Applied Mathematical Sciences, Vol. 16, 2022, no. 6, 273 - 278
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2022.916784

The Origin Moment of q-Gaussian Process

Liu Limin, Liu Xiaojie

College of Mathematics and Information Science
Henan Normal University
Henan Provience, 453007, China

Li Haoran

Center for Economics, Finance and Management Studies
Hunan University
Hunan Province, 410000, China

Abstract

In this paper, we discuss the origin moment of the q-Gaussian process, which is used to describe anomalous correlated diffusion. The origin moment are obtained using Itô formula and the definition of the q-gaussian process. Results of q-Gaussian process are compared with the standard Brownian motion.

Keywords: q-Gaussian process, Itô formula, The origin moment

1 Introduction

In 1998, C.Tsallis [1], the famous statistician, generalized the classical statistical mechanics and proposed the concept of non-extensive statistical mechanics from the perspective of generalized entropy, and discovered a series of non-extensive probability distribution families, shortened as q-distribution. Some scholars maximize the Tsallis derived distribution, that is D.A. Stariolo (1996) [2] study the nonlinear Fokker-Planck stochastic differential equation corresponding to drift term is 0 derived from the non-extensive distribution, called the q-Gaussian distribution.

Previously, Tsallis statistical theory was mainly devoted to the field of statistical physics, and it was recognized that it could make the basic concepts of thermodynamic statistical physics more generalized. Later, non-extended statistical mechanics is also widely used in financial markets. L. Borland (2007) [3] use q-Gaussian distribution to describe the distribution characteristics of stock price fluctuations, and many scholars study the properties of multiple q-Gaussian distributions. F.Michael, M.D. Johnson (2003) [4] study the maximum Tsallis entropy distribution under the condition of average distribution of maximize Tsallis entropy, deduced the q-Gaussian distribution density function. C. Vignat, A. Plastino (2007) [5] studied the problem of estimating the appropriate value of parameter q in q-Gaussian distribution. L. Borland (1998) [6] from the perspective of microscopic dynamics, discussed the process of martensite and the long-range correlation in this paper, Liu Limin et al.(2020) [7] proved the non-Markov property of q-Gaussian processes by numerical simulation.

2 Main Results

We assume as given a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions. Let W be a standard Brownian motion and the $(\mathcal{F}_t)_{t\geq 0}$ is the underlying filtration for the Brownian motion W.

Definition 2.1. *The process $\Omega = (\Omega(t))_{t\geq 0}$ is called a q-gaussian process if $\Omega(t)$ satisfy the following stochastic differential equation*

$$\begin{cases} d\Omega(t) = p(\Omega(t), t)^{\frac{1-q}{2}} dW(t), \\ \Omega(0) = 0, \end{cases} \tag{1}$$

where

$$p(x,t) = \frac{1}{Z_q(t)}[1-\beta(t)(1-q)x^2]^{\frac{1}{1-q}},$$

$$\beta(t) = c^{\frac{1-q}{3-q}}[(2-q)(3-q)t]^{\frac{-2}{3-q}},$$

$$Z_q(t) = [(2-q)(3-q)ct]^{\frac{1}{3-q}},$$

$$c = \frac{\pi}{q-1}\frac{\Gamma^2(\frac{1}{q-1}-\frac{1}{2})}{\Gamma^2(\frac{1}{q-1})},$$

and $\Gamma(.)$denote as a Gamma function.

Remark: q is a parameter reflecting the degree of diffusion. When $q = 1$, the system is normal diffusive and the process is Brownian motion. When $q > 1$, the system is superdiffusive and when $q < 1$, the system is subdiffusive.

The main results of this paper are given by the next theorem.

Theorem 2.2. *Let $\Omega(t)$ be a q-gaussian process. Then*

$$E(\Omega^{2n}(t)) = \frac{(2n-1)!!}{(5-3q)\times ...\times((2n+3)-(2n+1)q)\beta^n(t)}, n=1,2,... \qquad (2)$$

and

$$E(\Omega^{2n-1}(t)) = 0, n=1,2,... \qquad (3)$$

Proof. First, applying the Itô formula, we have

$$\begin{aligned} d\Omega^2(t) &= 2\Omega(t)d\Omega(t) + p^{1-q}(\Omega(t),t)dt \\ &= 2\Omega(t)p^{\frac{1-q}{2}}(\Omega(t),t)dW(t) + Z_q(t)^{q-1}[1-\beta(t)(1-q)\Omega^2]dt, \end{aligned}$$

Taking expectations we have

$$dE\Omega^2(t) = Z_q(t)^{q-1}[1-\beta(t)(1-q)E(\Omega^2(t))]dt.$$

Therefore

$$\begin{aligned} E\Omega^2(t) = \exp&\left(-\int(Z_q(t))^{q-1}\beta(t)(1-q)dt\right) \\ &\times\left\{C+\int[(Z_q(t))^{q-1}\exp\{\int(Z_q(t))^{q-1}\beta(t)(1-q)\}dt]dt\right\}. \end{aligned}$$

Since

$$\int(Z_q(t))^{q-1}\beta(t)(1-q)dt = \frac{(1-q)}{(2-q)(3-q)}\ln t, \qquad (4)$$

we conclude that

$$\begin{aligned} E\Omega^2(t) &= t^{-\frac{(1-q)}{(2-q)(3-q)}}\times\left\{C+\int[(2-q)(3-q)ct]^{-\frac{1-q}{3-q}}t^{\frac{(1-q)}{(2-q)(3-q)}}dt\right\} \\ &= t^{-\frac{(1-q)}{(2-q)(3-q)}}\times\left\{C+[(2-q)(3-q)c]^{-\frac{1-q}{3-q}}\frac{1}{\frac{2}{3-q}+\frac{(1-q)}{(2-q)(3-q)}}t^{\frac{2}{3-q}+\frac{(1-q)}{(2-q)(3-q)}}\right\} \\ &= Ct^{-\frac{(1-q)}{(2-q)(3-q)}} + t^{\frac{2}{3-q}}[(2-q)(3-q)]^{\frac{2}{3-q}}c^{\frac{-(1-q)}{3-q}}\frac{1}{5-3q} \\ &= Ct^{-\frac{(1-q)}{(2-q)(3-q)}} + \frac{1}{\beta(t)(5-3q)}. \end{aligned}$$

Substituted $E\Omega^2(0)=0$ yields

$$E\Omega^2(t) = \frac{1}{\beta(t)(5-3q)}.$$

Next, using the Itô formula, we have

$$d\Omega^4(t) = 4\Omega(t)^3p^{\frac{1-q}{2}}(\Omega(t),t)dW(t) + 6\Omega^2(t)(Z_q(t))^{q-1}[1-\beta(t)(1-q)\Omega^2(t)]dt,$$

Taking expectations we have

$$\begin{aligned} dE\Omega^4(t) &= 6E\Omega^2(t)(Z_q(t))^{q-1}dt - 6(Z_q(t))^{q-1}\beta(t)(1-q)E\Omega^4(t)dt \\ &= \frac{6}{\beta(t)(5-3q)}(Z_q(t))^{q-1}dt - 6(Z_q(t))^{q-1}\beta(t)(1-q)E\Omega^4(t)dt, \end{aligned}$$

Combing the same argument and the integral of (4) yields

$$\begin{aligned} E(\Omega^4(t)) &= \left(\exp\int -6(Z_q(t))^{q-1}\beta(t)(1-q)dt\right) \\ &\quad \left\{\int 6\frac{1}{\beta(t)(5-3q)}(Z_q(t))^{q-1}\left(\exp\int 6(Z_q(t))^{q-1}\beta(t)(1-q)dt\right)dt\right\}. \\ &= t^{-\frac{6(1-q)}{(2-q)(3-q)}}\int 6\left(\frac{1}{\beta(t)(5-3q)}(Z_q(t))^{q-1}t^{\frac{6(1-q)}{(2-q)(3-q)}}\right)dt, \\ &= \frac{6}{5-3q}t^{-\frac{6(1-q)}{(2-q)(3-q)}}\int\left(c^{\frac{-2(1-q)}{3-q}}[(2-q)(3-q)]^{\frac{1-q}{3-q}}t^{\frac{2}{3-q}+\frac{6(1-q)}{(2-q)(3-q)}}\right)dt, \\ &= \frac{6}{5-3q}c^{\frac{-2(1-q)}{3-q}}[(2-q)(3-q)]^{\frac{4}{3-q}}\frac{1}{(14-10q)}t^{\frac{4}{3-q}}, \\ &= \frac{3}{(5-3q)(7-5q)\beta(t)^2}. \end{aligned}$$

A similar calculation can be obtained

$$E\Omega^6(t) = \frac{3}{(5-3q)(7-5q)(9-7q)\beta(t)^3}.$$

Therefore

$$E\Omega^{2n}(t) = \frac{(2n-1)!!}{(5-3q)\times ...\times((2n+3)-(2n+1)q)\beta^n(t)}, n = 1, 2, ...$$

Second, since $\Omega(t)$ is a martingale, we conclude $E\Omega(t) = 0$. Therefore

$$\begin{aligned} d\Omega^3(t) &= 3\Omega(t)^2 p^{\frac{1-q}{2}}(\Omega(t),t)dW(t) + 3\Omega(t)p^{1-q}(\Omega(t),t)dt \\ &= 3\Omega(t)^2 p^{\frac{1-q}{2}}(\Omega(t),t)dW(t) + 3\Omega(t)(Z_q(t))^{q-1}[1-\beta(t)(1-q)\Omega^2(t)]dt, \end{aligned}$$

Taking expectations yields

$$\begin{aligned} dE\Omega^3(t) &= 3E\Omega(t)(Z_q(t))^{q-1}dt - 3(Z_q(t))^{q-1}\beta(t)(1-q)E(\Omega^3(t))dt \\ &= -3(Z_q(t))^{q-1}\beta(t)(1-q)E(\Omega^3(t)dt \end{aligned}$$

Thus

$$E\Omega^3(t) = C\exp\left(\int \left(-3(Z_q(t))^{q-1}\beta(t)(1-q)\right)dt\right) = Ct^{\frac{3(q-1)}{(2-q)(3-q)}}.$$

Since $\Omega(0) = 0$, then $E\Omega^3(0) = 0$. Therefore

$$E\Omega^3(t) = 0.$$

Using the Itô formula, we have

$$d\Omega^5(t) = 5\Omega(t)^4 p^{\frac{1-q}{2}}(\Omega(t),t)dW(t) + 10\Omega^3(t)(Z_q(t))^{q-1}[1-\beta(t)(1-q)\Omega^2(t)]dt,$$

Taking expectations yields

$$\begin{aligned} dE\Omega^5(t) &= 10E\Omega^3(t)(Z_q(t))^{q-1}dt - 10(Z_q(t))^{q-1}\beta(t)(1-q)E(\Omega^5(t))dt \\ &= -10(Z_q(t))^{q-1}\beta(t)(1-q)E(\Omega^3(t)dt \end{aligned}$$

The same argument yields

$$E\Omega^5(t) = 0.$$

Therefore

$$E(\Omega^{2n-1}(t)) = 0, n = 1, 2, ...$$

□

Remark: In order to ensure the variance of stochastic process is exist, we assume that $1 \leq q < \frac{5}{3}$. If $q = 1$, then $\beta(t) = (2t)^{-1}$. Thus the origin moment of the q-gaussian process is given by

$$E(\Omega^{2n}(t)) = (2n-1)!!t^n, \quad E(\Omega^{2n-1}(t)) = 0, n = 1, 2, ...,$$

which is the same results with the standard Brownian motion $W(t)$.

References

[1] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, *Journal of Statistical Physics*, **52** (1988), 479-487. https://doi.org/10.1007/bf01016429

[2] D.A. Stariolo, Aging in models of non-linear diffusion, *Physica Review E Statistical Physics Plasmas Fluids and Related Interdisciplinary Topics*, **55** (1996), 4806-4809. https://doi.org/10.1103/physreve.55.4806

[3] L. Borland, J.P. Bouchaud and Q. Finance et.al, A non-Gaussian option pricing model with skew, *Quantitative Finance*, **7** (2007), 703. https://doi.org/10.1080/14697680701790014

[4] F. Michael, M.D. Johnson, Financial Market Dynamics, *Physica A Statistical Mechanics and Its Applications*, **320** (2003), 525-534. https://doi.org/10.1016/s0378-4371(02)01558-3

[5] C. Vignat, A. Plastino, Scale invariance and related properties of q-Gaussian systems, *Physics Letters A*, **365** (2007), 370-375. https://doi.org/10.1016/j.physleta.2007.02.003

[6] L. Borland, Microscopic dynamics of the nonlinear Fokker-Planck equation: A phenomenological model, *Physical review E, Statistical physics, plasmas, fluids, and related interdisciplinary topics*, **57** (1998), 6634-6642. https://doi.org/10.1103/physreve.57.6634

[7] L. Limin, C. Yingying and X. Jie, The non-Markovian property of q-Gaussian process, *Computers Mathematics with Applications*, **79** (2020), 1802-1812. https://doi.org/10.1016/j.camwa.2019.10.006

Received: June 21, 2022; Published: July 14, 2022