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Quadratic and Cubic Equations for Four-Point Curves and Rectangles

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Abstract

Four positive numbers in a rectangular array can be interpolated by the bilinear equation. Two new bi-quadratic equations for this array are illustrated. By the sum of squares of deviations test, one of them is usually better than the bilinear equation. Exponential interpolation equations are recommended instead of most third- and higher-power polynomial equations. A method for estimating the degree of four-point curves, and certain four-point rectangles, is illustrated by examples.

Mathematics Subject Classification: 65D05, 65D07, 65D17

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1. Introduction

Classical methods for the design of experiments apply a four-point rectangle with a positive number at each vertex of the rectangle. See rectangle ACIG in Fig. 1. The coordinate system is x = –1 .. 1, y = –1 .. 1. It is desirable to have different interpolation equations for rectangles so that experimental data can be analyzed in different ways. This permits alternative interpretations of laboratory results.

G	I
A	C

Fig. 1. A four-point rectangle

The familiar, standard method for interpolating Fig. 1 is the bilinear equation. That equation does not render curvature estimates. This deficiency can often be overcome by means of a bi-quadratic equation originally derived by operational methods [1]. Other alternatives to the bilinear equation are desirable.

2. Two equations for the four-point rectangle

The bilinear equation, Eq. (1), is commonly used to interpolate Fig. 1.

$$z = (A+C+G+I)/4 + (I+C-A-G)x/4 + (I+G-A-C)y/4 + (I+A-C-G)xy/4 \tag{1}$$

The operation of factoring emphasizes division and multiplication to form ratios and products of other expressions. A dictionary defines the word ‘parse’ as follows: To examine closely or subject to detailed analysis, especially by breaking into parts or components. Factoring and parsing need not be the same operations.

Parsing of Eq. (1) can be effected by means of a bi-quadratic, polynomial equation for an eight-point cube [2]. Find the limit, corner-by-corner, of the top plane of the cube as it approaches the bottom plane of the cube. By this process, the bilinear equation can be parsed into two new equations: Eq. (2) and Eq. (3).

$$\begin{aligned} z &= (A + I)/2 + (C + I - A - G)(x/4) + (G + I - A - C)(y/4) \\ &\quad + ((C + G - A - I)/4)(x^2/2 - xy + y^2/2) \end{aligned} \tag{2}$$

$$\begin{aligned} z &= (C + G)/2 + (C + I - A - G)(x/4) + (G + I - A - C)(y/4) \\ &\quad + ((A - C - G + I)/4)(x^2/2 + xy + y^2/2) \end{aligned} \tag{3}$$

The sum of the right-hand sides of Eqs. (2) and (3), divided by two, is the familiar bilinear equation on the right-hand side of Eq. (1). This verifies the parsing of Eq. (1) into two new equations, Eqs. (2) and (3), for Fig. 1. Now let A = 1, C = 3, G = 7, I = 9. Equations (2) and (3) both yield $z = 5+x+3y$. The same result is obtained by the bilinear equation. Let new data in Fig. 1 be A = 1, C = 9, G = 49, I = 81.

Now the right-hand sides of Eq. (1), Eq. (2) and Eq. (3) are Eqs. (4), (5), and (6), respectively. The new equations reproduce the new numbers in Fig. 1. This test verifies the proper parsing of Eq. (1) into Eqs. (2) and (3).

$$z = 35 + 10x + 30y + 6xy \tag{4}$$

$$z = 41 + 10x + 30y - 3x^2 + 6xy - 3y^2 \tag{5}$$

$$z = 29 + 10x + 30y + 3x^2 + 6xy + 3y^2 \tag{6}$$

Trial data are prepared by applying the functions in the first column of Table 1 to the integers 1, 3, 7, 9 assigned as vertices A, C, G, I in Fig. 1, respectively. (The coordinate system $x = -1 .. 1$, $y = -1 .. 1$ applies to Fig. 1.) Table 1 lists the sums of the squares of the deviations of three numerical equations interpolating the trial data. In the first case, all of Eqs. (1), (2), and (3) render the bilinear equation,
$z = 5 + x + 3y$, denoted by the letter ‘z’ in Table 1. That is the correct equation in this case so the sums of the squared deviations are all zero. See the first row of numbers in Table 1.

Table 1. Approximate sums of squared deviations of three interpolation equations from true functions

Functions	Eq. (1)	Eq. (2)	Eq. (3)
z	0	0	0
z^2	207	512	42.7
z^3	47550	116200	10590
2^z	49650	119870	14363
$100/z$	1400	3316	467
$100/z^2$	2830	6416	1147
$\sin(10z^o)$	0.0263	0.0640	0.00581

When A, C, G, I are 1^2, 3^2, 7^2, 9^2, respectively, the equation for the surface AGCI is $z = (5+x+3y)^2$. This operation is denoted by “z^2” in the first column of Table 1. Equation (3) now renders the lowest sum of squared deviations (≈ 42.7) from the generating function. The results in Table 1 indicate that Eq. (3) is the best choice for interpolating Fig. 1. That equation has the lowest sum of squared deviations in every case in Table 1. Equation parsing is a new and potentially useful method.

3. Estimated degrees of four-point curves

Four equidistant x-coordinate numbers $[x_1, x_2, x_3, x_4]$ are [1, 2, 3, 4], respectively.

These numbers generate four y-coordinate numbers so that [A,C,G,I] are 1^2, 2^2, 3^2, 4^2] = [1,4,9,16], respectively. In this case, three of Eqs. (7)–(10) render the degree of the data as N = 2. The equation of the curve is $y = x^N = x^2$. This method applies when $A<C<G<I$ lie on a curve or on Fig. 1 above.

$$(G^{(1/N)} + C^{(1/N)} - I^{(1/N)})^N - A = 0 \tag{7}$$

$$(A^{(1/N)} + I^{(1/N)} - G^{(1/N)})^N - C = 0 \tag{8}$$

$$(I^{(1/N)} + A^{(1/N)} - C^{(1/N)})^N - G = 0 \tag{9}$$

$$(C^{(1/N}) + G^{(1/N)} - A^{(1/N)})^N - I = 0 \tag{10}$$

Let [A,C,G,I] = [1,27,125,343]. These numbers are on the y-axis of a new curve. In this case Eq. (7) fails but Eqs. (8)–(10) render N = 3. The equation for the new curve is Eq. (11). These methods often apply when [A,C,G,I] are an increasing sequence of positive numbers derived from a simple polynomial function.

$$y = x^N = x^3 \tag{11}$$

Now let the equation of a new curve be represented by Eq. (12).

$$z = x^2 + 100x \tag{12}$$

The x-axis numbers are an equidistant sequence of positive integers. The ordinate numbers of a new curve are [A,C,G,I] = [101,309,525,749]. Equations (7)–(10) now render the degree of the new curve as $N \approx 1.0671$. The term 100x, a linear term contributes numerically more to the new data than the term x^2. Hence, the degree of the new curve is closer to the number 1 than to the number 2.

4. Interpolation of a four-point rectangle

A method for interpolating four bilinear numbers in a rectangular array is the bilinear equation, Eq. (1). It applies in the x = –1 .. 1, y = –1 .. 1 coordinate system. A bi-quadratic for the same array is Eq. (7) in [1] or Eq. (10) in [3]. See Eq. (13).

$$z = (Px + Qy + Rxy + S)^2 \tag{13}$$

Equation (13) can be rewritten as Eq. (14).

$$z = (P/(x) + Q/(y) + R/(xy) + S)^2 \tag{14}$$

Four positive, bilinear numbers in a rectangular array can be treated by Eq. (14). For this purpose, use a new coordinate system: x = 1 .. 3, y = 1 .. 3. Equation (14) in this system renders four new equations: Eqs. (15) – (18). They apply to Fig. 1.

$$(P + Q + R + S)^2 - A = 0 \tag{15}$$

$$(P/3 + Q + R/3 + S)^2 - C = 0 \tag{16}$$

$$(P + Q/3 + R/3 + S)^2 - G = 0 \tag{17}$$

$$(P/3 + Q/3 + R/9 + S)^2 - I = 0 \tag{18}$$

Given the numerical values of A, C, G, I, Eqs. (15)–(18) are as a set of four simultaneous equations in the four unknowns P, Q, R, and S. For example, let A = 2, C = 4, G = 8, I = 10. Equations (15)–(18) can be solved as a set of four simultaneous equations for the terms P, Q, R, and S. Equations (15)–(18) are thereby transformed into a new set of four numerical equations. They can be rounded and abbreviated as Eqs. (19)–(22).

$$z = ((8.4331442 - 10.554465x - 3.3118239y + 6.8473578xy)^2)/(x^2y^2) \tag{19}$$

$$z = ((2.0691832 - 8.4331442x - 1.1905035y + 6.1402510xy)^2)/(x^2y^2) \tag{20}$$

$$z = ((-6.9308168 + 0.5668558x + 1.8094965y + 3.1402510xy)^2)/(x^2y^2) \tag{21}$$

$$z = (-0.56685578 - 1.5544646x - 0.31182388y + 3.8473578xy)^2)/(x^2y^2) \tag{22}$$

Equations (19)–(22) are now converted to the x = –1 .. 1, y = –1 ..1 coordinate system. This conversion restates Eqs. (19)–(22) into new equations: Eqs. (23)–(26). The conversion is effected by changing (x) into (x+2) and (y) into (y+2).

$$z = (-19.2994336 - 10.554465x - 3.3118239y + 6.8473578(x+2)(y+2))^2 \\ / ((x+2)^2(y+2)^2) \tag{23}$$

$$z = (-17.1781122 - 8.4331442x - 1.1905035y + 6.1402510(x+2)(y+2))^2 \\ / ((x+2)^2(y+2)^2) \tag{24}$$

$$z = (-2.17811224 + 0.56685578x + 1.8094965y + 3.1402510(x+2)(y+2))^2 \\ /((x+2)^2(y+2)^2) \tag{25}$$

$$z = (-4.29943274 - 1.5544646x - 0.31182388y + 3.8473578(x+2)(y+2))^2 \\ /((x+2)^2(y+2)^2) \tag{26}$$

When Eqs. (23)–(26) are substituted with (x,y) = (–1,–1), or (x,y) = (1,–1), or (x,y) = (–1,1), or (x,y) = (1,1) the cited equations render the original data (A,C,G,I) = (2,4,8,10), respectively, for rectangle ACIG in Fig. 1. This remark applies within the limits of the rounded numerical coefficients in Eqs (23)–(26). When substituted with (x,y) = (0,0), Eqs. (23)–(26) render alternative estimates at the center point of the rectangle ACIG in Fig. 1. The estimates are $z \approx 4.090504$, $z \approx 3.406693$, $z \approx 6.737778$, and $z \approx 7.686754$, respectively. The predictions differ from the predictions of the bilinear equation and the biquadratic equation for the center point of the rectangle ACIG in the above Fig. 1. Note that the ordering of Eqs. (23)–(26) depends on the computer software used for the calculations.

5. Cubic equations for rectangles of four positive numbers

Let [A,C,G,I] in Fig. 1 be [1,3,7,9], respectively. These numbers are bilinear so they can be interpolated by the bilinear equation as in Eq. (27).

$$z = (5 + x + 3y) \tag{27}$$

When the cited numbers are squared they become [1,9,49,81], respectively. There are eight different interpolation equations for these numbers: Eqs. (15)–(22) in [3]. The product of Eq. (27) and Eq. (15) in [3] is Eq. (28). This equation is bicubic in the variables 'x' and 'y'.

$$z = (5 + x + 3y)(xy - 3x - 7y - 9)^2/4 \tag{28}$$

Equations (29), (30), and (31) are alternative examples of this easy method for generating bicubic equations for certain rectangles as illustrated in [3].

$$z = (5 + x + 3y)(xy + 5y + 3)^2 \tag{29}$$

$$z = (xy - 3x - 7y - 9)^4 / (16(5 + x + 3y)) \tag{30}$$

$$z = (5xy + 3x + y)^4 / (5 + x + 3y) \tag{31}$$

Many four-point interpolation problems involve positive numbers in a rectangular array. Such numbers can be interpolated by the bilinear equation. They can also be interpolated by means of a biquadratic equation [1]. Sections (2), (4), and (5) in [7] address this problem by different methods.

For many years, a biquadratic equation for a four-point rectangle of positive numbers was regarded as impossible. The original form of a biquadratic equation for a rectangle defined by four positive numbers appears as Eq. (7) in [1]. An alternative form of this equation appears as Eq. (32). Within round-off errors, both equations render the same results when substituted with the same numerical data. Note the alternative form of the constant term in Eq. (32).

$$z = (A + C + G + I)/4 - x2c - y2c + (I + C - A - G)x/4 + (G + I - A - C)(y/4) + (I + A - C - G)(xy/4) + (x2c)x^2 + (y2c)y^2 \quad (32)$$

$$x2c = (I + A - C - G)(I + C - G - A) / (8(G + I - A - C)) \quad (33)$$

$$y2c = (I + A - C - G)(G + I - A - C) / (8(I + C - A - G)) \quad (34)$$

Future applications of four-point interpolation equations cannot be foreseen so diversity is desirable. For example, one form of an interpolation equation might be superior to another form when the equations are differentiated or integrated. The advantage of alternative forms of equations is suggested by Eqs. (28)–(31).

Equations (7)–(10) illustrate four equations for estimating the degree of a four-point, polynomial-type curve. One of those equations could be useful for estimating the degree of a polynomial interpolation equation for a four-point rectangle provided the four data are positive numbers and $A<C<G<I$ as in Fig. 1. Methods designed for curves are not necessarily applicable to rectangles so this application remains to be examined and evaluated. However, unforeseen applications of mathematical methods often deserve to be noticed. This is particularly true in laboratory work. Some early methods for are summarized in [4,5].

A simplified form of the expression for the center point (cpt) of a four-point rectangle appears as Eq. (4) in [6]. See Eq. (35). That ratio represents an easier form of the first term of Eq. (7) of [1]. The abbreviation ‘cpt’ represents ‘center point’ of four-point rectangle such as in Fig. 1.

$$cpt = ((C + G)(I - A)^2 - (I + A)(C - G)^2) / (2((I - A)^2 - (C - G)^2)) \quad (35)$$

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