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On Retrospective Premium in Insurance: the Expected Value and Variance

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Abstract

In this paper we tackle the issue of analysing the mean value and variance of the retrospective premium, premium adopted by insurance companies and characterised by a random component. In particular, we study the role of the rating parameters. The case of a loss distribution approximated by a translated gamma distribution enables us to highlight some new analytical aspects.

Mathematics Subject Classification: 62P05

Keywords: retrospective insurance premium, expected retrospective premium, retrospective premium variance, rating parameters, translated gamma distribution

1 Introduction

Premiums having a random component are frequently used in the insurance industry. In reinsurance contracts, for example, when the reinsurer generates a profit above a specific level, he often pays a commission to the insurer. As a result, the direct insurer's effective cost of reinsurance coverage has a random component: given the assumed *ex ante* reinsurance price, the reinsurer's reimbursements must also be considered.

The retrospective premium concept expresses another example of a random (re-)insurance premium. In the United States, this technique of computation is extensively employed in liability insurance and worker's compensation [6, 8]. Retrospectively rated insurance is an insurance policy with a premium that varies accordings to the losses suffered by the covered firm, rather than according to industry-wide loss experience. After the insurance has expired, an initial premium is payable, and modifications are made on a regular basis. Because the price of the insurance is expected to fall if the insured is able to restrict risk exposure, this strategy provides as an incentive to the insured firm to control its losses. The premium is adjustable within a defined range of values, with a minimum and maximum amount. Few studies in the actuarial literature have investigated this common technique from an analytical perspective (see [4, 6, 8, 9, 10]).

Starting from the retrospective approach, illustrated in [8, 5, 9, 4, 2] and already studied in [3], in this paper we analyze the behavior of the mean value and the variance of the retrospective premium.

In Section 2 we focus on the expected retrospective premium and on its variance and, through some equivalent representations, we describe some their properties. The Section 3 analyzes the retrospective premium in the particular case of a risk with translated gamma distribution: this representation enables to highlight some interesting analytical aspects. Section 4 contains some observations and research suggestions.

2 On retrospective premium

The insurance practice proposes a rating plan in which the premium takes into account the current claim experience and falls between a minimum and a maximum value (see [4, 9]). Now, we refer to the following definition of retrospective premium for a non-negative risk Y, with $E[Y] < \infty$:

$$\Pi(Y) = \min\left\{(B_Y + LY)T,\ H_Y\right\} \tag{1}$$

where:
$L \geq 0$ is the loss conversion factor that covers the loss adjustment expenses;
$T > 1$ is the tax multiplier including premium tax;
B_Y, $B_Y T$ and H_Y are, respectively, the basic, the minimum and the maximum premium, all are non-negative.

In literature, L, T, B_Y, H_Y are called *rating parameters*; they are related, in fact by definition (1) it is $B_Y T \leq \Pi(Y) \leq H_Y$.
To avoid the case of a deterministic premium $\Pi(Y)$, from now on it is assumed $B_Y T < H_Y$ and $L > 0$.

Let us denote y_M as follows

$$y_M = \frac{H_Y - B_Y T}{LT}. \tag{2}$$

Then (1) updates as:

$$\Pi(Y) = B_Y T + LT \min\{Y,\ y_M\}. \tag{3}$$

Let us consider the Limited Expected Value Function (LEVF) N_Y (see [7, 5]):

$$N_Y(u) = E[\min(Y, u)] = \int_0^u (1 - F_Y(y))dy. \qquad (u \geq 0) \tag{4}$$

We study the expected retrospective premium $E[\Pi(Y)]$ as a function of y_M, let us denote it by $\phi(y_M)$. Then it is

$$\phi(y_M) = B_Y T + LT\, N_Y(y_M) \tag{5}$$

and it holds

$$\lim_{y_M \to \infty} \phi(y_M) = B_Y T + LT E[Y]. \tag{6}$$

If the distribution function F_Y is assumed to be absolutely continuous and f_Y denotes the related density function, then it is

$$\phi'(y_M) = B_Y T + LT(1 - F_Y(y_M)) \qquad\qquad \phi''(y_M) = -LT f_Y(y_M) \tag{7}$$

Note that $\phi'(y_M) \geq 0$ and $\phi''(y_M) \leq 0$.

Let us denote by $\psi(y_M)$ the variance of the retrospective premium $Var[\Pi(Y)]$; then, it is

$$\psi(y_M) = L^2 T^2\, Var[\min\{Y,\ y_M\}] \tag{8}$$

moreover,

$$\lim_{y_M \to \infty} \psi(y_M) = L^2 T^2 Var[Y]. \tag{9}$$

Given the absolute continuity of the function F_Y and the definition (4), we obtain the following explicit expression for the variance $Var[\Pi(Y)]$:

$$\psi(y_M) = L^2 T^2 \left[y_M^2 - 2\int_0^{y_M} y F_Y(y)dy - N_Y^2(y_M) \right] \tag{10}$$

and, this way, we write the first and second order derivatives as follows

$$\psi'(y_M) = 2L^2 T^2 (1 - F_Y(y_M)) \left[\int_0^{y_M} F_Y(y)dy \right] \tag{11}$$

$$\psi''(y_M) = 2L^2T^2\left[F_Y(y_M) - F_Y^2(y_M) - f_Y(y_M)\int_0^{y_M} F_Y(y)dy\right]. \tag{12}$$

Let us observe that $\psi'(y_M) \geq 0$. It highlights that once the values of the parameters L and T are assumed, the variability (measured by variance) of the retrospective premium $\Pi(Y)$ changes in relation to variability of the minimum and maximum premia (see (2)). It increases as the maximum premium H_Y increases, when B_Y is given; it decreases as the minimum premium B_YT increases, when H_Y is given.
We note the variability of the $\psi''(y_M)$ sign which requires further investigation: in the next paragraph some results will be presented to highlight the variability of this sign.

3 The case of translated gamma distribution

We refer to the collective risk model (see [5]) to study the distribution of the random loss Y. Let us consider a random variable Z following a gamma distribution with positive parameters α e β. As it is well known, $F_Z(z) = \Gamma(\alpha;\beta z)$ where $\Gamma(\alpha;\beta z)$ is the incomplete gamma function, i.e.

$$\Gamma(\alpha;\beta z) = \frac{1}{\Gamma(\alpha)}\int_0^{\beta z} t^{\alpha-1}e^{-t}dt. \tag{13}$$

We make the assumption that Y has the same distribution of the random variable $Z + z_0$, where z_0 is constant (see [3]), so, it is $F_Y(y) = \Gamma(\alpha;\beta(y - z_0))$, where $y \geq z_0$.

Let μ_Y, σ_Y^2 and γ_Y denote the mean, variance and coefficient of skewness of Y, respectively. The parameters α, β and z_0 are chosen so that $Z + z_0$ and Y have the same first three moments:

$$\mu_Y = z_0 + \frac{\alpha}{\beta}, \qquad \sigma_Y^2 = \frac{\alpha}{\beta^2}, \qquad \gamma_Y = \frac{2}{\sqrt{\alpha}} \tag{14}$$

then, the parameters satisfy the conditions:

$$\alpha = \frac{4}{\gamma_Y^2}, \qquad \beta = \frac{2}{\gamma_Y\sigma_Y} \qquad \text{and} \qquad z_0 = \mu_Y - \frac{2\sigma_Y}{\gamma_Y}. \tag{15}$$

The LEVF of Z is given by:

$$N_Z(u) = \frac{\alpha}{\beta}\Gamma(\alpha+1;\beta u) + u[1 - \Gamma(\alpha;\beta u)], \quad u \geq 0 \tag{16}$$

and it is a differentiable and increasing function with $N_Z'(u) = 1 - \Gamma(\alpha;\beta u)$.

Furthermore, given the assumptions on Y and Z and the properties of the LEVF (see, e.g., [5]), we obtain the following results:

$$N_Y(u) = N_{Z+z_0}(u) = N_Z(u - z_0) + z_0, \quad u \geq z_0 \tag{17}$$

$$\lim_{u \to z_0} N_Y(u) = N_Y(z_0) = z_0 \qquad \lim_{u \to \infty} N_Y(u) = z_0 + \frac{\alpha}{\beta}. \tag{18}$$

Therefore, referring to (5) and (17), the expected retrospective premium admits the following formulation:

$$\phi(y_M) = B_Y T + LT(N_Z(y_M - z_0) + z_0) \tag{19}$$

where $y_M \geq z_0$. Given the result (6) and the hypothesis (14), it follows:

$$\lim_{y_M \to \infty} \phi(y_M) = B_Y T + LT\left(z_0 + \frac{\alpha}{\beta}\right). \tag{20}$$

Following some steps, the variance (10) can be so written

$$\begin{aligned}\psi(y_M) &= L^2T^2[y_M^2 - (y_M^2 - z_0^2)\Gamma(\alpha; \beta(y_M - z_0)) \\ &+ \frac{\alpha(\alpha+1)}{\beta^2}\Gamma(\alpha + 2; \beta(y_M - z_0)) \\ &+ \frac{2z_0\alpha}{\beta}\Gamma(\alpha + 1; \beta(y_M - z_0)) - N_Y^2(y_M)]\end{aligned}$$

with

$$N_Y(y_M) = \frac{\alpha}{\beta}\Gamma(\alpha + 1; \beta(y_M - z_0)) + (y_M - z_0)[1 - \Gamma(\alpha; \beta u)] + z_0$$

Referring to (9) and (14), it is possible to prove that

$$\lim_{y_M \to \infty} \psi(y_M) = L^2T^2\frac{\alpha}{\beta^2}. \tag{21}$$

Some graphs have been generated[1] to study the trend of the expected retrospective premium $\phi(y_M)$ and variance $\psi(y_M)$.
Figures 1 and 2 point out different representations of the expected retrospective premium $\phi(y; \alpha, \beta) := \phi(y_M)$ (y plays the role of y_M) with reference to different values of the parameters α and β. Analogously, Figures 3 and 4 show some representations of the variance $\psi(y; \alpha, \beta) := \psi(y_M)$ for different values of the parameters α and β. More precisely, in Figures 1 and 3, α changes between 5 and 9 and $\beta = 2$; in Figures 2 and 4, $\alpha = 9$ and β varies between 2 and 6.

The asymptotic behavior of the expected retrospective premium is in line with the result in (20). For fixed values of y (therefore for fixed values of

[1] Wolfram Research, Inc., Mathematica, Version 13.0.0, Champaign, IL (2022).

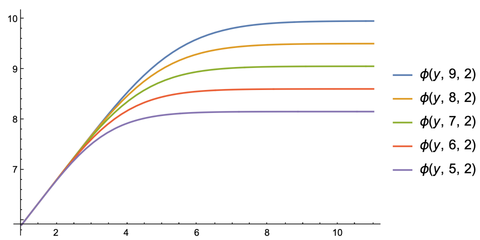

Figure 1: $\phi(y;\alpha,2)$

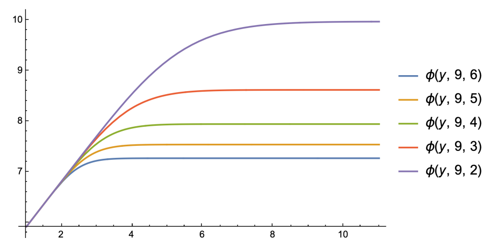

Figure 2: $\phi(y;9,\beta)$

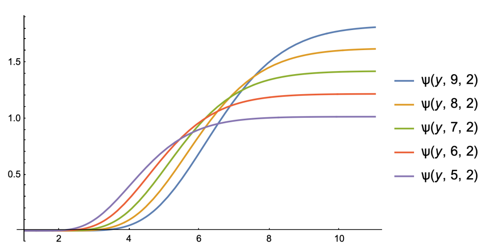

Figure 3: $\psi(y;\alpha,2)$

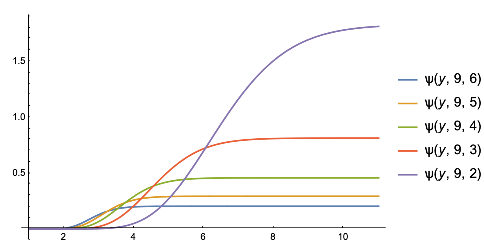

Figure 4: $\psi(y;9,\beta)$

the basic and of the maximum premia, B_Y and H_Y respectively) the expected retrospective premium increases as α increases and decreases when β increases: this is in accordance with the increasing behavior of the expected value $\mu_Y = z_0 + \frac{\alpha}{\beta}$ when α increases or β decreases.

Note that in Figure 3 and 4 the depicted graphs intersect. In particular, in Figure 3 the variance of the premium for assigned values of y is not ordered with respect to different values of α (β is fixed).
Let us denote by $y^*_{(\alpha_1,\alpha_2)}$ the solution of the equation

$$\psi(y;\alpha_1,\beta) = \psi(y;\alpha_2,\beta) \qquad \text{with } \alpha_1 < \alpha_2,$$

then it is

$$\begin{cases} \psi(y;\alpha_1,\beta) \geq \psi(y;\alpha_2,\beta) & \text{if } y \leq y^*_{(\alpha_1,\alpha_2)} \\ \psi(y;\alpha_1,\beta) \leq \psi(y;\alpha_2,\beta) & \text{if } y \geq y^*_{(\alpha_1,\alpha_2)} \end{cases}$$

Moreover,

$$y^*_{(\alpha_1,\alpha_2)} \leq y^*_{(\alpha_1,\alpha_3)} \qquad \text{if} \quad \alpha_1 \leq \alpha_2 \leq \alpha_3.$$

Analogous observations hold if we look at Figure 4, where β varies while α is fixed. So, the variability of the retrospective premium represented by its variance, is not ordered with respect to different values of α or β, unlike what happens for the variance σ^2_Y of the risk Y (see (14)).

4 Concluding remarks

The retrospective premium is an example of a random component insurance premium: its value and variance are relevant to the counterparties in an insurance (or reinsurance) contract and depend on the loss Y distribution. If the expected value of retrospective premium behaves predictably (increases as μ_Y increases), its variance behaves differently. If variance is considered an indicator of risk, the results presented here clearly suggest that a thorough and careful assessment is needed to guide insurer and policyholder choices in setting rating parameters values.

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