition 6. Let (X, a) be a V pretopological space. Let $A \subset X$. If one of the two following conditions is fulfilled :

- X is a finite set
- a is of V_S type

then $F_a(A) = \bigcup_{n \geq 0} a^n(A)$.

Remark 7. If a is of V type then a^n and F_a also are of V type. If a is of V_S type then a^n and F_a are also of V_S type.

Definition 8. Let (X, a) be a V pretopological space. Let $A \subset X$. We define the induced pretopology on A by a , denoted a_A , by :

$\forall C \subset A, a_A(C) = a(C) \cap A$.

(A, a_A) (or more simply A) is said pretopological subspace of (X, a) .

We note F_{a_A} the closing according to a_A and $(F_a)_A$ the closing obtained by restriction of closing F_a on A . $(F_a)_A$ is such as $\forall C \subset A, (F_a)_A(C) = F_a(C) \cap A$.

4 Strong connectivity in (X, a) (see [2], [4], [6], [7], [10])

Definition 9. Let (X, a) be a V pretopological space.

(X, a) is strongly connected if and only if $\forall C \subset X, C \neq \emptyset, F_a(C) = X$.

Definition 10. Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty.

A is a strongly connected subset of (X, a) if and only if A endowed with $(F_a)_A$ is strongly connected.

A is a strongly connected component of (X, a) if and only if A is a strongly connected subset of (X, a) and $\forall B, A \subset B \subset X$ with $A \neq B$, B is not a strongly connected subset of (X, a) .

(A, a_A) (or more simply A) is a strongly connected subspace of (X, a) if and only if (A, a_A) , as a pretopological space, is strongly connected.

A is a greatest strongly connected subspace of (X, a) if and only if (A, a_A) is a strongly connected subspace of (X, a) and $\forall B, A \subset B \subset X$ and $A \neq B$, (B, a_B) is not a strongly connected subspace of (X, a) .

Proposition 11 ([2]). *Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty.*

i- If A is a strongly connected subspace of (X, a) then A is a strongly connected subset of (X, a) .

ii- If one of the following three conditions is fulfilled :

** A is a closed subset of X for a*

** complementary of A in X is closed for a and (X, a) is a V_D space*

** a is idempotent*

then A is a strongly connected subspace of $(X, a) \Leftrightarrow A$ is a strongly connected subset of (X, a) .

Proposition 12 ([11]). *Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty.*

i- If (X, a) is strongly connected then A is a strongly connected subset of (X, a) .

ii- If A is a strongly connected subset of (X, a) then $\forall B \neq \emptyset, B \subset A$, B is a strongly connected subset of (X, a) .

Proposition 13 ([4], [10]). *Let (X, a) be a V_S pretopological space. Let $A \subset X$ with A non empty.*

A is a strongly connected component of $(X, a) \Leftrightarrow A$ is a greatest strongly connected subspace of (X, a) .

Definition 14 ([4], [7]). Let X a non empty set. Let a_1 and a_2 two pretopologies on X .

a_1 is thinner than a_2 if and only if $\forall C \subset X, a_1(C) \subset a_2(C)$.

Proposition 15 ([7]). Let X a non empty set. Let a_1 and a_2 two V type pretopologies on X such as a_1 thinner than a_2 . Let $A \subset X$ with A non empty.

If A is a strongly connected subspace of (X, a_1) then A is a strongly connected subspace of (X, a_2) .

5 Articulation point and weak point in a pretopological space : the case of the Strong Connectivity

In graph Theory, we call an articulation point of a strongly connected graph a vertex which, if we remove it, loses the strong connectivity (the induced subgraph without this vertex is not strongly connected) ([3]).

Taking into account the Definition 10 and the Proposition 12, we cannot claim to decompose a strongly connected component A of a pretopological space (X, a) by finding its articulation points (where b of A is said articulation point of A if $A - \{b\}$ is not a strongly connected subset of (X, a)). Indeed, the set of articulation points of A would be empty.

The generalization in Pretopology of the notion of articulation point can only be done from pretopological subspaces. We will therefore seek to apply these notions to the greatest strongly connected subspaces.

Remark 16. Let (X, a) be a V pretopological space. Let $A \subset X$. Let $B \subset X$ with $A \subset B$.

i- $(a_B)_A = a_A$.

ii- $\forall C \subset A, C \subset F_{aA}(C) \subset F_{aB}(C) \subset B$ and $C \subset (F_a)_A(C) = F_a(C) \cap A \subset F_a(C) \cap B = (F_a)_B(C) \subset B$.

Proof.

i- $\forall C \subset A, (a_B)_A(C) = a_B(C) \cap A = a(C) \cap B \cap A = a(C) \cap A$ (because $A \subset B$)

And then $(a_B)_A(C) = a_A(C)$.

ii- See [2].

Proposition 17. Let (X, a) be a V pretopological space. Let $A \subset X$. Let $B \subset X$ with $A \subset B$.

A is a strongly connected subspace of $(X, a) \Leftrightarrow A$ is a strongly connected subspace of (B, a_B) .

Proof.

We note $F_{(a_B)A}$ the closing according to $(a_B)_A$.

A is a strongly connected subspace of (B, a_B)

$\Leftrightarrow \forall C \subset A, C \neq \emptyset, F_{(a_B)A}(C) = A$ (by definition)

$\Leftrightarrow \forall C \subset A, C \neq \emptyset, F_{aA}(C) = A$ (Remark 16-i)

$\Leftrightarrow A$ is a strongly connected subspace of (X, a) .

Definition 18. Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty and A strongly connected subspace of (X, a) . Let $b \in A$ with $A \neq \{b\}$.

i- b is an articulation point of A in (X, a) if and only if $(A - \{b\}, a_{A-\{b\}})$ is not a strongly connected subspace of (X, a) .

ii- Let k a natural number with $k \neq 0$. Let b an articulation point of A in (X, a) .

b is a k order articulation point of A in (X, a) if and only if the smallest of the greatest strongly connected subspaces of $(A - \{b\}, a_{A-\{b\}})$ has k as cardinal.

Remark 19. Let (X, a) be a V pretopological space. Let $A \subset X$. Let $B \subset X$ with $A \subset B$. Let k a natural number with $k \neq 0$.

i- b is an articulation point of A in $(X, a) \Leftrightarrow b$ is an articulation point of A in (B, a_B) .

ii- b is a k order articulation point of A in $(X, a) \Leftrightarrow b$ is a k order articulation point of A in (B, a_B) .

Proof.

i- and ii- Obvious by Remark 16-i and Proposition 17.

Remark 20. Let (X, a) be a V pretopological space. Let $A \subset X$. Let $B \subset X$ with $A \subset B$. Let k a natural number with $k \neq 0$.

We note $AP(A)$ the set of all articulation points of A in (B, a_B) and k - $AP(A)$ the set of all k order articulation points of A in (B, a_B) .

We have k - $AP(A) \subset AP(A)$ (by definition).

Definition 21. Let (X, a) be a V pretopological space. Let $A \subset X$ with A non empty and A strongly connected subspace of (X, a) . Let $c \in A$.

c is a weak point of A in (X, a) if and only if it exists $b \in A - \{c\}$ with $b \in 1$ - $AP(A)$ in (X, a) and $\{c\}$ greatest strongly connected subspace of $(A - \{b\}, a_{A-\{b\}})$.

Remark 22. Let (X, a) be a V pretopological space. Let $A \subset X$. Let $B \subset X$ with $A \subset B$.

c is a weak point of A in $(X, a) \Leftrightarrow c$ is a weak point of A in (B, a_B) .

Proof.

Obvious by Remark 16-i and Proposition 17.

Remark 23. Let (X, a) be a V pretopological space. Let $A \subset X$. Let $B \subset X$ with $A \subset B$.

We note $WP(A)$ the set of all weak points of A in (B, a_B) .

We have $1-AP(A) \neq \emptyset \Leftrightarrow WP(A) \neq \emptyset$ (by definition).

6 Articulation point and weak point in a network in pretopology : the case of the Strong Connectivity

We will seek here to establish the relation between the set of articulation points (respectively the set of weak points) of a greatest strongly connected subspace of a network studied by the union of the pretopologies (or by the composition of pretopologies) and the set of articulation points (respectively the set of weak points) of the greatest strongly connected subspaces of spaces constituting the network.

Definition 24. Let X a non empty set. Let I a countable family of indices. The family $\{ (X, a_i), i \in I \}$ of pretopological spaces is a network on X .

Definition 25. Let X a non empty set. For any pretopologies a_1 and a_2 defined on X , for any subset A of X , we define the two following mappings :

$(a_1 \cup a_2)(A) = a_1(A) \cup a_2(A)$ [union of pretopologies]

$(a_1 \odot a_2)(A) = a_1(a_2(A))$ [composition of pretopologies]

More generally, in a network $\{ (X, a_i), i \in I \}$ such as for any $i \in I$, a_i is of V type, we note $F_{\cup}$ the closure according to $\bigcup_{i \in I} a_i$.

We define the mapping, denoted $\prod_{i \in I} a_i$, from $P(X)$ onto $P(X)$ by :

$\forall A \subset X, \prod_{i \in I} a_i(A) = \{ x \in X / \text{there exists } n \in I \text{ such as } x \in a_n(a_{n-1}(\dots (a_1(A)) \dots)) \}$ and we denote $F_{\prod}$ the closure according to $\prod_{i \in I} a_i$.

For any subset A of X , we note $F_{(\cup)A}$ the closure according to $(\bigcup_{i \in I} a_i)_A$, $F_{(\prod)A}$ the closure according to $(\prod_{i \in I} a_i)_A$, $F_{\cup A}$ the closure according to $\bigcup_{i \in I} a_{iA}$.

Proposition 26 ([7]). *Let $\{ (X, a_i), i \in I \}$ be a network on X such as for any $i \in I$, a_i is of V type. Let $A \subset X$ with A non empty.*

If A is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ then A is a strongly connected subspace of $(X, \prod_{i \in I} a_i)$.

Proposition 27 ([7]). Let $\{ (X, a_i), i \in I \}$ be a network on X such as for any $i \in I$, a_i is of V type. Let $A \subset X$ with A non empty.

If there exists $i \in I$ such as A strongly connected subspace of (X, a_i) then A is strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$.

Proposition 28 ([7]). Let $\{ (X, a_i), i \in I \}$ be a network on X such as for any $i \in I$, a_i is of V type. Let $A \subset X$.

$$\bigcup_{i \in I} a_{iA} = (\bigcup_{i \in I} a_i)_A.$$

Proposition 29. Let $\{ (X, a_i), i \in I \}$ be a network on X such as for any $i \in I$, a_i is of V type. Let E greatest strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$). Let $b \in E$ with $E \neq \{ b \}$.

Let $\{ C_k, k \in K \}$ a family of subsets non empty of X such as :

$$1- \bigcup_{k \in K} C_k = X$$

2- $\forall k \in K$, there exists $\{ A_j, j \in J \}$ a family of subsets non empty of X such as :

$$2-1- C_k = \bigcup_{j \in J} A_j$$

2-2- $\forall j \in J$, there exists $i \in I$, A_j greatest strongly connected subspace of (X, a_i)

2-3- $\forall j \in J, \forall j' \in J$, there exists a sequence $j_0 \dots j_r$ of elements of J such as $j_0 = j$, $j_r = j'$ and $\forall l = 0, \dots, r-1, A_{j_l} \cap A_{j_{l+1}} \neq \emptyset$

2-4- $\forall A' \subset X, A' \notin \{ A_j, j \in J \}$, if there exists $i \in I$ such as A' greatest strongly connected subspace of (X, a_i) then $A' \cap C_k = \emptyset$.

If $b \in AP(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then we have

i - it exists $i \in I$, it exists $A \subset E$ with A greatest strongly connected subspace of (X, a_i) and $b \in AP(A)$ in (X, a_i)

Or ii - it exists $A \subset E$, it exists $A' \subset E$, it exists $i \in I$, it exists $i' \in I$ with A greatest strongly connected subspace of (X, a_i) and $b \in A$ and $A \neq \{ b \}$ and A' greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ and $A' \neq \{ b \}$ and $A \cap A' = \{ b \}$

Or iii - it exists $k \in K$ with $C_k \subset E$ and $C_k \neq \{ b \}$ and $F_{\bigcup_{i \in I} a_i - \{ b \}}(C_k - \{ b \}) \neq E - \{ b \}$ (respectively $F_{\prod_{i \in I} a_i - \{ b \}}(C_k - \{ b \}) \neq E - \{ b \}$).

Proof.

Let us show that if (i), (ii) and (iii) are not satisfied then $b \notin AP(E)$.

Let $b \in E$.

It exists $k \in K, C_k \subset E$ and $b \in C_k$ (Proposition 9-i of [10] and Proposition 42 of [7], Proposition 26)

So it exists $i \in I$, it exists $A \subset C_k$, A greatest strongly connected subspace of (X, a_i) with $b \in A$

Two cases are possible :

1- It exists $i \in I$, it exists $A \subset C_k$, A greatest strongly connected subspace of (X, a_i) with $b \in A$ and $A \neq \{b\}$

(i) is not satisfied so

$\forall i \in I, \forall A \subset E$, A greatest strongly connected subspace of (X, a_i) with $b \in A$ implies $b \notin AP(A)$ in (X, a_i)

Then $A - \{b\}$ is a strongly connected subspace of (X, a_i) (by definition of $AP(A)$)

So $A - \{b\}$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively $A - \{b\}$ is a strongly connected subspace of $(X, \prod_{i \in I} a_i)$) (Propositions 26 and 27).

Also (ii) is not satisfied so

$\forall A \subset E, \forall A' \subset E, \forall i \in I, \forall i' \in I$,

A greatest strongly connected subspace of (X, a_i) with $b \in A$ and $A \neq \{b\}$ and A' greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ and $A' \neq \{b\}$ implies $A \cap A' \neq \{b\}$.

But $A - \{b\}$ and $A' - \{b\}$ are strongly connected subspaces of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$)

As $A \cap A' \neq \{b\}$, we have $(A - \{b\}) \cap (A' - \{b\}) \neq \emptyset$

So $(A - \{b\}) \cup (A' - \{b\})$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$) (the union of two strongly connected subspaces which have an intersection non empty is a strongly connected subspace [2]).

Then $\forall k \in K$ with $C_k \subset E$, $C_k - \{b\}$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$).

But (iii) is not satisfied so :

$\forall k \in K, C_k \subset E$ with $C_k \neq \{b\}$ implies $F_{\cup E - \{b\}}(C_k - \{b\}) = E - \{b\}$ (respectively $F_{(\prod) E - \{b\}}(C_k - \{b\}) = E - \{b\}$).

But $\forall k \in K$ with $C_k \subset E$ and $C_k \neq \{b\}$, $C_k - \{b\}$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$)

Then $\forall k \in K$ with $C_k \subset E$ and $C_k \neq \{b\}$, we have

$\forall C \subset C_k - \{b\}, C \neq \emptyset, F_{\cup C_k - \{b\}}(C) = C_k - \{b\} \subset F_{\cup E - \{b\}}(C)$ (respectively $\forall C \subset C_k - \{b\}, C \neq \emptyset, F_{(\prod) C_k - \{b\}}(C) = C_k - \{b\} \subset F_{(\prod) E - \{b\}}(C)$) ($C_k - \{b\}$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$), Remark 16-ii and Proposition 28)

So $\forall k \in K$ with $C_k \subset E$ and $C_k \neq \{b\}$, we have

$\forall C \subset C_k - \{b\}, C \neq \emptyset, F_{\cup E - \{b\}}(C_k - \{b\}) \subset F_{\cup E - \{b\}}(C) \subset E - \{b\}$ (respectively $F_{(\prod) E - \{b\}}(C_k - \{b\}) \subset F_{(\prod) E - \{b\}}(C) \subset E - \{b\}$)

Then $\forall k \in K, C_k \subset E$ and $C_k \neq \{b\}$, we have

$\forall C \subset C_k - \{b\}, C \neq \emptyset, F_{\cup E - \{b\}}(C) = E - \{b\}$ (respectively $F_{(\prod) E - \{b\}}(C) = E - \{b\}$) ((iii) is not satisfied)

And then $\forall C \subset E - \{b\}$, $C \neq \emptyset$, $F_{\cup E - \{b\}}(C) = E - \{b\}$ (respectively $F_{(\prod)E - \{b\}}(C) = E - \{b\}$)

Then $E - \{b\}$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$)

And then $b \notin AP(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$).

2- $\forall i \in I$, $\forall A \subset C_k$, A greatest strongly connected subspace of (X, a_i) with $b \in A$ implies $A = \{b\}$

In this case, $C_k = A = \{b\}$.

But (iii) is not satisfied so

$\forall k \in K$, $C_k \subset E$ with $C_k \neq \{b\}$ implies $F_{\cup E - \{b\}}(C_k - \{b\}) = E - \{b\}$ (respectively $F_{(\prod)E - \{b\}}(C_k - \{b\}) = E - \{b\}$).

then $E - \{b\}$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$) (see case 1-)

And then $b \notin AP(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$).

Proposition 30. *Let $\{(X, a_i), i \in I\}$ be a network on X such as for any $i \in I$, a_i is of V type. Let E greatest strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$). Let $b \in E$ with $E \neq \{b\}$.*

Let $\{C_k, k \in K\}$ a family of subsets non empty of X which satisfied the conditions of the Proposition 29.

If $b \in 1-AP(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then we have

1- it exists $i \in I$, it exists $A \subset E$ with A greatest strongly connected subspace of (X, a_i) and $b \in 1-AP(A)$ in (X, a_i)

Or 2- it exists $A \subset E$, it exists $A' \subset E$, it exists $i \in I$, it exists $i' \in I$ with A greatest strongly connected subspace of (X, a_i) and $b \in A$ and $\text{card}(A) = 2$ and A' greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ and $\text{card}(A') \geq 2$ and $A \cap A' = \{b\}$

Or 3- it exists $c \in E - \{b\}$, it exists $k \in K$, with $C_k = \{c\}$ and $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE - \{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E - \{b\}})$).

Or 4- it exists $i \in I$, it exists $A \subset E$ with A greatest strongly connected subspace of (X, a_i) and $b \in A$ and $\text{card}(A) = 2$ and it exists $k \in K$, with $C_k = A$ and $A - \{b\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE - \{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E - \{b\}})$).

Proof.

Let us show that if (1), (2), (3) and (4) are not satisfied then $b \notin 1-AP(E)$.

Let $b \in E$.

It exists $k \in K$, $C_k \subset E$ and $b \in C_k$ (Proposition 9-i of [10] and Proposition 42 of [7], Proposition 26)

So it exists $i \in I$, it exists $A \subset C_k$, A greatest strongly connected subspace of (X, a_i) with $b \in A$

Three cases are possible :

1- $\text{Card}(A) > 2$

We have $b \notin 1\text{-AP}(A)$ in (X, a_i) ((1) is not satisfied)

So $\forall c \in A - \{b\}$, $\{c\}$ is not greatest strongly connected subspace of $(A - \{b\}, a_{iA-\{b\}})$ (by definition)

Then $\forall c \in A - \{b\}$, it exists $C \subset A - \{b\}$ with $\{c\} \subset C$ and $\{c\} \neq C$ and C greatest strongly connected subspace of $(A - \{b\}, a_{iA-\{b\}})$

Then $\forall c \in A - \{b\}$, it exists $C \subset A - \{b\}$ with $\{c\} \subset C$ and $\{c\} \neq C$ and C strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Propositions 17, 26, 27, 28)

So $\forall c \in A - \{b\}$, $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$)

Finally, if $b \in 1\text{-AP}(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then it exists $c \in E - A$, $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

Then it exists $k' \in K$, $C_{k'} \subset E$ and $c \in C_{k'}$.

1-1 If $k' = k$

Then $c \in C_k - A$ so it exists $i' \in I$, it exists $A' \subset C_k$, A' greatest strongly connected subspace of $(X, a_{i'})$ with $c \in A'$ and $A' \neq \{c\}$.

If $b \notin A'$ then A' is strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$) (Propositions 26 and 27)

So A' is a strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Propositions 17 and 28)

And $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$)

So $b \in 1\text{-AP}(E)$ implies $b \in A'$.

If $\text{card}(A') > 2$ then as $b \notin 1\text{-AP}(A')$ in $(X, a_{i'})$ ((1) is not satisfied), we have $\forall c' \in A' - \{b\}$, $\{c'\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

So we have $\text{card}(A') = 2$

Then $b \in 1\text{-AP}(E)$ implies $\text{card}(A') = 2$ and $A \cap A' = \{b\}$.

And the result because (2) is not satisfied.

1-2 Si $k' \neq k$

In this case, if $\text{card}(C_{k'}) > 1$ then, as $C_{k'}$ is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$), $C_{k'}$ is a strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Propositions 17, 28)

So $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$)

Then $b \in 1\text{-AP}(E)$ implies $C_{k'} = \{c\}$ and $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

And the result because (3) is not satisfied.

2- Card(A) = 2

2-1 If $A = C_k$

In this case, $b \in 1\text{-AP}(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) implies that it exists $c \in E - \{b\}$, $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Proposition 28).

Then it exists $k' \in K$, $C_{k'} \subset E$ and $c \in C_{k'}$.

2-1-1 If $k' \neq k$, we can progress as in 1-2.

2-1-2 If $k' = k$

In this case, $\{c\} = A - \{b\}$

So $b \in 1\text{-AP}(E)$ implies $\{c\}$ is greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

And the result because (4) is not satisfied.

2-2 - If $A \subset C_k$ and $A \neq C_k$

2-2-1 If it exists $i' \in I$, it exists $A' \subset E$ with A' greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ and $\text{card}(A') > 2$

Then $b \notin 1\text{-AP}(E)$ (see the case 1- applied to A').

2-2-2 $\forall A' \subset E, \forall i' \in I, A'$ greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ implies $\text{card}(A') \leq 2$

If $b \in 1\text{-AP}(E)$ then it exists $c \in E - \{b\}$, $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Proposition 28).

Then it exists $k' \in K$, $C_{k'} \subset E$ and $c \in C_{k'}$.

2-2-2-1 If $k' = k$

- If $c \in C_k - A$

It exists $i'' \in I$, it exists $A'' \subset C_k$, A'' greatest strongly connected subspace of $(X, a_{i''})$ with $c \in A''$ and $A'' \neq \{c\}$.

If $b \notin A''$ then $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (see the case 1-1)

So $b \in 1\text{-AP}(E)$ implies $\text{card}(A'') = 2$ and $A \cap A'' = \{b\}$.

And the result because (2) is not satisfied.

- If $c \in A$

Then $\{c\} = A - \{b\}$.

If it exists $i' \in I$, it exists $A' \subset C_k$ with $A' \not\subset A$ and A' greatest strongly connected subspace of $(X, a_{i'})$ and $A - \{b\} \subset A'$

Then $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (see the case 1-1 because $b \notin A'$)

So if $b \in 1\text{-AP}(E)$ then

$\forall i' \in I, \forall A' \subset C_k$ with $A' \not\subset A$, if A' greatest strongly connected subspace of $(X, a_{i'})$ then $A - \{b\} \not\subset A'$

Then, as $A \subset C_k$ and $A \neq C_k$, it exists $i' \in I$, it exists $A' \subset C_k$ with $A' \not\subset A$ and A' greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ and $\text{card}(A') = 2$

Finally, if $b \in 1\text{-AP}(E)$ then it exists $i' \in I$, it exists $A' \subset C_k$ with A' greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ and $\text{card}(A') = 2$ and $A \cap A' = \{b\}$.

And the result because (2) is not satisfied.

2-2-2-2 If $k' \neq k$, we can progress as in 1-2.

3- $\text{Card}(A) = 1$

In this case, $A = \{b\}$.

Two cases are possible :

3-1 $A = C_k$

If $b \in 1\text{-AP}(E)$ then it exists $k' \in K, k' \neq k, C_{k'} \subset E$ and $c \in C_{k'}$ and $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE - \{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E - \{b\}})$) (Proposition 28).

In this case, if $\text{card}(C_{k'}) > 1$ then $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE - \{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E - \{b\}})$) (see the case 1-2)

So if $b \in 1\text{-AP}(E)$ then $C_{k'} = \{c\}$ and $\{c\}$ is a greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE - \{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E - \{b\}})$).

And the result because (3) is not satisfied.

3-2 If $A \subset C_k$ and $A \neq C_k$

In this case, it exists $i' \in I$, it exists $A' \subset C_k$ with $\text{card}(A') \geq 2$ and $A \subset A'$ and A' greatest strongly connected subspace of $(X, a_{i'})$

And the result accordind to the cases 1 and 2 applied to A' .

Proposition 31. *Let $\{(X, a_i), i \in I\}$ be a network on X such as for any $i \in I, a_i$ is of V type. Let E greatest strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$). Let $c \in E$.*

Let $\{C_k, k \in K\}$ a family of subsets non empty of X which satisfied the conditions of the Proposition 29.

If $c \in \text{WP}(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then we have

1- it exists $i \in I$, it exists $A \subset E$ with A greatest strongly connected subspace of (X, a_i) and $c \in A$ and $c \in \text{WP}(A)$ in (X, a_i)

Or 2- it exists $A \subset E$ with $A = \{b, c\}$, it exists $A' \subset E$, it exists $i \in I$, it exists $i' \in I$ with A greatest strongly connected subspace of (X, a_i) and A' greatest strongly connected subspace of $(X, a_{i'})$ and $b \in A'$ and $\text{card}(A') \geq 2$ and $A \cap A' = \{b\}$

Or 3- it exists $i \in I$, it exists $A \subset E$ with $A = \{ b, c \}$ and A greatest strongly connected subspace of (X, a_i) and it exists $k \in K$, with $A = C_k \subset E$ and $\{ c \}$ greatest strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$.

Or 4- it exists $k \in K$, with $C_k = \{ c \}$ and it exists $b \in E - \{ c \}$ with $\{ c \}$ greatest strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$.

Proof.

Let us show that if (1), (2), (3) and (4) are not satisfied then $c \notin \text{WP}(E)$.

Let $c \in E$.

It exists $k \in K$, $C_k \subset E$ and $c \in C_k$ (Proposition 9-i of [10] and Proposition 42 of [7], Proposition 26)

So it exists $i \in I$, it exists $A \subset C_k$, A greatest strongly connected subspace of (X, a_i) with $c \in A$

Three cases are possible :

1- $\text{Card}(A) > 2$

(1) is not satisfied so $c \notin \text{WP}(A)$ in (X, a_i) .

So $\forall b \in A - \{ c \}$, $\{ c \}$ is not greatest strongly connected subspace of $(A - \{ b \}, a_{iA-\{b\}})$

Then $\forall b \in A - \{ c \}$, it exists $C \subset A - \{ b \}$ with $\{ c \} \subset C$ and $\{ c \} \neq C$ and C greatest strongly connected subspace of $(A - \{ b \}, a_{iA-\{b\}})$

So $\forall b \in A - \{ c \}$, it exists $C \subset A - \{ b \}$ with $\{ c \} \subset C$ and $\{ c \} \neq C$ and C strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$ (Propositions 17, 26, 27, 28)

Then $\forall b \in A - \{ c \}$, $\{ c \}$ is not greatest strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$.

Finally, if $c \in \text{WP}(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then it exists $b \in E - A$, $\{ c \}$ greatest strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$.

But A is greatest strongly connected subspace of (X, a_i) with $c \in A$ and $A \neq \{ c \}$

So A is strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$ (Propositions 17, 26, 27, 28)

And $\{ c \}$ is not greatest strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$.

2- $\text{Card}(A) = 2$

2-1 If $A = C_k$

In this case, if $c \in \text{WP}(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then it exists $b \in E - \{ c \}$, $\{ c \}$ greatest strongly connected subspace of $(E - \{ b \}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{ b \}, (\prod_{i \in I} a_i)_{E-\{b\}}))$.

Then it exists $k' \in K$, $C_{k'} \subset E$ and $b \in C_{k'}$.

2-1-1 Si $k' \neq k$

In this case, as C_k is a strongly connected subspace of $(X, \bigcup_{i \in I} a_i)$ (respectively of $(X, \prod_{i \in I} a_i)$), C_k is strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Propositions 17 and 28)

So $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

2-1-2 If $k' = k$

In this case, $\{b\} = A - \{c\}$ and $\{c\}$ is greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

And the result because (3) is not satisfied.

2-2 - If $A \subset C_k$ and $A \neq C_k$

2-2-1 If it exists $i' \in I$, it exists $A' \subset E$ with A' greatest strongly connected subspace of $(X, a_{i'})$ and $c \in A'$ and $\text{card}(A') > 2$

Then $c \notin \text{WP}(E)$ (see the case 1- applied to A').

2-2-2 If $\forall A' \subset E, \forall i' \in I, A'$ greatest strongly connected subspace of $(X, a_{i'})$ and $c \in A'$ implies $\text{card}(A') \leq 2$

In this case, let $b \in E - \{c\}$.

If $b \notin A$ then, as A is greatest strongly connected subspace of (X, a_i) , A is strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Propositions 17, 26, 27, 28)

So $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

Finally, if $c \in \text{WP}(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then it exists $b \in A - \{c\}$, $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$).

If it exists $i' \in I$, it exists $A' \subset C_k$ with $A' \not\subset A$ and A' greatest strongly connected subspace of $(X, a_{i'})$ and $c \in A'$

Then $\text{card}(A') = 2$ and $A \cap A' = \{c\}$ and $b \notin A'$

So A' strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$) (Propositions 17, 26, 27, 28)

Then $\{c\}$ is not greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}})$)

So if $c \in \text{WP}(E)$ then

$\forall i' \in I, \forall A' \subset C_k$ with $A' \not\subset A$, if A' greatest strongly connected subspace of $(X, a_{i'})$ then $c \notin A'$

Then, as $A \subset C_k$ and $A \neq C_k$, it exists $i' \in I$, it exists $A' \subset C_k$ with A' greatest strongly connected subspace of $(X, a_{i'})$ and $A \cap A' = \{b\}$ and $\text{card}(A') \geq 2$.

And the result because (2) is not satisfied.

3- $\text{Card}(A) = 1$

In this case, $A = \{c\}$.

Two cases are possible :

3-1 If $A = C_k$

If $c \in \text{WP}(E)$ in $(X, \bigcup_{i \in I} a_i)$ (respectively in $(X, \prod_{i \in I} a_i)$) then it exists $b \in C_{k'}$, $k' \in K$, $C_{k'} \subset E$ and $\{c\}$ greatest strongly connected subspace of $(E - \{b\}, \bigcup_{i \in I} a_{iE-\{b\}})$ (respectively of $(E - \{b\}, (\prod_{i \in I} a_i)_{E-\{b\}}))$.

And the result because (4) is not satisfied.

3-2 If $A \subset C_k$ and $A \neq C_k$

In this case, it exists $i' \in I$, it exists $A' \subset C_k$ with $\text{card}(A') \geq 2$ and $A \subset A'$ and A' greatest strongly connected subspace of $(X, a_{i'})$

And the result according to the cases 1 and 2 applied to A' .

Remark 32. In general, the converses of the Propositions 29, 30 and 31 are not true.

Example 33. Let $\{(X, a_i), i \in I\}$ be a network with $X = \{a, b, c\}$, $I = \{1, 2\}$, a_1 and a_2 pretopologies of descendants defined respectively by the following graph 1 and graph 2 :

x	$R(x)$
a	$\{b\}$
b	$\{c\}$
c	$\{a\}$

Graph 1

x	$R(x)$
a	$\{c\}$
b	$\{a\}$
c	$\{b\}$

Graph 2

X is greatest strongly connected subspace of (X, a_1) and X is greatest strongly connected subspace of (X, a_2) with $b \in 1\text{-AP}(X)$ in (X, a_1) and $b \in 1\text{-AP}(X)$ in (X, a_2) .

So X is greatest strongly connected subspace of $(X, a_1 \cup a_2)$ (respectively of $(X, a_1 \odot a_2)$) but b is not articulation point of X in $(X, a_1 \cup a_2)$ (respectively of $(X, a_1 \odot a_2)$). Indeed, $\{a, c\}$ is strongly connected subspace of $(X, a_1 \cup a_2)$ (respectively of $(X, a_1 \odot a_2)$).

Then, $a \in \text{WP}(X)$ in (X, a_1) and $a \in \text{WP}(X)$ in (X, a_2) but $a \notin \text{WP}(X)$ in $(X, a_1 \cup a_2)$ (respectively in $(X, a_1 \odot a_2)$).

7 Conclusion

The concepts of articulation point and weak point make it possible to propose a decomposition of the greatest strongly connected subspaces of a space or a network in Pretopology ([1], [4], [5], [12]) by progressing from the periphery (the weakest positions) towards the more central parts (admitting neither weak points nor articulation point) of the space or of the network.

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