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Unicyclic Graphs with the Total Domination Number Twice the Distance-2 Domination Number

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Abstract

The distance between two vertices u and v in a graph equals the length of a shortest path from u to v. The distance-2 domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a vertex subset where every vertex not belonging to the set is within distance two from some element of the set. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a vertex subset where every vertex of G is adjacent to an element of the set. For any nontrivial connected graph G, we can see that $\gamma_t(G) \geq 2$. A unicyclic graph is a connected graph containing exactly one cycle. Here we focus on the unicyclic graphs. For $n \geq 1$, let $\mathscr{H}(n)$ be the set of unicyclic graphs H satisfying $\gamma_t(H) = 2\gamma_2(H) = 2n$. In this paper, we provide a constructive characterization of $\mathscr{H}(n)$ for all $n \geq 1$.

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Keywords: unicyclic graph, distance-2 domination number, total domination number

1 Introduction

One of the fastest growing areas within graph theory is the study of domination and related subset problems. The literature on the domination parameters in graphs has been detailed in the two books ([14],[15]). The decision problem

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of determining the domination number of a graph G is NP-complete even if G is bipartite [14]. We want to study two domination parameters, including distance-2 domination and total domination.

The concept of distance dominating set was initiated by Slater [21]. The distance domination problem is NP-complete for general graphs [5]. The distance dominating set is an important property used in the allocation of finite resources to massively parallel architecture. It also helps in sharing resources amongst the nodes and thereby lays the framework for designing alternate parallel paths should one or more of the nodes fail. Here we discuss the distance-2 domination. A set D of vertices is a distance-2 dominating set (D2DS) if every vertex not belonging to D is within distance two from some element of D. The distance-2 domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a distance-2 dominating set in G. A D2DS D of G is called a γ_2-set of G if $|D| = \gamma_2(G)$. Sridharan, Subramanian and Elias [22] obtain various upper bounds for $\gamma_2(G)$ and characterize the classes of graphs attaining these bounds. Bibi, Lakshmi and Jothilakshmi [3] presented an algorithm for finding a minimal and minimum distance-2 dominating sets of graph. They also explored on the applications of distance-2 dominating sets in networks [4].

A set D of vertices in a graph G is called a total dominating set (TDS) if every vertex of G is adjacent to an element of D. The total domination number of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set in G. For any nontrivial connected graph G, we can see that $\gamma_t(G) \geq 2$. A TDS D of G is called a γ_t-set of G if $|D| = \gamma_t(G)$. A γ_t-set D of G is called a $(\gamma_t, -L)$-set of G if D contains no leaves of T. The problem of determining the total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [8]. The problem of determining the total domination in graphs is now well studied in graph theory ([1],[2],[7],[9],[10],[11], [12],[13],[16],[19]). Laskar, Pfaff, Hedetniemi and Hedetniemi [20] presented a linear time algorithm to determine minimum total dominating sets of a tree and showed that for undirected path graphs the problem remains NP-complete. More details on the total domination problems can be found in survey papers by Henning [17].

For $n \geq 1$, let $\mathscr{H}(n)$ be the set of unicyclic graphs T satisfying $\gamma_t(H) = 2\gamma_2(H) = 2n$. In this paper, we provide a constructive characterization of $\mathscr{H}(n)$ for all $n \geq 1$.

2 Notations and preliminary results

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. The (open) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *closed neighborhood* $N_G[v]$ is $N_G[v] =$

$N_G(v) \cup \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G(A) = \bigcup_{v \in A} N_G(v)$ and $N_G[A] = \bigcup_{v \in A} N_G[v]$. The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. A vertex v is an *isolated vertex* if $\deg_G(x) = 0$. For two sets A and B, the *difference of A and B*, denoted by $A - B$, is the set of all the elements of A that are not elements of B. For a subset $A \subseteq V(G)$, the *deletion of A from G* is the graph $G - A$ obtained by removing all vertices in A and all edges incident to these vertices. A *u-v path* $P : u = v_1, v_2, \ldots, v_k = v$ of G is a sequence of k vertices in G such that $v_i v_{i+1} \in E(G)$ for $i = 1, 2, \ldots, k - 1$. For any two vertices u and v in G , the *distance between u and v*, denoted by $dist_G(u, v)$, is the minimum length of the u-v paths in G. Denote by P_n a *n-path* with n vertices. The length of P_n is n-1. For any vertex v of a graph G, the *distance-2 closed neighborhood of* v is $N_G^2[v] = \{u : dist_G(u, v) \leq 2\}$ and the *distance-2* (*open*) *neighborhood of* v is $N_G^2(v) = N_G^2[v] - \{v\}$. For any subset $A \subseteq V(G)$, denote $N_G^2(A) = \bigcup_{v \in A} N_G^2(v)$ and $N_G^2[A] = \bigcup_{v \in A} N_G^2[v]$. A *forest* is a graph with no cycles, and a *tree* is a connected forest. A tree T is called *nontrival* if $T| \geq 2$. A *complete bipartite graph $K_{m,n}$* is a graph whose vertices can be divided into two disjoint and independent sets A and B, where $|A| = m \geq 1$ and $|B| = n \geq 1$, such that every vertex of A is adjacent to every vertex of B. The *star* $S(v_1, k)$ is the graph $K_{1,k-1}$, where v_1 is a center and $V(S(v_1, k)) = \{v_1, v_2, \ldots, v_k\}$ for $k \geq 2$. The *induced subgraph* $\prec A \succ_G$ induced by $A \subseteq V(G)$ is the graph with vertex set A and the edge set $E(\prec A \succ_G) = \{uv \in E(G) : u, v \in A\}$. The *union* $G = G_1 \cup G_2$ is the graph with the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2)$. For other undefined notions, the reader is referred to [6] for graph theory.

We begin with the following two straightforward observations.

Observation 2.1. *If D is a D2DS of a graph G, then $N_G^2[D] = V(G)$.*

Observation 2.2. *If D is a TDS of a graph G, then $N_G(D) = V(G)$.*

The following lemmas are useful.

Lemma 2.3. [18] *Let G be a connected graphs of order $|G| \geq 2$. If D is a TDS of G, then $\prec D \succ_G$ has no isolated vertex.*

Proof. If v is an isolated vertex of the subgraph $\prec D \succ_G$, then $N_G(v) \cap D = \emptyset$ and $v \notin N_G(D)$. So D is not a TDS of G, this is a contradiction. □

Lemma 2.4. *Let T be a tree of order $|T| \geq 2$ and S be a γ_t-set of T. Suppose S can be partitioned into $S_1, S_2, \ldots, S_k$ such that each $\prec S_i \succ_T$ is nontrival , where $i = 1, \ldots, k$, and k is as large as possible. For $i = 1, \ldots, k$, let u_i be a center of the induced subgraph $\prec S_i \succ_T$. Then $D = \{u_1, u_2, \ldots, u_k\}$ is a D2DS of T and $2\gamma_2(T) \leq \gamma_t(T)$.*

Proof. Since k is as large as possible, this means that every $\prec S_i \succ_T$ is a star and $|S_i| \geq 2$. So $2k \leq |S|$.
Claim. $N_T^2[D] = V(T)$. Let $v \in V(T)$. Since S is a γ_t-set of T, this implies $v \in N_T(w)$ for some $w \in S$. Suppose $w \in S_j$ for some j and u_j is a center of the induced subgraph $\prec S_j \succ_T$. Since $\prec S_j \succ_T$ is a star, then $dist_T(v, u_j) \leq 2$. Hence $N_T^2[D] = V(T)$.

By Claim, D is a D2DS of T and $2\gamma_2(T) \leq 2|D| = 2k \leq |S| = 2\gamma_t(T)$. We complete the proof. □

3 Characterization

In this section, we characterize the set $\mathscr{H}(n)$ for all $n \geq 1$.

Lemma 3.1. *Let T be a tree and $\gamma_2(T) \geq 1$. Suppose $H = T + u_1u_2$, where $u_1u_2 \notin E(T)$, then $\gamma_2(H) \leq \gamma_2(T)$ and $\gamma_t(H) \leq \gamma_t(T)$.*

Proof. Let S be a γ_2-set of T and and D γ_t-set of T, respectively. Then S is a D2DS of H, so $\gamma_2(H) \leq |D| = \gamma_2(T)$. The set D is a TDS of T, so $\gamma_t(H) \leq |D| = \gamma_t(T)$. □

Lemma 3.2. *Let T be a tree and $\gamma_t(T) \geq 4$. Suppose $H = T + u_1u_2$, where $u_1u_2 \notin E(T)$, then $\gamma_t(H) \geq \gamma_t(T) - 2$.*

Proof. Suppose, by contradiction, $\gamma_t(H) \leq \gamma_t(T) - 3$. Then there exists a γ_t-set S of T such that $S' = S - \{w_1, w_2, w_3\}$ is a TDS of H, where $w_1, w_2, w_3 \in S$ are three distinct vertices. Then $w_i \in N_T[\{u_1, u_2\}]$ for $i = 1, 2, 3$. We consider two cases.
Case 1. $u_i = w_j$ for some i and j. Say $u_1 = w_1$. Let $A = N_H(u_1) - N_H(S - \{u_1\})$. Then $A = \emptyset$. Let $B = N_T(u_1) - N_T(S - \{u_1\})$. Then $B = (N_H(u_1) - \{u_2\}) - N_H(S - \{u_1\}) \subseteq A = \emptyset$, so $B = \emptyset$. Hence $S_1 = S - \{u_1\}$ is a TDS of T, this is a contradiction.
Case 2. $u_i \neq w_j$ for all i and j. Since $w_i \in N_T[\{u_1, u_2\}]$ for $i = 1, 2, 3$, we suppose $w_1, w_2 \in N_T(u_1)$ and $w_3 \in N_T(u_2)$. Hence $N_T(w_i) - N_T(S - \{w_i\}) = \{u_1\}$ for $i = 1, 2$. Let $S_2 = S - \{w_2\}$. Then $N_T(S_2) = V(T)$, so S_2 is a TDS of T with cardinality $|S_2| = |S| - 1 = \gamma_t(T) - 1$. This is a contradiction.

By case 1 and case 2, we complete the proof. □

We first define some collections.

$$\mathscr{T}^{+0}(n) = \{\text{T: T is a tree and } \gamma_t(T) = 2\gamma_2(T) = 2n\}$$

$$\mathscr{T}^{+1}(n) = \{\text{T: T is a tree and } \gamma_t(T) = 2\gamma_2(T) + 1 = 2n + 1\}$$

$$\mathscr{T}^{+2}(n) = \{\text{T: T is a tree and } \gamma_t(T) = 2\gamma_2(T) + 2 = 2n + 2\}$$

Let T be a tree. If $uv \notin E(T)$ and $\gamma_t(T+uv) = \gamma_t(T) - i$, where $i = 0, 1, 2$, then uv is called $R^{-i} - paired$ in T. Let $\widetilde{\mathscr{T}}(n)$ be the collection of the trees in $\mathscr{T}^{+i}(n)$ for all $i = 0, 1, 2$. Suppose $H = T + uv$ is a uncyclic graph obtained from $T \in \widetilde{\mathscr{T}}(n)$, where $uv \notin E(T)$, by one of the following Operations.

Operation 1. Assume $T \in \mathscr{T}^{+0}(n)$ and uv is R^{-0}-paired in T.

Operation 2. Assume $T \in \mathscr{T}^{+0}(n+1)$ and uv is R^{-2}-paired in T.

Operation 3. Assume $T \in \mathscr{T}^{+1}(n)$and uv is R^{-1}-paired in T.

Operation 4. Assume $T \in \mathscr{T}^{+2}(n)$ and uv is R^{-2}-paired in T.

Theorem 3.3. *Suppose $H \in \mathscr{H}(n)$ and $H = T + uv$, where T is a tree and $uv \notin E(T)$. Then H is obtained from $T \in \tilde{\mathscr{T}}(n)$ by one of the Operation 1∼4.*

Proof. By Lemma 3.1 and Lemma 3.2, we have $2n \leq \gamma_t(T) \leq 2n + 2$. We consider three cases.

<u>Case 1.</u> $\gamma_t(T) = 2n$. By Lemma 2.4, then $2n = 2\gamma_2(H) \leq 2\gamma_2(T) \leq \gamma_t(T) = 2n = \gamma_t(H)$, the equalities all hold. Thus uv is R^{-0}-paired in T and $\gamma_2(T) = n$. Hence H is obtained from $T \in \mathscr{T}^{+0}(n)$ by the Operation 1.

<u>Case 2.</u> $\gamma_t(T) = 2n+1$. By Lemma 2.4, then $2n = 2\gamma_2(H) \leq 2\gamma_2(T) \leq \gamma_t(T) = 2n + 1 = \gamma_t(H) + 1$. Thus uv is R^{-1}-paired in T and $\gamma_2(T) = n$. Hence H is obtained from $T \in \mathscr{T}^{+1}(n)$ by the Operation 3.

<u>Case 3.</u> $\gamma_t(T) = 2n+2$. By Lemma 2.4, then $2n = 2\gamma_2(H) \leq 2\gamma_2(T) \leq \gamma_t(T) = 2n + 2 = \gamma_t(H) + 2$. Thus uv is R^{-2}-paired in T and $n \leq \gamma_2(T) \leq n + 1$. If $\gamma_2(T) = n$, then H is obtained from $T \in \mathscr{T}^{+2}(n)$ by the Operation 4. If $\gamma_2(T) = n + 1$, then H is obtained from $T \in \mathscr{T}^{+0}(n+1)$ by the Operation 2.

By Case 1, Case 2 and Case 3, we have that H is obtained from $T \in \tilde{\mathscr{T}}(n)$ by one of the Operation 1∼4. □

By Theorem3.3, we provide a characterization $\mathscr{H}(n)$ for all $n \geq 1$.

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