Applied Mathematical Sciences, Vol. 16, 2022, no. 5, 251 - 260
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2022.916795

Directional Derivatives in Non-Hausdorff TVS: Topological Filter Techniques without Metric Structures

Francesco Colasanto

Università degli Studi di Firenze
Dipartimento di Matematica e Informatica "Ulisse Dini", Italy

Luca Grilli

Università degli Studi di Foggia
Dipartimento di Economia, Management e Territorio, Italy
Corresponding author

Domenico Santoro

Università degli Studi di Bari Aldo Moro
Dipartimento di Economia e Finanza, Italy

Abstract

In this work, we introduce and give some results about directional derivatives in non-Hausdorff Topological Vector Space over general Topological Division Ring. Through the paper, we use some topological filter techniques that are needful for the development of the theory because of the total lack of a metric structure of the spaces, but most of all for the non-uniqueness of the limits due to the absence of the T2 bond.

Mathematics Subject Classification: 58B25, 57N20, 57N17

Keywords: Directional derivatives, Non-Hausdorff, TVS, Filterbase

1 Introduction and preliminary definitions

This section gives some preliminary definitions of Topological Vector Spaces and Filter Theory. For a deeper treatment of these topics, see [5] and [6]. We recall that our work is strongly subordinate to the notion of filter since the general setting.

Definition 1.1. *(Topological Division Ring)*
$(\mathbb{K}, +, \cdot, \beta)$ *is a Topological Division Ring if and only if* $(\mathbb{K}, +, \cdot)$ *is a Division Ring and* β *a topology on* $\mathbb{K}$ *such that*

- $+ : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ *is continuous respect to* β
- $\cdot : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ *is continuous respect to* β
- $inv : \mathbb{K} \setminus \{0_{\mathbb{K}}\} \to \mathbb{K} \setminus \{0_{\mathbb{K}}\}$ *is continuous respect to* β, *where* $inv(t) = t^{-1} \quad \forall t \in \mathbb{K} \setminus \{0_{\mathbb{K}}\}$.

Notation 1.2. *We simply use* $\mathbb{K}$ *to indicate Valued Division Ring, instead of* $(\mathbb{K}, +, \cdot, |\cdot|)$.

Definition 1.3. *(TVS on Valued Division Ring)*
Let $\mathbb{K}$ *be a Valued Division Ring, then* $(E, +, \cdot, \tau)$ *is a Topological Vector Space on* $\mathbb{K}$ *if and only if* $(E, +, \cdot)$ *is a Left-*$\mathbb{K}$*-module and* τ *a topology on* E *such that*

- $+ : E \times E \to E$ *is continuous respect to* τ
- $\cdot : \mathbb{K} \times E \to E$ *is continuous respect to* τ

Notation 1.4. *We simply use* E *to indicate a Topological Vector Space on* $\mathbb{K}$, *instead of* $(E, +, \cdot, \tau)$.

Remark 1.5. *In the previous definition, some authors add the hypothesis of separation axiom for the topology* τ *(T2 Hausdorff condition). In a non-Hausdorff TVS* E, *the most crucial topological aspect is the subspace* $\overline{\{0_E\}}$ *whose size gives a magnitude of the "Hausdorffness" of the space* E. *Moreover, we can easily show that the quotient space* $E/\overline{\{0_E\}}$ *is a Hausdorff TVS over* $\mathbb{K}$. *In particular, for the differentiability of linear maps between non-Hausdorff spaces, we can easily pass to the quotient and use the classical theory, but for nonlinear maps there are some different aspect that we have to consider.*

Lemma 1.6. *Let* $\mathbb{K}$ *be a Topological Division Ring,* $(E, +, \cdot, \tau)$ *a non-Hausdorff TVS on* $\mathbb{K}$, *then we have that for all* $x \in \overline{\{0_E\}}$ *and for all* $V \in \tau$ *such that* $x \in V$ *we have that*

$$\overline{\{0_E\}} \subseteq V.$$

Proof. We just note that the translation is a homeomorphism and $\overline{\{0_E\}}$ is a closed vector subspace of E. In fact, if we fix $x \in \overline{\{0_E\}}$, then by the previous argument we have that $x + \overline{\{0_E\}} = \overline{\{0_E\}}$ for the subspace condition and $x + \overline{\{0_E\}} = \overline{\{x\}}$ for the homeomorphism condition. In particular, we have that

$$\overline{\{x\}} = \overline{\{y\}} \qquad\qquad \forall x, y \in \overline{\{0_E\}}.$$

□

Definition 1.7. *(Affine Space on a group)*
Let $(G, *)$ be a group, $(\mathbb{A}, \Phi)$ is an Affine space on $(G, *)$ if and only if $Phi : G \times \mathbb{A} \to \mathbb{A}$ is left, free, transitive action of G on $\mathbb{A}$.

Remark 1.8. *Let $(\mathbb{A}, \Phi)$ be an affine space on a group $(G, *)$, then by definition we have that for all $u, v \in \mathbb{A}$ there exists $g \in G$ such that $g * u := \Phi(g, u) = v$ because the action is transitive and this g is unique because Φ is free.*

Notation 1.9. *Let $(\mathbb{A}, \Phi)$ be an affine space on a group $(G, *)$, then for all $w, u \in \mathbb{A}$ and for all $g \in G$ we indicate with $w - u$ the unique $g \in G$ such that $g * u = \Phi(g, u) = v$.*

Definition 1.10. *(Filter)*
Let X be a non-empty set, $\mathcal{F}$ is a Filter on X if and only if:

- $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$
- $A \cap B \in \mathcal{F} \qquad\qquad \forall A, B \in \mathcal{F}$
- $A \subseteq B \Rightarrow B \in \mathcal{F} \qquad\qquad \forall A \in \mathcal{F}, \forall B \subseteq X.$

Notation 1.11. *Let (Y, ν) a topological space then for all $y \in Y$ we use $\mathfrak{U}^{\nu}(y)$ to indicate the filter of neighbourhood of y respect to ν or only $\mathfrak{U}(y)$ if is clear the topology we use.*

Definition 1.12. *(Filterbase)*
Let X be a non-empty set, $\mathcal{F}$ is a Filterbase on X if and only if:

- $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$
- $\forall A, B(A, B \in \mathcal{F} \Rightarrow \exists C \in \mathcal{F} : C \subseteq A \cap B).$

Definition 1.13. *(Push-forward of a Filterbase)*
Let X, Y be non-empty sets, $\mathcal{F}$ a filterbase on X, $f : X \to Y$, then we define

$$f(\mathcal{F}) = \{U \subseteq Y \mid \exists W \subseteq X : f(W) = U\}.$$

Remark 1.14. *Let X, Y be non-empty sets, $\mathcal{F}$ a filterbase on X, $f : X \to Y$, then we easily verify that $f(\mathbb{F})$ is a filterbase on Y.*

Definition 1.15. *(Preorder on the family of Filterbases of a set)*
Let X be a non-empty set, $\mathcal{F}, \mathcal{G}$ two filterbase of X, then we say that $\mathcal{G} \leq_f \mathcal{F}$ if and only if

$$\forall A (A \in \mathcal{G} \Rightarrow \exists B \in \mathcal{F} : B \subseteq A).$$

Definition 1.16. *(Limit set respect to a Filterbase)*
Let X be a non-empty set, (Y, ν) a topological space, $\mathcal{F}$ a filterbase on X and $f : X \to Y$, then we define

$$\lim_{\mathcal{F}}{}^S f = \{y \in Y \ \mid \mathfrak{U}^\nu(y) \leq_f f(\mathcal{F})\}$$

Remark 1.17. *Let X be a non-empty set, (Y, ν) a topological Hausdorff space, $\mathcal{F}$ a filterbase on X and $f : X \to Y$, then there are two disjoint possibilities:*

- $\lim_{\mathcal{F}}{}^S f = \emptyset$
- $\exists y_0 \in Y$ *such that* $\lim_{\mathcal{F}}{}^S f = \{y_0\}$

Lemma 1.18. *Let X, Y be non-empty set, (Z, ν) a topological space, $\mathcal{F}, \mathcal{G}$ two filterbase respectively on X and Y, $g : X \to Y$, $f : Y \to Z$ such that $\mathcal{G} \leq_f g(\mathcal{F})$ then we have that*

$$\lim_{\mathcal{G}}{}^S f \subseteq \lim_{\mathcal{F}}{}^S f \circ g$$

Proof. We fix $z \in \lim_{\mathcal{G}}{}^S f$, then for all $V \in \mathfrak{U}(a)$ there exists $G \in \mathcal{G}$ such that $f(G) \subseteq V$, and by hypothesis there exists $F \in \mathcal{F}$ such that $g(F) \subseteq G$, in particular we have that

$$f \circ g(F) = f(g(F)) \subseteq f(G) \subseteq V$$

and so we can conclude that $z \in \lim_{\mathcal{F}}{}^S f \circ g$. □

Lemma 1.19. *Let X be a non-empty set, (Y, ν),(Z, μ) two topological spaces, $\mathcal{F}$ a filter on X, $g : X \to Y$, $f : Y \to Z$ a continuous map, then we have that*

$$f(\lim_{\mathcal{F}}{}^S g) \subseteq \lim_{\mathcal{F}}{}^S f \circ g.$$

In particular if f is an homeomorphism then we have the equality

$$f(\lim_{\mathcal{F}}{}^S g) = \lim_{\mathcal{F}}{}^S f \circ g. \tag{1}$$

Proof. We fix $y \in \lim_{\mathcal{F}}{}^S g$, so by definition we have that for all $V \in \mathfrak{U}^\nu(y)$ there exists $F \in \mathcal{F}$ such that $g(F) \subseteq V$. We want to verify that $f(y) \in \lim_{\mathcal{F}}{}^S f \circ g$. So we fix $O \in \mathfrak{U}^\mu(f(y))$. By continuity of f, there exists $G \in \mathfrak{U}(y)$ such that

$f(G) \subseteq O$, by the previous remark, there exists $F \in \mathcal{F}$ such that $g(F) \subseteq G$ and so

$$f \circ g(F) = f(g(F)) \subseteq f(G) \subseteq O,$$

in particular $f(a) \in \lim_{\mathcal{F}}{}^S f \circ g$. If f is an omeomorphism by the prevoius argument we have that

$$f^{-1}(\lim_{\mathcal{F}}{}^S f \circ g) \subseteq \lim_{\mathcal{F}}{}^S f^{-1} \circ f \circ g = \lim_{\mathcal{F}}{}^S g$$

so by invertibility of we obtain

$$\lim_{\mathcal{F}}{}^S f \circ g \subseteq f(\lim_{\mathcal{F}}{}^S g).$$

and we can conclude. □

2 Directional Derivatives

Definition 2.1. *(Directional filterbase respect to a point)*
Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, then we define

$$\mathcal{F}_v^{A,x_0} = \{\{t \in \mathbb{K} \mid t \neq 0_{\mathbb{K}},\ x_0 + tv \in A\} \cap H \mid H \in \mathfrak{U}(0_{\mathbb{K}})\}$$

Notation 2.2. *For simplicity we indicate $\mathcal{F}_v^{A,x_0}$ only with $\mathcal{F}_v$ and $\{t \in \mathbb{K} \mid t \neq 0_{\mathbb{K}},\ x_0 + tv \in A\}$ only with $\Sigma_v^A(x_0)$.*

Remark 2.3. *By the previous definition is clear that $\emptyset \neq \mathcal{F}_v^{A,x_0} \subseteq \mathcal{P}(\Sigma_v^A(x_0))$, but is not necessary true that $\emptyset \notin \mathcal{F}_v^{A,x_0}$ and therefore that $\mathcal{F}_v^{A,x_0}$ is a filterbase of $\Sigma_v^A(x_0)$.*

Definition 2.4. *Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, then we define $\phi_f^{A,x_0,v} : \Sigma_v^A(x_0) \to F$ in this way*

$$\phi_f^{A,x_0,v}(t) = t^{-1}(f(x_0 + tv) - f(x_0)) \qquad \forall t \in \Sigma_v^A(x_0).$$

Remark 2.5. *We recall that S_1 and S_2 are affine spaces, so when we write $x_0 + tv$ we indicate $\Phi_1(tv, x_0)$ where Φ_1 is the action of E on S_1, and when we write $f(x_0+tv)-f(x_0)$ we indicate the unique $w \in F$ such that $\Phi_2(w, f(x_0)) = f(x_0 + tv)$, where Φ_2 is the action of F on S_2.*

Definition 2.6. *(Directional Derivatives)*
Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, then we say that f is derivable respect to v in x_0 over the set A if and only if

- $\emptyset \notin \mathcal{F}_v^{A,x_0}$
- $\lim_{\mathcal{F}_v^{A,x_0}} {}^S\phi_f^{A,x_0,v} \neq \emptyset$

Definition 2.7. *(Class of derivable function 1)*
Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $A \subseteq E$, $x_0 \in A$, then we define $D_{\mathbb{K}}(A, v, x_0)$ in this way:

$$D_{\mathbb{K}}(A, v, S_2, x_0) := \{f : A \to S_2 \mid \emptyset \notin \mathcal{F}_v^{A,x_0}, \lim_{\mathcal{F}_v^{A,x_0}} {}^S\phi_f^{A,x_0,v} \neq \emptyset\}$$

Notation 2.8. *Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, then we indicate $\lim_{\mathcal{F}_v^{A,x_0}} {}^S\phi_f^{A,x_0,v}$ with $\frac{\partial^S f}{\partial v}(x_0)$*

Remark 2.9. *Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, then by the previous definition we have that*

$$\frac{\partial^S f}{\partial v}(x_0) \subseteq F.$$

Moreover if $f \in D_{\mathbb{K}}(A, v, S_2, x)$ then thanks to Lemma 1.6 *we have that*

$$\frac{\partial^S f}{\partial v}(x_0) = y + \overline{\{0_F\}} \qquad \forall y \in \frac{\partial^S f}{\partial v}(x_0)$$

and so $\frac{\partial^S f}{\partial v}(x_0)$ is an affine subspace of F parallel to $\{0_F\}$. In particular, the case $f \in D_{\mathbb{K}}(A, v, S_2, x)$ for all $x \in A$ is very interesting, because is well defined $\frac{\partial f}{\partial v} : A \to F/\overline{\{0_F\}}$ where the target space is T2. Thus we can understand that in non-Hausdorff setting the first order theory is different from the classical case, but for the higher order derivability we recover the T2 axiom passing to the quotient and also the classical theory.

Remark 2.10. *Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $L : S_1 \to S_2$ an affine function associated to the continuous linear mapping $l : E \to F$ then we have that*

$$L \in D_{\mathbb{K}}(S_1, v, S_2, x) \qquad \forall v \in E \ \ \forall x \in S_1$$

and

$$\frac{\partial^S L}{\partial v}(x) = \overline{\{l(v)\}} \qquad \forall v \in E \ \ \forall x \in S_1.$$

Proof. We fix $x \in S_1$ and $v \in E$. Certainly we have that $\mathcal{F}_v^{S_1,x}$ is a filterbase, moreover by the properties of affine functions

$$\phi_L^{S_1,x,v}(t) = t^{-1}(L(x+tv) - L(x)) = t^{-1}l(tv) = l(v) \qquad \forall t \in \mathbb{K} \setminus \{0_{\mathbb{K}}\}$$

and so we can conclude by previous remark. □

Proposition 2.11. *(Composition with affine function I)*
Let E, F, G be TVS on the Topological Division Ring $\mathbb{K}$, S_1, S_2, S_3 affine spaces respectively on E, F, G, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, $L : S_2 \to S_3$ an affine function associated to the continuous linear mapping $l : F \to G$, then if $f \in D_{\mathbb{K}}(A, v, S_2, x_0)$ we have that

- $L \circ f \in D_{\mathbb{K}}(A, v, x_0)$
- $\frac{\partial^S L\circ f}{\partial v}(x_0) \supseteq l\left(\frac{\partial^S f}{\partial v}(x_0)\right)$

Proof. First of all we notice that $\mathcal{F}_v^{A,x_0}$ is a filterbase because $f \in D_{\mathbb{K}}(A, v, x_0)$. Then, by affinity of L, we have that

$$\phi_{L\circ f}^{A,x_0,v}(t) = t^{-1}(L \circ f(x_0+tv) - L \circ f(x_0)) = t^{-1}l(f(x_0+tv) - f(x_0)) =$$

$$= l(t^{-1}(f(x_0+tv) - f(x_0))) = l \circ \phi_f^{A,x_0,v}(t) \qquad \forall t \in \Sigma_v^A(x_0).$$

so by Lemma 1.19 and the previous argument we have that

$$l\left(\frac{\partial^S f}{\partial v}(x_0)\right) = l\left(\lim_{\mathcal{F}_v^{A,x_0}} {}^S \phi_f^{A,x_0,v}\right) \subseteq \lim_{\mathcal{F}_v^{A,x_0}} {}^S l \circ \phi_f^{A,x_0,v} = \frac{\partial^S L\circ f}{\partial v}(x_0).$$

□

Proposition 2.12. *(Composition with affine function II)*
Let D, E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_0, S_1, S_2 affine spaces respectively on D, E, F, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, $L : S_0 \to S_1$ an affine function associated to the continuous linear mapping $l : D \to E$ such that $L^{-1}(\{x_0\}), l^{-1}(\{v\}) \neq \emptyset$. If $y_0 \in L^{-1}(\{x_0\})$, $w \in l^{-1}(\{v\})$, $H = L^{-1}(A)$ and $f \in D_{\mathbb{K}}(A, v, S_2, x_0)$ we have that

- $f \circ L \in D_{\mathbb{K}}(H, w, S_2, y_0)$,
- $\frac{\partial^S f}{\partial v}(x_0) = \frac{\partial^S f\circ L}{\partial w}(y_0)$.

Proof. First of all we have to show that $\mathcal{F}_w^{H,y_0}$ is really a filterbase and we can do this by the following equality

$$\Sigma_w^H(y_0) = \Sigma_v^A(x_0)$$

and in particular we have that $\mathcal{F}_w^{H,y_0} = \mathcal{F}_v^{A,x_0}$. In fact, if we fix $t \in \Sigma_w^H(y_0)$, then by definition and the properties of affine functions we have

$$x_0 + tv = L(y_0) + l(tw) = L(y_0 + tw) \in A \tag{2}$$

and so $t \in \Sigma_v^A(x_0)$. Conversely if we fix $t \in \Sigma_v^A(x_0)$, thanks to (2) and by definition of H, we obtain that $y_0 + tw \in H$ and so $t \in \Sigma_w^H(y_0)$. Moreover by (2) we can check that

$$\phi_{f \circ L}^{H,y_0,w}(t) = \phi_f^{A,x_0,v}(t) \qquad \forall t \in \Sigma_v^A(x_0)$$

so we conclude with

$$\frac{\partial^S f \circ L}{\partial w}(y_0) = \lim_{\mathcal{F}_w^{H,y_0}} {}^S\phi_{f\circ L}^{H,y_0,w} = \lim_{\mathcal{F}_v^{A,x_0}} {}^S\phi_f^{A,x_0,v} = \frac{\partial^S f}{\partial v}(x_0).$$

□

Remark 2.13. *Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$. If $H = \Sigma_v^A(x_0) \cup \{0_{\mathbb{K}}\}$, $g : H \to S_2$ is the function defined as $g(t) = f(x_0 + tv)$ for all $t \in H$, then we have that*

- $g \in D_{\mathbb{K}}(H, 1_{\mathbb{K}}, S_2, 0_{\mathbb{K}}) \iff f \in D_{\mathbb{K}}(A, v, S_2, x_0)$
- *If is valid one of the previous condition:* $g'(0_{\mathbb{K}}) := \frac{\partial^S g}{\partial 1_{\mathbb{K}}}(0_{\mathbb{K}}) = \frac{\partial^S f}{\partial v}(x_0)$.

Proof. We notice that $\{t \in \mathbb{K} \mid t \neq 0_{\mathbb{K}}, t \in H\} = \Sigma_v^A(x_0)$ and so $\mathcal{F}_{1_{\mathbb{K}}}^{H,0_{\mathbb{K}}} = \mathcal{F}_v^{A,x_0}$. Moreover, by a simple calculation we have that

$$\phi_g^{H,0_{\mathbb{K}},1_{\mathbb{K}}}(t) = \phi_f^{A,x_0,v}(t) \qquad \forall t \in \Sigma_v^A(x_0)$$

and thus we can conclude by previous remark. □

Proposition 2.14. *Let E, F be TVS on the Topological Division Ring $\mathbb{K}$, S_1 an affine space on E, S_2 an affine space on F, $v \in E$, $\xi \in \mathbb{K} \setminus \{0_{\mathbb{K}}\}$, $A \subseteq E$, $x_0 \in A$, $f : A \to S_2$, then we have that*

$$f \in D_{\mathbb{K}}(A, v, S_2, x_0) \iff f \in D_{\mathbb{K}}(A, \xi v, S_2, x_0)$$

and if is valid one of the previous condition

$$\frac{\partial^S f}{\partial \xi v}(x_0) = \xi \frac{\partial^S f}{\partial v}(x_0).$$

Proof. If $v = 0_E$ is clear, so we can suppose $v \neq 0_E$. First of all we have to verify that $\mathcal{F}_{\xi v}^{A,x_0}$ is really a filterbase on $\Sigma_{\xi v}^A(x_0)$, so we just check, thanks to the previous argument, that

$$\Sigma_{\xi v}^A(x_0) \cap H \neq \emptyset \qquad\qquad \forall H \in \mathfrak{U}(0_{\mathbb{K}}).$$

So we fix $H \in \mathfrak{U}(0_{\mathbb{K}})$. Since $\xi \neq 0_{\mathbb{K}}$, then the homothety $y \mapsto \xi y$, $y \mapsto \xi^{-1} y$ are homeomorphism, being $\mathbb{K}$ a division ring, so $\xi H \in \mathfrak{U}(0_{\mathbb{K}})$. Moreover we can easily see that

$$\Sigma_{\xi v}^A(x_0) = \xi^{-1} \Sigma_v^A(x_0).$$

In particular

$$\Sigma_{\xi v}^A(x_0) \cap H = \xi^{-1} \Sigma_v^A(x_0) \cap \xi^{-1} \xi H = \xi^{-1}(\Sigma_v^A(x_0) \cap \xi H) \neq \emptyset$$

because $\mathcal{F}_v^{A,x_0}$ is a filterbase on $\Sigma_v^A(x_0)$. To conclude the proof we have to verify that

$$\lim_{\mathcal{F}_{\xi v}^{A,x_0}} {}^S \phi_f^{A,x_0,\xi v} \neq \emptyset.$$

First of all we notice that for all $t \in \Sigma_{\xi v}^A(x_0)$ we have thanks to the associativity of $\mathbb{K}$

$$\phi_f^{A,x_0,\xi v}(t) = t^{-1}(f(x_0 + t\xi v) - f(x_0)) = (\xi \xi^{-1}) t^{-1}(f(x_0 + t\xi v) - f(x_0)) =$$

$$= \xi (t\xi)^{-1}(f(x_0 + t\xi v) - f(x_0)) = \xi \phi_f^{A,x_0,v}(t\xi)$$

So if we call $\rho_\xi^{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$ the right-homothety on $\mathbb{K}$, and $\lambda_\xi^F : F \to F$ the left-homothety on F, we have that

$$\phi_f^{A,x_0,\xi v}(t) = \lambda_\xi^F \circ \phi_f^{A,x_0,v} \circ \rho_\xi^{\mathbb{K}}(t) \qquad\qquad \forall t \in \Sigma_{\xi v}^A(x_0).$$

We can see easily that $\rho_\xi^{\mathbb{K}}(\Sigma_{\xi v}^A(x_0)) = \Sigma_v^A(x_0)$ and since $\mathbb{K}$ is a topological division ring so $\rho_\xi^{\mathbb{K}}$ is a homeomorphism with $\rho_\xi^{\mathbb{K}}(0_{\mathbb{K}}) = 0_{\mathbb{K}}$ and so by definition of continuity we have that

$$\mathcal{F}_v^{A,x_0} \leq_f \rho_\xi^{\mathbb{K}}(\mathcal{F}_{\xi v}^{A,x_0})$$

so thanks to Lemma 1.18 we have that

$$\lim_{\mathcal{F}_v^{A,x_0}} {}^S \phi_f^{A,x_0,v} \subseteq \lim_{\mathcal{F}_{\xi v}^{A,x_0}} {}^S \phi_f^{A,x_0,v} \circ \rho_\xi^{\mathbb{K}}$$

and thanks to Lemma 1.19 and being λ_ξ^F an homeomorphism we obtain that

$$\lim_{\mathcal{F}_{\xi v}^{A,x_0}} {}^S \phi_f^{A,x_0,\xi v} = \lim_{\mathcal{F}_{\xi v}^{A,x_0}} {}^S \lambda_\xi^F \circ \phi_f^{A,x_0,v} \circ \rho_\xi^{\mathbb{K}} = \lambda_\xi^F \left(\lim_{\mathcal{F}_{\xi v}^{A,x_0}} {}^S \phi_f^{A,x_0,v} \circ \rho_\xi^{\mathbb{K}} \right) = \xi \left(\lim_{\mathcal{F}_{\xi v}^{A,x_0}} {}^S \phi_f^{A,x_0,v} \circ \rho_\xi^{\mathbb{K}} \right) \supseteq$$

$$\supseteq \xi \left(\lim_{\mathcal{F}_v^{A,x_0}} {}^S\phi_f^{A,x_0,v} \right) \neq \emptyset,$$

in particular we have shown that $f \in D_{\mathbb{K}}(A, \xi v, S_2, x_0)$ and

$$\xi \frac{\partial^S f}{\partial v}(x_0) \subseteq \frac{\partial^S f}{\partial \xi v}(x_0),$$

therefore applying again the argument used above, we have that

$$\xi^{-1} \frac{\partial^S f}{\partial \xi v}(x_0) \subseteq \frac{\partial^S f}{\partial v}(x_0)$$

and multiplying by ξ, we finish.

□

Remark 2.15. *In the Proposition* 2.14 *is crucial the associativity of the division ring* $\mathbb{K}$*. In particular, this result is not necessary true for TVS over general topological division algebras.*

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Received: June 29, 2022; Published: July 12, 2022