Applied Mathematical Sciences, Vol. 16, 2022, no. 7, 313 - 327
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2022.916797

Use of Variational Method to Evaluate Well-Posedness of an Advection-Diffusion Reaction Equation of Solute Transport

Elias Mwakilama

Department of Mathematics
Pan African University Institute for Basic Sciences Technology and Innovation (PAUISTI)
Jomo Kenyatta University of Agriculture & Technology
Nairobi, Kenya

Duncan Gathungu

Department of Pure & Applied Mathematics
Jomo Kenyatta University of Agriculture & Technology
Nairobi, Kenya

Vusi Magagula

Department of Mathematics
University of Eswatini
Kwaluseni, Matsapha, Eswatini

Abstract

Although various forms of the solute transport equations model the Spatio-temporal distribution of contaminants in soil or groundwater systems using the corresponding analytical or numerical solutions, the qualitative analysis of well-posed solution properties of these models is rarely considered. This paper investigates the existence and uniqueness of classical solutions to an advection-diffusion reaction equation (ADRE). We derive the findings by applying the method of variation formulation (VF) in the Sobolev space.

Mathematics Subject Classifications: 35D30, 35B09, 35K57

Keywords: Well-posedness, advection diffusion reaction equation, variation formulation, Sobolev space, Banach-Nečas-Babuška Theorem

1 Introduction

Traditionally, when modelling heat and mass transport in soil or groundwater systems, studies focus on developing numerical and analytical solutions for partial differential equations (PDEs) [1, 2]. Unlike the case of ordinary differential equations (ODEs), as applied to disease models [3, 4], qualitative analysis of the formulated PDEs to heat and mass transport is rarely considered. Instead, the focus is on the dimensional homogeneity principle [5] of the models formulated. Yet, for the cause-and-effect phenomena such as heat transport, the models are often inhomogeneous [6, 7]. Recently, Hasan-Zadeh [8] examined the existence and uniqueness of the weak solution to the time-dependent advection-diffusion equation (ADE) using advanced components of Sobolev spaces [9], weak solutions and some important integral inequalities, but for an ODE version of the original PDE. Alzate et al. [10] described the variational method to qualitatively study the existence, uniqueness and regularity of the solution to a PDE and applied it to the Poisson's equation but on the basis of the conditions of the Lax-Milgram Theorem [11, 12]. However, following Clason [13], the Banach-Nečas-Babuška Theorem [14, 15], which generalizes the Lax-Milgram Theorem (see Appendix 4), is particularly well suited for the study of the global existence of solutions to the parabolic PDEs. In this paper, we investigate the existence and uniqueness of solutions to the advection-diffusion reaction equation (ADRE) using the Banach-Nečas-Babuška Theorem in the Sobolev space. The novelty in this case is not conceptual; rather, it is in the manner we set up a transport problem to illuminate a relevant topic further while utilising abstract space analysis and outlining potential futures. The rest of the paper is organized as follows: in Section 2, we present the investigated PDE and list the mathematical preliminaries from the abstract space, relevant for the analysis framework. The result and analysis are presented in Section 3, and in Section 4 the conclusion is made.

2 Methodology

2.1 The model

A two-dimensional analytical solution to the ADRE of bilateral uniform flow with first-order decay, absorption, and sink to model the transport of dissolved solute in a homogeneous and isotropic non-fractured porous medium Ψ_p was developed by [16], but not qualitatively evaluated for the existence and uniqueness of solution properties. To evaluate the solution properties of the model [16]

$$R_m \frac{\partial C}{\partial t} + \phi^2 \left(u \frac{\partial C}{\partial x} + w \frac{\partial C}{\partial z} \right) = \frac{\theta_w}{Pe} \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial z^2} \right) + \bar{\zeta} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) + (\acute{\varrho}_3 + \acute{\varrho}_4) C + \acute{\varrho}_5, \tag{1}$$

where C and T are solute concentration and soil temperature, respectively; R_m is matrix retardation factor; $\bar{\zeta}$ is thermophoresis parameter; $\acute{\varrho}_3$, $\acute{\varrho}_4$ and $\acute{\varrho}_5$ are first-order reaction, adsorption and sink coefficients, respectively; ϕ is soil porosity, θ_w is soil water content, and Pe is Peclet number; together with the initial and boundary conditions

$$\left. \begin{array}{l} C(z, x, 0) = 0 \\ C(0, x, t) = 1 \\ \lim_{x \to \infty} \frac{\partial C}{\partial x} = 0 \\ C(\infty, x, t) = 0 \end{array} \right\}, \tag{2}$$

an investigation of well-posedness in abstract space was needed. Since (1)–(2) is a linear parabolic PDE, we can use the method of variational formulation (VF) [10, 17] or the energy method (EM) [18, 19] to achieve that. But, first, we assume $\bar{\zeta}$ is negligible, and transform (2) into Dirichlet boundary condition $C(z, x, t) = 0$ for $x, z \in \partial \Psi_p$. Furthermore, let $\bar{e_1} = \phi^2$ and $\bar{e_2} = \theta_w / Pe$.

Conventionally, studies widely employ the VF because of its simplicity. The EM, however, has an advantage over the VF since it does not only describe the mechanism for testing the existence of the unique solution but also helps to identify the type of boundary conditions to use to ensure a unique solution. We employ the VF method but blend it with the EM technique and term the approach 'the VF-EM' technique. To use the VF technique, we transform the given PDE (Eq.1) from its original domain Ψ_p into some fixed reference domain Ψ and apply the abstract techniques from functional analysis [20]. With the EM, however, we need to show that the energy integral

$$\acute{E}(t) = \frac{\partial}{\partial t} \int_{\Psi} u^2, \tag{3}$$

is a decreasing function for every real-valued function $u \in L^2(0, T; \mathcal{V})$. In the VF approach, we employ the Banach-Nečas-Babuška Theorem [13, 20], a generalized Lax-Milgram theorem [12], to show that the model is well posed by investigating existence of unique, positive, and bounded weak solutions [21]. But, first, we recall some definitions and facts from functional analysis [22, 23, 24].

2.2 Preliminaries

Lemma 2.1. *Hölders inequality [22]. For* $p, q \in \mathbb{R}$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $\S_j, \zeta_j \in \mathbb{C}$,

$$\sum_{j=1}^{n} |\S_j \zeta_j| \leq \left(\sum_{j=1}^{n} |\S_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |\zeta_j|^q \right)^{\frac{1}{q}}.$$

Lemma 2.2. *Cuachy-Schwarz inequality [22]. Let* $\S, \zeta \in L^2(\Psi)$*; then,*

$$|\left(\S, \zeta\right)| \leq \|\S\|_{L^2(\Psi)} \|\zeta\|_{L^2(\Psi)}.$$

Definition 2.3. *Let* $T > 0$ *be some fixed time and* $\Psi \subset \mathbb{R}^n$ *be the given space domain. Set* $\mathcal{Q} = (0, T) \times \Psi$ *and let* $u(t, \vec{x})$ *be a real-valued function of* t *on* $\mathcal{Q}$ *defined in a Banach space*[1] $\mathcal{V}$*. Suppose* $\mathcal{V}$ *consists of functions dependent on* $\vec{x}$ *only, then*

$$u : (0, T) \to \mathcal{V}, \quad \text{and} \quad t \mapsto u(t) \in \mathcal{V}.$$

Definition 2.4. *Hölder spaces [12, 13].*

Let $\mathbb{N}$ *be the set of non-negative integers. Let* $\bar{\mathcal{Q}}$ *be an open bounded set in* $\mathbb{R}^n$ *and let* $k \in \mathbb{N}$*. Then, for* $k \geq 0$, $C^k(0, T; \mathcal{V})$ *is the set of all* $\mathcal{V}$*-valued functions on* $[0, T]$ *which are* k *times continuously differentiable on* $\bar{\mathcal{Q}}$ *with respect to* t*. Further, if* $d_t^j u$ *is denoted as the* j*th derivative of* u*, then the Hölder space*[2] $C^k(0, T; \mathcal{V})$ *is a Banach space when equipped with the norm*

$$\|u\|_{C^k(0,T;\mathcal{V})} = \sum_{0 \leq j \leq k} \sup_{t \in [0,T]} \|d_t^j u\|_{\mathcal{V}}. \tag{4}$$

Definition 2.5. *Lebesgue/Bochner spaces [12, 13]. For* $1 \leq p \leq \infty$*, let* $L^p(0, T; \mathcal{V})$ *be defined as the space of all* $\mathcal{V}$*-valued functions on* $(0, T)$ *for which* $t \mapsto \|u(t)\|_{\mathcal{V}}$ *is a function in* $L^p(0, T)$*. Then the Lebesgue space*[3] $L^p(0, T)$*, with*

$$\int_0^T |u(t)|_{\mathcal{V}}^p dt < \infty,$$

is a Banach space when equipped with the norm

[1] A complete normed space, i.e. for $u(t, \vec{x}) \in \mathcal{V}$, $\|u\|_{\mathcal{V}} = \max|u_t|$, $t \in (0, T)$ whenever $\mathcal{V}$ is a space of continuous functions [24]

[2] also defined as space of continuous function in [12]

[3] also called space of integrable functions in [12]

$$\|u(t)\|_{L^p(0,T;\mathcal{V})} = \begin{cases} \left(\int_0^T |u(t)|_{\mathcal{V}}^p dt\right)^{\frac{1}{p}} & if \quad p < \infty, \\ \\ ess\sup_{t\in(0,T)} |u(t)|_{\mathcal{V}} & if \quad p = \infty. \end{cases}$$

Moreover, if $p = 2$ and the space $L^2(0,T)$ is endowed with an inner product $(\cdot,\cdot)$, then $L^p(0,T)$ is a Hilbert space[4] $\mathcal{H}$.

Then, we need to construct a weak formulation of the model using some standard results and definitions based on the theory of parabolic PDEs in the Sobolev space.

Definition 2.6. *Weak derivative [20].*

Let $\mathcal{V}$ and $\mathcal{H}$ be Hilbert spaces[5] and let $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$ such that $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$ is a Hilbert triple. Then $u \in L^2(0,T;\mathcal{V})$ is said to have a weak derivative $d_t u \in L^2(0,T;\mathcal{V}^*)$ if there exists $\omega \in L^2(0,T;\mathcal{V}^*)$ such that

$$\int_0^T \omega'(t)\,(u(t),v)_{\mathcal{H}} = -\int_0^T \omega(t)\langle u(t), v\rangle_{\mathcal{V}^*,\mathcal{V}} \qquad \forall\omega \in \mathcal{D}(0,T), v \in \mathcal{V}. \tag{5}$$

Definition 2.7. *Sobolev spaces [12, 13]. Let $u \in L^p(0,T;\mathcal{V})$ have a weak derivative $d_t u \in L^p(0,T;\mathcal{V})$. Then $u \in \mathcal{W}_k^p(0,T;\mathcal{V})$, is defined as a Sobolev space of order k, i.e.,*

$$\mathcal{W}_1^p(0,T;\mathcal{V}) = \{u \in L^p(0,T;\mathcal{V}) : d_t u \in L^p(0,T;\mathcal{V})\}.$$

It is then called a Banach space[6] if endowed with the norm

$$\|u\|_{\mathcal{W}_1^p(0,T;\mathcal{V})} = \|u\|_{L^p(0,T;\mathcal{V})} + \|d_t u\|_{L^p(0,T;\mathcal{V}^*)}.$$

Lemma 2.8. *Hilbert space [20]. The space*

$$\mathcal{W}_1^p(0,T;\mathcal{V},\mathcal{V}^*) = \left\{u \in L^2(0,T;\mathcal{V}) : d_t u \in L^2(0,T;\mathcal{V}^*)\right\}.$$

equipped with an inner product,

$$(u,v)_{\mathcal{W}_1^p(0,T;\mathcal{V},\mathcal{V}^*)} = \int_0^T (u(t),v(t))_{\mathcal{V}} + \int_0^T (d_t u, d_t v)_{\mathcal{V}^*}, \tag{6}$$

is a Hilbert space, $\mathcal{H}_0^1$.

---

[4]in general, any vector space $\vec{X}$, equipped with an inner product $(\cdot,\cdot)_{\vec{X}}$ and the associated norm $\|u\|_{\vec{X}} = (\cdot,\cdot)^{\frac{1}{2}}$ is called a Hilbert space if there exists a Cauchy sequence $\{u_m\}$ in $\vec{X}$.

[5]any vector space $\vec{X}$, equipped with inner product $(\cdot,\cdot)_{\vec{X}}$ and associated with norm $\|u\|_{\vec{X}} = (\cdot,\cdot)^{\frac{1}{2}}$ for a Cauchy sequence $\{u_m\}$ in $\vec{X}$ [24].

[6]A complete normed space, i.e. for $u(t,\vec{x}) \in \mathcal{V}$, $\|u\|_{\mathcal{V}} = \max|u_t|$, $t \in (0,T)$ whenever $\mathcal{V}$ is a space of continuous functions [24]

Lemma 2.9. *Poincaré-Friedrich's inequality. Let Ψ be an open bounded set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Psi$. Let $u \in \mathcal{H}_0^1(\Psi)$, then there exists a constant $k_*(\Psi)$, independent of u, so that*

$$\int_\Psi u^2(x)dx \le k_* \sum_{i=1}^{n} \int_\Psi \left|\frac{\partial u}{\partial x_i}(x)\right|^2 dx.$$

Then

- $\mathcal{H}_0^1(\Psi) = \left\{u \in L^2(\Psi) : d_x u \in L^2(\Psi), u = 0 \quad on \quad \partial\Psi\right\}$,
- $\mathcal{X} = \mathcal{H}_0^1(0,T) = \left\{u \in L^2(0,T;\mathcal{V}) : d_t u \in L^2(0,T;\mathcal{V}^*), u_0 = 0 \quad in \quad \mathcal{X}\right\}$.

Lemma 2.10. *Weak-formulation [13] or Transport Theorem [20].*
Let $(\mathcal{V},\mathcal{H},\mathcal{V}^*)$ be a Hilbert triple such that for every $u, v \in \mathcal{W}(\mathcal{V},\mathcal{V}^*)$, the map

$$\boxed{t \mapsto (t; u(t), v(t))_{\mathcal{H}},}$$

is absolutely continuous on $t \in (0,T)$ and for almost every $t \in (0,T)$,

$$\int_0^T \langle d_t u(t), v(t)\rangle_{\mathcal{V}^*,\mathcal{V}} dt = \langle u(T), v(T)\rangle_{\mathcal{H}} - \langle u(0), v(0)\rangle_{\mathcal{H}} - \int_0^T \left\{\langle d_t v(t), u(t)\rangle_{\mathcal{V}^*,\mathcal{V}} - a\left(t; u(t), v(t)\right)\right\} dt. \quad (7)$$

Using Theorem .1 in Appendix and the above mathematical preliminaries, we establish that there exists a unique solution of (1)–(2). In compressed form, let (1)–(2) be

$$R_m \partial_t C(\vec{x},t) - \bar{e}_2 \nabla^2 C(\vec{x},t) + b_i \nabla C(\vec{x},t) + \tilde{g} C(\vec{x},t) = f(\vec{x},t), \quad \text{for} \quad (t,\vec{x}) \quad \text{in} \quad (0,T)\times\Psi, \quad (8)$$

$$C(\vec{x},0) = 0, \quad (9a)$$
$$C(\vec{x},t) = 0, \quad \text{on} \quad \partial\Psi, \quad (9b)$$

where $\vec{x} = (x,z)$, $b_i = \{\bar{e}_1 u, \bar{e}_1 v\}$, $\tilde{g} = (\acute{\varrho}_3 + \acute{\varrho}_4)$, and $f = \acute{\varrho}_5$. Let $a(t;\cdot,\cdot) : \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ bilinear on $\mathcal{V}\times\mathcal{V}$, $\tilde{g} \in L^\infty(0,T;\mathcal{V})$, $b_i, \bar{e}_2 \in \mathcal{H}_0^1$, and $f \in L^2(0,T;\mathcal{V}^*)$ be a continuous linear function so that $C_0 \in \mathcal{H}$.

Then, since "every classical solution of (8)–(9) is a weak solution of (1)–(2)", we show that (8)–(9) has a unique ***weak*** solution $c \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*)$ so that

$$\left.\begin{aligned} \langle d_t c(t), v(t)\rangle_{\mathcal{V}^*,\mathcal{V}} + a\left(t; c(t), v(t)\right) = \langle f(t), v(t)\rangle_{\mathcal{V}^*,\mathcal{V}}, \forall v \in \mathcal{V}, \forall t \in (0,T)\,, \\ c_0(\vec{x}) = 0, \end{aligned}\right\} \tag{10}$$

where v's are test functions whose collection generates a Banach space. Eventually (1)–(2) has a unique solution.

3 Result Analysis

Proposition 3.1. *Let $c \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*)$ be a unique **weak** solution of (8)–(9). If every property of Theorem .1 is satisfied, then (1)–(2) has a unique classical solution.*

Proof. To show that there exists a weak solution that satisfies (8)–(9), we first recast (10) into a Banach space. Thus, we seek the solution

$$\boxed{c \in \mathcal{X} \quad \text{such that} \quad a(c, v) = \mathcal{L}(v), \quad \forall v \in \mathcal{Y}}$$

where the bi-linear function

$$a(c, v) = \int_{t=0}^{T} \left\{\langle d_t c(t), v(t)\rangle_{\mathcal{V}^*,\mathcal{V}} + a\left(t; c(t), v(t)\right)\right\} dt, \tag{11}$$

$$\mathcal{L}(v) = \langle f, v\rangle_{\mathcal{Y}^*,\mathcal{Y}} = \int_0^T \langle f(t), v(t)\rangle_{\mathcal{V}^*,\mathcal{V}} dt, \tag{12}$$

$a(c, v) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, $\mathcal{X} = \{c \in \mathcal{W}(\mathcal{V}, \mathcal{V}^*) : c(0) = 0\}$, $\mathcal{Y} = L^2(0, T; \mathcal{V})$, and $\mathcal{Y}^* = L^2(0, T; \mathcal{V}^*)$.

First, we show that Eq. (8) satisfies property (i) of the Theorem (.1). Re-arranging Eq. (8), we have

$$a\left(t; c(t), v(t)\right) = -\sum_{i,j} \int_0^T \frac{\partial}{\partial t}\left(a_{i,j}\frac{\partial c(t)}{\partial t}\right) v(t) dt + \sum_i \int_0^T \left(b_i \frac{\partial c(t)}{\partial t}\right) v(t) dt + \int_0^T \tilde{g} c(t) v(t) dt. \tag{13}$$

Since the functions to the RHS of Eq. (13) and their derivatives are continuous and integrable over $L^2(0, T; \mathcal{V})$, then by definitions (2.4) and (2.5), the mapping $t \mapsto a\left(t; c(t), v(t)\right)$ is Lebesgue measurable for each $t \in [0, T]$.

Then, we show that **boundedness** property (ii) of Theorem (.1) is satisfied. Integrating the first integral on the RHS of Eq. (13) by parts, and using the initial condition $c(0) = 0$ (see 9a), yields

$$a\,(t;\cdot,\cdot) = \sum_{i,j} \int_0^T a_{i,j} \frac{\partial c(t)}{\partial t} \frac{\partial v(t)}{\partial t} dt + \sum_i \int_0^T \left(b_i \frac{\partial c(t)}{\partial t} \right) v(t) dt + \int_0^T \tilde{g} c(t) v(t) dt. \tag{14}$$

Taking the modulus of (14), we obtain

$$|a\,(t; c(t), v(t))| \leq \bar{m} \left\{ \sum \left| \int_0^T \frac{\partial c(t)}{\partial t} \frac{\partial v(t)}{\partial t} dt \right| + \sum \left| \int_0^T \frac{\partial c(t)}{\partial t} v(t) dt \right| + \left| \int_0^T c(t) v(t) dt \right| \right\}, \tag{15}$$

where $\bar{m} = \max \left\{ \max_{i,j \leq n; t \in [0,T]} |a_{i,j}(t)|, \max_{i \leq n; t \in [0,T]} |b_i(t)|, \max_{t \in [0,T]} |\tilde{g}(t)| \right\}$. Applying the Hölders inequality (lemma 2.1), for $p = 2$, into (15) gives

$$\begin{aligned} |a\,(t; c(t), v(t))| \leq \bar{m} & \left\{ \sum \left(\int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \left| \frac{\partial v(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \right\} \\ + \bar{m} & \left\{ \sum \left(\int_0^T \left| \frac{\partial c(t)}{\partial t} \right|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |v(t)|^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T |c(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |v(t)|^2 dt \right)^{\frac{1}{2}} \right\}. \end{aligned} \tag{16}$$

Algebraically, (16) can be expressed as

$$\boxed{|a\,(t; c(t), v(t))| \leq \bar{m} \left\{ \tilde{x}\tilde{y} + \tilde{x}\tilde{c} + \tilde{v}\tilde{c} + \tilde{v}\tilde{y} \right\},}$$

where the term $\tilde{v}\tilde{y} = 0$. Since it can be shown, using the Poincar$\acute{e}$-Friedrich's inequality (lemma 2.9), that

$$\tilde{v}\tilde{y} = \left(\int_0^T |c(t)|^2 dt \right)^{\frac{1}{2}} \sum \left(\int_0^T \left| \frac{\partial v(t)}{\partial t} \right| dt \right)^{\frac{1}{2}} \geq \frac{\|c(t)\|}{\sqrt{k_*}} \left(\int_0^T |v(t)|^2 dt \right)^{\frac{1}{2}} = 0, \tag{17}$$

when $c(t) = 0$ on $\partial\Psi$ (see 9b), substituting (17) into (16) leads to

$$|a\left(t;c(t),v(t)\right)| \leq \bar{m}\left\{\left(\int_0^T |c(t)|^2 dt\right)^{\frac{1}{2}} + \sum\left(\int_0^T \left|\frac{\partial c(t)}{\partial t}\right|^2 dt\right)^{\frac{1}{2}}\right\} \times \left\{\left(\int_0^T |v(t)|^2 dt\right)^{\frac{1}{2}} + \sum\left(\int_0^T \left|\frac{\partial v(t)}{\partial t}\right|^2 dt\right)^{\frac{1}{2}}\right\}. \quad (18)$$

By majorization [25], it can be shown that

$$\boxed{\tilde{w}^{\frac{1}{2}} + \sum_{j=1}^{n} (\S_j)^{\frac{1}{2}} \leq \sqrt{2}\left(\tilde{w} + \sum_{j=1}^{n} \S_j\right)^{\frac{1}{2}}, \forall \tilde{w} \in \mathbb{R}, \S_j \in \mathbb{R}^n,}$$

so that the RHS of (18) simplifies to

$$\leq 2\bar{m}\left\{\int_0^T |c(t)|^2 dt + \sum \int_0^T \left|\frac{\partial c(t)}{\partial t}\right|^2 dt\right\}^{\frac{1}{2}} \times \left\{\int_0^T |v(t)|^2 dt + \sum \int_0^T \left|\frac{\partial v(t)}{\partial t}\right|^2 dt\right\}^{\frac{1}{2}}. \quad (19)$$

Employing the Cuachy-Schwarz inequality (lemma 2.2) under the normed Banach space for $p = 2$ onto (19), and letting $\alpha_0 = 2\bar{m}$, the property (ii) of Theorem (.1)

$$|a\left(t;c(t),v(t)\right)| \leq \alpha_0 \|c(t)\|_{\mathcal{V}} \|v(t)\|_{\mathcal{V}}, \quad (20)$$

is satisfied.

To establish **positive definiteness** property (iii) of Theorem (.1), we use the bilinear form

$$\boxed{a\left(t;\cdot,\cdot\right) = \sum_{i,j} \int_0^T a_{i,j} \frac{\partial c(t)}{\partial t}\frac{\partial c(t)}{\partial t} dt + \sum_i \int_0^T \left(b_i \frac{\partial c(t)}{\partial t}\right) c(t) dt + \int_0^T \tilde{g} c(t) c(t) dt.}$$

Using the uniform ellipticity condition [12],

$$\boxed{\sum_{i,j=1} a_{i,j}(x)\xi_i\xi_j \geq \tilde{d}\sum_{i=1} \xi_i^2, \forall \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n, x \in \bar{\Psi}}$$

where $\tilde{d} > 0$ is constant, independent of x and ξ, we obtain

$$a\left(t;c(t),c(t)\right) \geq \tilde{d}\sum\int_0^T\left|\frac{\partial c(t)}{\partial t}\right|^2 dt+\sum_i\int_0^T\left(b_i\frac{\partial c(t)}{\partial t}\right)c(t)dt+\int_0^T\tilde{g}c(t)^2dt. \quad (21)$$

Since, by the chain rule, we have

$$\boxed{\int_0^T\frac{\partial c(t)}{\partial t}c(t)dt=\int_0^T\frac{1}{2}\frac{\partial}{\partial t}c(t)^2dt,}$$

then applying integration by parts to the above expression and substituting the result into (21) yields

$$a\left(t;c(t),c(t)\right) \geq \tilde{d}\sum\int_0^T\left|\frac{\partial c(t)}{\partial t}\right|^2 dt+\int_0^T\left\{\tilde{g}-\frac{1}{2}\sum_i\frac{\partial b_i(t)}{\partial t}\right\}|c(t)|^2dt. \quad (22)$$

Because b_i and $\tilde{g}$ are non-negative advection, and reaction and adsorption coefficients, respectively

$$\boxed{\tilde{g}-\frac{1}{2}\sum_i\frac{\partial b_i(t)}{\partial t}\geq 0, \forall i=1,2,\cdots,n; \quad \forall\vec{x}.}$$

Therefore,

$$a\left(t;c(t),c(t)\right) \geq \tilde{d}\sum\int_0^T\left|\frac{\partial c(t)}{\partial t}\right|^2 dt. \quad (23)$$

But, by the Poincaré-Friedrich's inequality (lemma 2.9),

$$\boxed{\sum\int_0^T\left|\frac{\partial c(t)}{\partial t}\right|^2 dt \geq \frac{1}{k_*}\int_0^T|c(t)|^2dt,}$$

so that

$$k_*a\left(t;c(t),c(t)\right) \geq \tilde{d}\int_0^T|c(t)|^2dt. \quad (24)$$

Summing (23) and (24) yields

$$a\left(t;c(t),c(t)\right) \geq \frac{\tilde{d}}{1+k_*}\left\{\int_0^T|c(t)|^2dt+\sum\int_0^T\left|\frac{\partial c(t)}{\partial t}\right|^2 dt\right\}, \quad (25)$$

so that

$$a\,(t;c(t),c(t)) \geq \alpha_1 \|c(t)\|_{\mathcal{V}}^2, \tag{26}$$

where $\alpha_1 = \dfrac{\tilde{d}}{1+k_*}$. Therefore, the property (iii) of Theorem (.1) is satisfied.

Then, to show **uniqueness**, by the Lax-Milgram Theorem (.2), (26) implies that

$$\alpha_1 \|c(t)\|_{\mathcal{V}}^2 \leq a\,(t;c(t),c(t)) = \mathcal{L}(c(t)) = \langle f(t), c(t)\rangle \, \forall f(t) \in \mathcal{X}, \tag{27}$$

whenever $\mathcal{V} = \mathcal{W}\,(\mathcal{V},\mathcal{V}^*)$.

Since $\mathcal{L}(c(t))$ is linear functional on $\mathcal{X}$, applying the Reisz Representation Theorem[7] (RRT) [22], we have

$$\langle f(t), c(t)\rangle \leq |\langle f(t), c(t)\rangle| \leq \|f(t)\|_{L^2(0,T;\mathcal{V}^*)} \|c(t)\|_{L^2(0,T;\mathcal{V})},$$

so that

$$\|c(t)\|_{\mathcal{W}(\mathcal{V},\mathcal{V}^*)} \leq \frac{1}{\alpha_1} \|f(t)\|_{L^2(0,T;\mathcal{V}^*)} = \frac{1}{\alpha_1} \|f(t)\|_{\mathcal{Y}^*}. \tag{28}$$

Moreover, since the variable $c \in L^2\,(0,T;\mathcal{V})$, we can use Lemma (2.10) to show that

$$-2 \int_0^T \frac{1}{2} \frac{\partial}{\partial t} \|c(t)\|^2 dt = \int_0^T a\,(t;c(t),c(t))\, dt - \|c(T)\|^2. \tag{29}$$

Hence, substituting the result (26) into Eq. (29) results in

$$-\frac{\partial}{\partial t} \int_0^T \|c(t)\|^2 dt \geq \alpha_1 \int_0^T \|c(t)\|^2 dt - \|c(T)\|^2 \geq 0, \tag{30}$$

so that

$$\frac{\partial}{\partial t} \int_0^T \|c(t)\|^2 dt \leq 0. \tag{31}$$

□

Remarks

- Since the result (31) is a condition (3) set by the Energy method (EM), we have used the VF-EM approach to show that (8) is well posed.
- Results (20), (26), and (28) respectively, mean (8) has a non-negative, bounded, and unique solution.

[7] any bounded linear function on a Hilbert space $\mathcal{H}$ can be represented as an inner product with some unique vector in $\mathcal{H}$

4 Conclusions

In this paper, we have investigated the existence and uniqueness of a solution to the advection-diffusion reaction equation in a Sobolev space for the problem of modelling transport of contaminants in porous medium. The main result is proved by using the Banach-Nečas-Babuška theorem based on identifying weak solutions in the Sobolev space. By using the same methodology and concepts as deliberated in this manuscript, we can extend the results to partial differential equations of heat transport transport equations. In future, will explore the use of the Galerkin's method to prove the existence of solutions of coupled PDEs, and approximate their solutions [26].

Acknowledgements. This work was produced as part of PhD study of the corresponding author at Pan African University-Institute for Basic Sciences, Technology & Innovation (PAUISTI), funded by the African Union (AU). We are therefore grateful to both AU and PAUISTI. Sincere thanks are also due to Mr. John A. Chirwa for language editing and proofreading of the manuscript.

Conflict of interest. The authors declare no conflict of interest.

Appendix

Theorem .1. *Banach-Nečas-Babuška Theorem [14, 15]*
Assume that the bilinear form $a\,(t;\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ satisfies the following properties;

(i) The mapping $t \mapsto a\,(t;\cdot,\cdot)$ is measurable for all $u, v \in \mathcal{V}$,

(ii) There exists $\alpha_0 > 0$ such that $|a\,(t;u,v)\,| \leq \alpha_0 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}$ for almost every $t \in (0,T)$ and all $u, v \in \mathcal{V}$,

(iii) There exists $\alpha_1 > 0$ such that $a\,(t;u,u) \geq \alpha_1 \|u\|^2_{\mathcal{V}}$ for almost every $t \in (0,T)$ and all $u \in \mathcal{V}$.

Then, a given parabolic equation has a unique solution $u \in \mathcal{W}^p_1\,(\mathcal{V},\mathcal{V}^*)$ satisfying

$$\|u\|_{\mathcal{W}^p_1(\mathcal{V},\mathcal{V}^*)} \leq \frac{1}{\alpha_1} \|f\|_{\mathcal{Y}^*}.$$

Theorem .2. *Lax-Milgram Theorem [11, 12]*
Suppose that $\mathcal{V}$ is a real Hilbert space equipped with norm $\|\cdot\|_{\mathcal{V}}$. Let $a\,(t;\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a bilinear form on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{L}\,(\cdot)$ be a linear form on $\mathcal{V}$ such that;

(i) there exists $c_0 > 0$ *such that* $a(v,v) \geq c_0 \|v\|_{\mathcal{V}}^2$ *for all* $v \in \mathcal{V}$,

(ii) there exists $c_1 > 0$ *such that* $|a(v,w)| \leq c_0 \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}}$ *for all* $v, w \in \mathcal{V}$,

(iii) there exists $c_2 > 0$ *such that* $|\mathcal{L}(v)| \leq c_2 \|v\|_{\mathcal{V}}$ *for all* $v \in \mathcal{V}$.

Then, there exists a unique $u \in \mathcal{V}$ *satisfying*

$$\|u\|_{\mathcal{H}^1(\Psi)} \leq \frac{1}{c_0} \|f\|_{L^2(\Psi)}$$

such that

$$a(u,v) = \mathcal{L}(v) \, \forall v \in \mathcal{V}.$$

References

[1] J. Bear, *Modelling Phenomena of Flow and Transport in Porous Media,* New York, Springer, 2018.

[2] K.D. Malay, P.P. Mukherjee, K. Muralidhar, *Modeling Transport Phenomena in Porous Media with Applications*, New York, Springer, 2017.

[3] F.A. Rihan, M.N. Anwar, Qualitative Analysis of Delayed SIR Epidemic Model with a Saturated Incidence Rate, *International Journal of Differential Equations,* (2012), 1–13. https://doi.org/10.1155/2012/408637

[4] R. Almeida, M.T.T. Monteiro, E. Venturino, L. Machado, A qualitative analysis of a Mycoplasma genitalium epidemiological model, *Computational and Mathematical Methods,* **3** (6) (2021), e1199. https://doi.org/10.1002/cmm4.1199

[5] F.L.L.B., Carneiro, On the use, by Einstein, of the principle of dimensional homogeneity, in three problems of the physics of solids, *Anais da Academia Brasileira de Ciências,* **72** (4) (2000), 591–596. https://doi.org/10.1590/S0001-37652000000400009

[6] E.F. Adiutori, *A new view of dimensional homogeneity and its impact on the fundamental equations and parameters of heat transfer science*, General Papers in Heat Transfer. Ventuno Press, West Chester, Ohio., 1992. Available from: http://thenewengineering.com/Conference199208B.pdf

[7] E.F. Adiutori, An Alternate View of Dimensional Homogeneity, and Its Impact on Engineering Science, *The Open Mechanical Engineering Journal,* **12** (2018), 164–174. https://doi.org/10.2174/1874155X01812010164

[8] A. Hasan-Zadeh, Solving Advection-Diffusion Equations via Sobolev Space Notions, *International Journal of Partial Differential Equations and Applications,* **8** (1) (2020), 1–5. https://doi.org/10.12691/ijpdea-8-1-1

[9] V. Maźya, Sobolev space in Mathematics II: Applications in analysis and partial differential equation, *International Mathematical Series,* **9** (2009).

[10] P.P.C. Alzate, G.C. Vélez, F. Mesa, A Note on the Variational Formulation of PDEs and Solution by Finite Elements, *Advanced Studies in Theoretical Physics,* **12** (4) (2018), 173–179. https://doi.org/10.12988/astp.2018.8412

[11] P.D. Lax, A.N. Milgram, Parabolic Equations, *Ann. Math. Studies,* **33** (1954).

[12] E. Süli, *An Introduction to the Numerical Analysis of Partial Differential Equations*, Lecturer notes, University of Oxford., Feb 2005. Available from: http://people.maths.ox.ac.uk/suli/nspde.pdf

[13] C. Clason, *Numerical Partial Differential Equations. Lecturer notes*, Institute for Mathematics and Scientific Computing, Karl-Franzens-Universität Graz., July 2014. Available from: https://www.uni-due.de/ adf040p/skripte/NumPDENotes11.pdf

[14] I. Babuška, A.K. Aziz, *Survey Lectures on the Mathematical Foundation of the Finite Element Method* in: A.K. Aziz, (Ed.), *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* New York: Academic Press, 1972.

[15] I. Rosca, On the Babuška-Lax-Milgram theorem, *An. Univ. Bucuresti Mat.,* **38** (3) (1989), 61–65.

[16] E.P. Mwakilama, D.K. Gathungu, V.M. Magagula, Use of the Repeated Integral Transformation Method to Describe the Transport of Solute in Soil, *Heliyon,* Available at SSRN: https://ssrn.com/abstract=4082890 or https://doi.org/10.2139/ssrn.4082890

[17] D. Boffi, F. Brezzi, M. Fortin, *Variational Formulations and Finite Element Methods*, in: D. Boffi, F. Brezzi, M. Fortin, (Ed.), *Mixed Finite Element Methods and Applications. Springer Series in Computational Mathematics,* Berlin: Springer, 2013. https://doi.org/10.1007/978-3-642-36519-5_1

[18] S.N. Antontsev, S.I. Shmarev, The energy method. Application to PDEs of hydrodynamics with nonstandard growth, *J. Phys.: Conf. Ser.,* **894** (2017). https://doi.org/10.1088/1742-6596/894/1/012001

[19] A. Hocquet, M. Hofmanov, An energy method for rough partial differential equations, *Journal of Differential Equations,*
(2017). https://doi.org/10.1016/j.jde.2018.04.006

[20] A. Alphonse, C.M. Elliott, B. Stinner, *An abstract framework for parabolic PDEs on evolving spaces*, 2014. Available from: https://homepages.warwick.ac.uk/staff/C.M.Elliott/AlpEllSti14.pdf

[21] P. Sacks, *Techniques of Functional Analysis for Differential and Integral Equations,* New York, Elsevier Inc., 2017. https://doi.org/10.1016/C2016-0-01322-0

[22] E. Kreyszig, *Introductory Functional Analysis with Applications*, New York, John Wiley & Sons, Inc., 1989.

[23] R. Magnus, *Fundamental Mathematical Analysis*, 1 Eds., Switzerland: Springer International Publishing, 2020. https://doi.org/10.1007/978-3-030-46321-2

[24] J. Muscat, *Functional Analysis: An Introduction to Metric Spaces, Hilbert Spaces, and Banach Algebras*, 1 Eds., Switzerland: Springer International Publishing, 2014. https://doi.org/10.1007/978-3-319-06728-5

[25] A.W. Marshall, J. Olkin, B. Alnodi, *Inequalities: Theory of majorization and its applications*, 2 Eds., New York: Springer, 2011.

[26] P. Sacks, *Weak Solutions of Partial Differential Equations*, in: P. Sacks, (Ed.), *Techniques of Functional Analysis for Differential and Integral Equations*, Chp. 16, (2017), 269–293. https://doi.org/10.1016/B978-0-12-811426-1.00016-7

Received: July 1, 2022; Published: July 28, 2022