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Positive Solutions for a p-Laplacian Type Fractional Differential Equation with Mixed Boundary Conditions

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Abstract

In this paper, a class of p-Laplacian type fractional four-point boundary value problem is studied. Based on the monotone iterative method, we obtained the existence of positive solutions of the boundary value problem. An example is given to show the validity of our main theorem.

Keywords: fractional differential equation; monotone iterative method; p-Laplacian operator; four-point boundary value problem, positive solution

1 Introduction

In recent years, the boundary value problem of fractional differential equation are widely used in science and engineering fields[1-4]. There are abundant papers on fractional boundary value problem with different types of boundary conditions[5-8]. On the other hand, in order to study the one-dimensional variable turbulence of gas through porous media, the nonlinear diffusion equation was obtained and abstracted as p-Laplacian equation[9]. Many important results related to the boundary value problem of fractional differential equations with p-Laplacian operator have been obtained[10-13].

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In this paper, we consider the following fractional differential equations:

$$\begin{cases} {}^{C}D^{\beta}_{0+}(\Phi_p({}^{C}D^{\alpha}_{0+}u(t))) = f(t,u(t),{}^{C}D^{\gamma}_{0+}u(t)), \\ u(0)+u'(0)=0, \Phi_p({}^{C}D^{\alpha}_{0+}u(0)) + \Phi'_p({}^{C}D^{\alpha}_{0+}u(0)) = 0, \\ u(1) = r_1 u(\xi),\ \Phi_p({}^{C}D^{\alpha}_{0+}u(1)) = r_2\Phi_p({}^{C}D^{\alpha}_{0+}u(\eta)), \end{cases} \tag{1.1}$$

where $1<\alpha,\beta\le 2$, $0<\gamma<1$, $0<r_1,\ r_2,\ \xi,\ \eta<1$. ${}^{C}D^{\alpha}_{0+}$, ${}^{C}D^{\beta}_{0+}$, ${}^{C}D^{\gamma}_{0+}$ are the Caputo fractional derivative. Φ_p is the p-Laplacian operator such that $\Phi_p(s)=|s|^{p-2}s$, $p>1$ and $\Phi_p^{-1}(s)=\Phi_q(s)$, $\frac{1}{p}+\frac{1}{q}=1$. $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a given continuous function.

The purpose of this paper is to study the existence of positive solutions of boundary value problem (1.1) by monotone iterative method.

2 Preliminaries

Let $E=\{u : u\in C[0,1], {}^{C}D^{\gamma}_{0+}u\in C[0,1]\}$ be a Banach Space with the norm $\|u\|_\gamma=\|u\|_\infty+\|{}^{C}D^{\gamma}_{0+}u\|_\infty$, where $\|\cdot\|_\infty$ is the maximum norm. Set the cone $P\subset E$ by $P=\{u\in E \mid u(t)\ge 0,\ {}^{C}D^{\gamma}_{0+}u(t)\ge 0,\ t\in[0,1]\}$.

Definition 2.1. *([1]) The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $y:[0,\infty)\to\mathbb{R}$ is defined as follows*

$$I^{\alpha}_{0+}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,\ t>0,$$

provided the right side is point-wise defined on $(0,\infty)$.

Definition 2.2. *([1]) The Caputo derivative of fractional order $\alpha>0$ for a function $y:[0,\infty)\to\mathbb{R}$ is defined as follows*

$${}^{C}D^{\alpha}_{0+}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,\ t>0,\ n=-[-\alpha],$$

where $[\alpha]$ denotes the integral part of the real number α.

Lemma 2.3. *([1]) For $\alpha>0$. Assume that $y, {}^{C}D^{\alpha}_{0+}y(t)\in L[0,1]$. Then*

$$I^{\alpha}_{0+}({}^{C}D^{\alpha}_{0+}y)(t)=y(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_Nt^{\alpha-N},\ c_i\in\mathbb{R},\ i=0,1,2,\ldots,N,$$

where N is the smallest integer greater than or equal to α.

Lemma 2.4. *([1]) For $y\in L[0,1]$,*
(i) if $\rho>\sigma>0$, then ${}^{C}D^{\sigma}_{0+}I^{\rho}_{0+}y(t)=I^{\rho-\sigma}_{0+}y(t)$, ${}^{C}D^{\sigma}_{0+}I^{\sigma}_{0+}y(t)=y(t)$;
(ii) if $\rho>0$, $\sigma>0$, then ${}^{C}D^{\rho}_{0+}t^{\sigma-1}=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\rho)}t^{\sigma-\rho-1}$.

Lemma 2.5. *The boundary value problem (1.1) has an unique solution:*

$$u(t) = \int_0^1 G(t,s)\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds,$$

where

$$G(t,s) = \frac{1}{A}\begin{cases} t^{\alpha-2}(1-t)(1-s)^{\alpha-1} + r_1\xi^{\alpha-2}(1-\xi)(t-s)^{\alpha-1} - r_1t^{\alpha-2}(1-t)(\xi-s)^{\alpha-1}, \\ 0\le s\le t\le 1,\ s\le\xi, \\ t^{\alpha-2}(1-t)(1-s)^{\alpha-1} + r_1\xi^{\alpha-2}(1-\xi)(t-s)^{\alpha-1},\ 0\le s\le t\le 1,\ s\ge\xi, \\ t^{\alpha-2}(1-t)(1-s)^{\alpha-1} - r_1t^{\alpha-2}(1-t)(\xi-s)^{\alpha-1},\ 0\le t\le s\le 1,\ s\le\xi, \\ t^{\alpha-2}(1-t)(1-s)^{\alpha-1},\ 0\le t\le s\le 1,\ s\ge\xi, \end{cases}$$

$$H(t,s) = \frac{1}{B}\begin{cases} t^{\beta-2}(1-t)(1-s)^{\beta-1} + r_2\eta^{\beta-2}(1-\eta)(t-s)^{\beta-1} - r_2t^{\beta-2}(1-t)(\eta-s)^{\beta-1}, \\ 0\le s\le t\le 1,\ s\le\eta, \\ t^{\beta-2}(1-t)(1-s)^{\beta-1} + r_2\eta^{\beta-2}(1-\eta)(t-s)^{\beta-1},\ 0\le s\le t\le 1,\ s\ge\eta, \\ t^{\beta-2}(1-t)(1-s)^{\beta-1} - r_2t^{\beta-2}(1-t)(\eta-s)^{\beta-1},\ 0\le t\le s\le 1,\ s\le\eta, \\ t^{\beta-2}(1-t)(1-s)^{\beta-1},\ 0\le t\le s\le 1,\ s\ge\eta, \end{cases}$$

$A = r_1\Gamma(\alpha)\xi^{\alpha-2}(1-\xi),\ B = r_2\Gamma(\beta)\eta^{\beta-2}(1-\eta).$

Proof. Set $y = \Phi_p{}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}u(t)$, then ${}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}u(t) = \Phi_q(y(t))$.
Due to Lemma 2.3, we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Phi_q(y(s))ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}, \qquad (2.1)\\ u'(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\Phi_q(y(s))ds + c_1t^{\alpha-2}. \end{aligned}$$

From the boundary condition $u(0)+u'(0)=0,\ u(1)=r_1u(\xi)$, we derive

$$\begin{aligned} c_1 &= \frac{r_1\int_0^{\xi}(\xi-s)^{\alpha-1}\Phi_q(y(s))ds - \int_0^1(1-s)^{\alpha-1}\Phi_q(y(s))ds}{r_1\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}, \\ c_2 &= \frac{\int_0^1(1-s)^{\alpha-1}\Phi_q(y(s))ds - r_1\int_0^{\xi}(\xi-s)^{\alpha-1}\Phi_q(y(s))ds}{r_1\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}. \end{aligned} \qquad (2.2)$$

Thus, by (2.1) and (2.2), we can get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Phi_q(y(s))ds + \frac{t^{\alpha-2}(1-t)}{r_1\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}\int_0^1 (1-s)^{\alpha-1}\Phi_q(y(s))ds \\ &\quad - \frac{t^{\alpha-2}(1-t)}{\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}\int_0^{\xi}(\xi-s)^{\alpha-1}\Phi_q(y(s))ds \\ &= \int_0^1 G(t,s)\Phi_q(y(s))ds. \qquad (2.3) \end{aligned}$$

At the same way, we have

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s,u(s),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(s))ds \\ &\quad + \frac{t^{\beta-2}(1-t)}{r_2\Gamma(\beta)\eta^{\beta-2}(1-\eta)}\int_0^1 (1-s)^{\beta-1} f(s,u(s),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(s))ds \\ &\quad - \frac{t^{\beta-2}(1-t)}{\Gamma(\beta)\eta^{\beta-2}(1-\eta)}\int_0^\eta (\eta-s)^{\beta-1} f(s,u(s),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(s))ds \\ &= \int_0^1 H(t,s) f(s,u(s),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(s))ds = \Phi_p({}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}u(t)). \end{aligned} \tag{2.4}$$

Using of (2.3) and (2.4), the problem (1.1) has an unique solution:

$$u(t) = \int_0^1 G(t,s)\Phi_q\bigg(\int_0^1 H(s,\tau) f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds.$$

The proof is complete. □

Lemma 2.6. *The functions $G(t,s)$ and $H(t,s)$ satisfies:*
(i) $G(t,s)\ge 0$, $H(t,s)\ge 0$, for $t,\ s\in[0,1]$;
(ii) for $t,\ s\in[0,1]$, we have $G(t,s)\le \frac{2(1-s)^{\alpha-1}}{A}$, $H(t,s)\le\frac{2(1-s)^{\beta-1}}{B}$.

Proof. It's obvious that $A>0,\ B>0$. For $0\le t\le s\le 1,\ s\le\xi$, we have

$$\begin{aligned} AG(t,s) &= t^{\alpha-2}(1-t)(1-s)^{\alpha-1} - r_1 t^{\alpha-2}(1-t)(\xi-s)^{\alpha-1} \\ &\ge t^{\alpha-2}(1-t)(1-s)^{\alpha-1} - r_1 t^{\alpha-2}(1-t)(1-s)^{\alpha-1} \\ &\ge 0. \end{aligned}$$

For $0\le s\le t\le 1,\ s\ge\xi$, we get

$$\begin{aligned} AG(t,s) &= t^{\alpha-2}(1-t)(1-s)^{\alpha-1} + r_1\xi^{\alpha-2}(1-\xi)(t-s)^{\alpha-1} \\ &\le t^{\alpha-2}(1-t)(1-s)^{\alpha-1} + r_1\xi^{\alpha-2}(1-\xi)(1-s)^{\alpha-1} \\ &\le 2(1-s)^{\alpha-1}. \end{aligned}$$

Hence, $0\le G(t,s)\le\frac{2(1-s)^{\alpha-1}}{A},\ for\ t,\ s\in[0,1]$.

Similarly, we can get $0\le H(t,s)\le\frac{2(1-s)^{\beta-1}}{B},\ for\ t,\ s\in[0,1]$. □

3 The main existence result

In this section, we consider the existence of positive solutions for problem (1.1). Before proving the main result, we introduce the following hypothesis.

Assume there exists positive constants $\lambda_1,\ \lambda_2$ such that:
$(H1)$ $\max\limits_{0\le t\le 1} f(t,\lambda_1,\lambda_2)\le\Phi_p(\lambda M)$, where $M=\frac{\alpha A(\beta B)^{q-1}\Gamma(\alpha-\gamma+1)(1-\xi)}{2^{q-1}[2\Gamma(\alpha-\gamma+1)(1-\xi)+\alpha A(1-\xi)+\xi^{\alpha}A(\alpha-\gamma)]}$,

$\lambda=\lambda_1+\lambda_2$;
$(H2)$ $f(t,\theta_1,\omega_1)\le f(t,\theta_2,\omega_2)$, for $0\le\theta_1\le\theta_2\le\lambda_1$, $0\le\omega_1\le\omega_2\le\lambda_2$, $t\in[0,1]$;
$(H3)$ $f(t,0,0)>0$, for $t\in[0,1]$.

Define operator $T:P\to P$ as follow:

$$Tu(t)=\int_0^1 G(t,s)\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds. \qquad (3.1)$$

Lemma 3.1. *$T:P\to P$ is completely continuous.*

Proof. For any $u\in P$, by the continuity of f, $G(t,s)$ and $H(t,s)$, we have $T:P\to P$ and T is continuous. Let $\Omega\subset P$ be bounded, which is to say exists $K>0$ such that $\|u\|_\gamma\le K$, for all $u\in\Omega$. Set $L=\max\limits_{\substack{0\le t\le 1\\ u\in\Omega}} f(t,u,{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(t))+1$. From (2.1), (3.1) and Lemma 2.4, then, for $u\in\Omega$, $t\in[0,1]$, we have

$$\begin{aligned}
Tu(t)&=\int_0^1 G(t,s)\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&\le\int_0^1\frac{2(1-s)^{\alpha-1}}{A}\bigg(\int_0^1\frac{2(1-\tau)^{\beta-1}}{B}Ld\tau\bigg)^{q-1}ds\\
&=\frac{2^qL^{q-1}}{\alpha A(\beta B)^{q-1}}\\
&<\infty,
\end{aligned}$$

$$\begin{aligned}
{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}(Tu(t))&={}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}I_{0+}^{\alpha}\big(\Phi_q(y(t))\big)+c_1{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}t^{\alpha-1}=I_{0+}^{\alpha-\gamma}\big(\Phi_q(y(t))\big)+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}c_1t^{\alpha-\gamma-1}\\
&=\bigg|\frac{1}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&\quad-\frac{t^{\alpha-\gamma-1}}{r_1\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^1(1-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&\quad+\frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\bigg|\\
&\le\frac{1}{\Gamma(\alpha-\gamma)}\int_0^t(t-s)^{\alpha-\gamma-1}\bigg(\int_0^1\frac{2(1-\tau)^{\beta-1}}{B}Ld\tau\bigg)^{q-1}ds\\
&\quad+\frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}\bigg(\int_0^1\frac{2(1-\tau)^{\beta-1}}{B}Ld\tau\bigg)^{q-1}ds\\
&=\frac{(2L)^{q-1}t^{\alpha-\gamma}}{(\beta B)^{q-1}\Gamma(\alpha-\gamma+1)}+\frac{(2L)^{q-1}t^{\alpha-\gamma-1}\xi^2}{\alpha(\beta B)^{q-1}\Gamma(\alpha-\gamma)(1-\xi)}\\
&<\infty.
\end{aligned}$$

Hence, $T(\Omega)$ is uniformly bounded.

For $u\in P$, $0\le t_1\le t_2\le 1$, we have

$$|Tu(t_1)-Tu(t_2)|=\bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds$$

$$
\begin{aligned}
&+\frac{t_1^{\alpha-2}(1-t_1)}{r_1\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}\int_0^1(1-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&-\frac{t_1^{\alpha-2}(1-t_1)}{\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&-\frac{t_2^{\alpha-2}(1-t_2)}{r_1\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}\int_0^1(1-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&+\frac{t_2^{\alpha-2}(1-t_2)}{\Gamma(\alpha)\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\bigg|\\
\le{}&\frac{(2\beta)^{q-1}}{\Gamma(\alpha)(BL)^{q-1}}\bigg|\int_{t_2}^{t_1}(t_1-s)^{\alpha-1}ds+\int_0^{t_2}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]ds\\
&-\frac{(1-t_1)(t_1^{\alpha-2}-t_2^{\alpha-2})}{r_1\xi^{\alpha-2}(1-\xi)}\int_0^1(1-s)^{\alpha-1}ds+\frac{(1-t_1)(t_1^{\alpha-2}-t_2^{\alpha-2})}{\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}ds\bigg|\\
={}&\frac{(2\beta)^{q-1}}{\Gamma(\alpha+1)(BL)^{q-1}}\bigg[(t_1^\alpha-t_2^\alpha)-\frac{(1-t_1)(t_1^{\alpha-2}-t_2^{\alpha-2})}{r_1\xi^{\alpha-2}(1-\xi)}+\frac{(1-t_1)(t_1^{\alpha-2}-t_2^{\alpha-2})\xi^2}{(1-\xi)}\bigg]\\
={}&\varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
&\big|{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}(Tu(t_1))-{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}(Tu(t_2))\big|\\
={}&\bigg|\frac{1}{\Gamma(\alpha-\gamma)}\int_0^{t_1}(t_1-s)^{\alpha-\gamma-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&-\frac{t_1^{\alpha-\gamma-1}}{r_1\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^1(1-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&+\frac{t_1^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&-\frac{1}{\Gamma(\alpha-\gamma)}\int_0^{t_2}(t_2-s)^{\alpha-\gamma-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&+\frac{t_2^{\alpha-\gamma-1}}{r_1\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^1(1-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\\
&-\frac{t_2^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}\Phi_q\bigg(\int_0^1 H(s,\tau)f(\tau,u(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u(\tau))d\tau\bigg)ds\bigg|\\
\le{}&\frac{(2L)^{q-1}}{\Gamma(\alpha-\gamma)(\beta B)^{q-1}}\bigg|\int_{t_2}^{t_1}(t_1-s)^{\alpha-\gamma-1}ds+\int_0^{t_2}[(t_1-s)^{\alpha-\gamma-1}-(t_2-s)^{\alpha-\gamma-1}]ds\\
&-\frac{t_1^{\alpha-\gamma-1}-t_2^{\alpha-\gamma-1}}{r_1\xi^{\alpha-2}(1-\xi)}\int_0^1(1-s)^{\alpha-1}ds+\frac{t_1^{\alpha-\gamma-1}-t_2^{\alpha-\gamma-1}}{\xi^{\alpha-2}(1-\xi)}\int_0^\xi(\xi-s)^{\alpha-1}ds\bigg|\\
={}&\frac{(2L)^{q-1}}{\Gamma(\alpha-\gamma)(\beta B)^{q-1}}\bigg|(\alpha-\gamma)(t_1^{\alpha-\gamma}-t_2^{\alpha-\gamma})-\frac{t_1^{\alpha-\gamma-1}-t_2^{\alpha-\gamma-1}}{r_1\alpha\xi^{\alpha-2}(1-\xi)}+\frac{(t_1^{\alpha-\gamma-1}-t_2^{\alpha-\gamma-1})\xi^2}{\alpha(1-\xi)}\bigg|\\
={}&\varepsilon.
\end{aligned}
$$

Thus, $T(\Omega)$ is equicontinuous. We have $T: P\to P$ is completely continuous by Arzela-Ascoli theorem. □

Theorem 3.2. *Assume there exists positive constants λ_1, λ_2 such that $(H1)$, $(H2)$, $(H3)$ hold, then problem (1.1) has two positive solutions u^* and v^*.*

Proof. Set $P_\lambda = \{u \in P : \|u\|_\gamma \le \lambda, where\ \lambda = \lambda_1 + \lambda_2, \|u\|_\infty \le \lambda_1, \|{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u\|_\infty \le \lambda_2\}$.

We first prove $TP_\lambda \subseteq P_\lambda$. $\forall u \in P_\lambda$, we have $0 \le u(t) \le \|u\|_\infty \le \lambda_1$, $0 \le {}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u(t) \le \|{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u\|_\infty \le \lambda_2$. Using (H_1) and (H_2), then $0 \le f(t, u(t), {}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u(t)) \le f(t, \lambda_1, \lambda_2) \le \Phi_p(\lambda M)$. For any $u \in U$, we know that

$$\begin{aligned}
\|Tu\|_\gamma &= \|Tu\|_\infty + \|{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}(Tu)\|_\infty \\
&= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)\Phi_q\left(\int_0^1 H(s,\tau) f(\tau, u(\tau), {}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u(\tau))d\tau\right)ds\right| \\
&\quad + \max_{0\le t\le 1}\left|\frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\Phi_q\left(\int_0^1 H(s,\tau) f(\tau, u(\tau), {}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u(\tau))d\tau\right)ds\right. \\
&\quad - \frac{t^{\alpha-\gamma-1}}{r_1\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^1 (1-s)^{\alpha-1}\Phi_q\left(\int_0^1 H(s,\tau) f(\tau, u(\tau), {}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u(\tau))d\tau\right)ds \\
&\quad \left. + \frac{t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^\xi (\xi-s)^{\alpha-1}\Phi_q\left(\int_0^1 H(s,\tau) f(\tau, u(\tau), {}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u(\tau))d\tau\right)ds\right| \\
&\le \lambda M\int_0^1 \frac{2(1-s)^{\alpha-1}}{A}\left(\int_0^1 \frac{2(1-\tau)^{\beta-1}}{B}d\tau\right)^{q-1} ds \\
&\quad + \frac{\lambda M}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1}\left(\int_0^1 \frac{2(1-\tau)^{\beta-1}}{B}d\tau\right)^{q-1} ds \\
&\quad + \frac{\lambda M t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)\xi^{\alpha-2}(1-\xi)}\int_0^\xi (\xi-s)^{\alpha-1}\left(\int_0^1 \frac{2(1-\tau)^{\beta-1}}{B}d\tau\right)^{q-1} ds \\
&= \frac{2^q\lambda M}{\alpha A(\beta B)^{q-1}} + \frac{2^{q-1}t^{\alpha-\gamma}\lambda M}{(\beta B)^{q-1}\Gamma(\alpha-\gamma+1)} + \frac{2^{q-1}t^{\alpha-\gamma-1}\xi^2\lambda M}{\alpha(\beta B)^{q-1}\Gamma(\alpha-\gamma)(1-\xi)} \\
&\le \frac{2^{q-1}[2\Gamma(\alpha-\gamma+1)(1-\xi) + \alpha A(1-\xi) + \xi^\alpha A(\alpha-\gamma)]}{\alpha A(\beta B)^{q-1}\Gamma(\alpha-\gamma+1)(1-\xi)}\lambda M \\
&= \lambda.
\end{aligned}$$

Therefore, $Tu \in P_\lambda$. So that, $TP_\lambda \subseteq P_\lambda$.

Set $u_0(t) = \lambda, 0 \le t \le 1$, then $\|u\|_\gamma = \lambda$ and $u_0 \in P_\lambda$. Let $u_{n+1} = Tu_n = T^{n+1}u_0, n = 0, 1, 2, \cdots$. Since $TP_\lambda \subseteq P_\lambda$, we have $u_n \in P_\lambda, n = 0, 1, 2, \cdots$.

From (H_1) and (H_2), for $t \in [0, 1]$, one has

$$\begin{aligned}
u_1(t) = (Tu_0)(t) &= \int_0^1 G(t,s)\Phi_q\left(\int_0^1 H(s,\tau) f(\tau, u_0(\tau), {}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma} u_0(\tau))d\tau\right)ds \\
&\le \int_0^1 \frac{2(1-s)^{\alpha-1}}{A}\left(\int_0^1 \frac{2(1-\tau)^{\beta-1}}{B}\Phi_p(\lambda M)d\tau\right)^{q-1} ds \\
&= \frac{2^q\lambda M}{\alpha A(\beta B)^{q-1}} \\
&\le \lambda \\
&= u_0(t),
\end{aligned}$$

$$\begin{aligned} u_{n+1}(t) = (Tu_n)(t) &= \int_0^1 G(t,s)\Phi_q\Big(\int_0^1 H(s,\tau)f(\tau,u_n(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u_n(\tau))d\tau\Big)ds \\ &\le \int_0^1 G(t,s)\Phi_q\Big(\int_0^1 H(s,\tau)f(\tau,u_{n-1}(\tau),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\gamma}u_{n-1}(\tau))d\tau\Big)ds \\ &= u_n(t). \end{aligned}$$

Hence, we can obtain $u_{n+1} \le u_n, 0 \le t \le 1, n = 0, 1, 2, \cdots$.

Set $v_0 = 0, 0 \le t \le 1$, then $\|v\|_\gamma = 0$ and $v_0 \in P_\lambda$. Let $v_{n+1} = Tv_n = T^{n+1}v_0, n = 0, 1, 2, \cdots$. Since $TP_\lambda \subseteq P_\lambda$, we have $v_n \in P_\lambda, n = 0, 1, 2, \cdots$.

Same as above, we can get $v_{n+1} \ge v_n, 0 \le t \le 1, n = 0, 1, 2, \cdots$. Therefore, we obtain monotone sequences $\{u_n\}$ and $\{v_n\}$.

According to Lemma 3.1, we can know that $\{u_n\}$, $\{v_n\}$ have convergent subsequence $\{u_{n_k}\}$, $\{v_{n_k}\}$ and exists $u^*, v^* \in P_\lambda$ such that $u_{n_k} \to u^*$, $v_{n_k} \to v^*$. Thus, there exists $u^*, v^* \in P_\lambda$ such that $u_n \to u^*, v_n \to v^*$, i.e. $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$. Applying the continuity of T and $u_{n+1} = Tu_n, v_{n+1} = Tv_n$, we have $Tu^* = u^*, Tv^* = v^*$. From (H_3), we have $u^* > 0$, $v^* > 0$.

Then boundary value problem (1.1) has two positive solutions. □

4 Examples

We consider the following equation:

$$\begin{cases} {}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}(\Phi_{\frac{3}{2}}({}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(t))) = f(t,u(t),{}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{1}{2}}u(t)), \\ u(0) + u'(0) = 0, \Phi_{\frac{3}{2}}({}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(0)) + \Phi'_{\frac{3}{2}}({}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(0)) = 0, \\ u(1) = \frac{1}{2}u(\frac{1}{2}),\ \Phi_{\frac{3}{2}}({}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(1)) = \frac{1}{2}\Phi_{\frac{3}{2}}({}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(\frac{1}{2})), \end{cases} \tag{4.1}$$

where $\Phi_{\frac{3}{2}}({}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(t)) = \frac{{}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(t)}{\sqrt{|{}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(t)|}}$ if ${}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(t) \ne 0$, and $\Phi_{\frac{3}{2}}({}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(t)) = 0$ if ${}^{\mathrm{C}}\mathrm{D}_{0+}^{\frac{3}{2}}u(t) = 0$.

Assume that $f(t,\theta,\omega) = \frac{1}{10}(1+\theta t+\omega)$. By computation, we have $A = B \approx 0.3134$, $M \approx 0.27964$. Set $\lambda_1 = \lambda_2 = 5$, then $\lambda = 10$. So, $f(t,\theta,\omega)$ satisfies
$(H1)$ $\max_{0\le t\le 1} f(t,\lambda_1,\lambda_2) = \max_{0\le t\le 1} f(t,5,5) = 1.1 < \Phi_{\frac{3}{2}}(\lambda M) = \Phi_{\frac{3}{2}}(0.27964) \approx 1.67$;
$(H2)$ $f(t,\theta_1,\omega_1) \le f(t,\theta_2,\omega_2)$, for $0 \le \theta_1 \le \theta_2 \le 5$, $0 \le \omega_1 \le \omega_2 \le 5$, $t \in [0,1]$;
$(H3)$ $f(t,0,0) = 0.01 > 0$, for $t \in [0,1]$.

By Theorem 3.2, the problem (4.1) has two positive solutions.

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