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Existence and Uniqueness of Solutions for Boundary Value Problems of Fractional Differential Equations with Caputo-Hadamard Derivative[1]

Zheng Huang [2], Lijing Duan [3,*] and Jingli Xie [4]

[2] Mathematics and Computer Science
Normal College of Jishou University

[3,4] College of Mathematics and Statistics, Jishou University
Jishou, Hunan 416000, P.R. China
[*] Corresponding author

Abstract

In this paper, we are established the existence of positive solutions and uniqueness of a general class of nonlinear Caputo-Hadamard fractional differential equations in Banach's space. To find the solution for the problem, we use the Krasnosel'skii fixed point theorem and Banach fixed point theorem. The fractional differential equation is converted into an alternative integral structure using the Hadamard fractional integral operator. We provide Some examples with specific parameters and assumptions to show the results of the proposal.

Mathematics Subject Classifications: 34B15, 34B18, 34B37, 58E30

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1 Introduction

In this paper, we consider the following boundary value problem of fractional differential equations with Caputo-Hadamard derivative

$$\begin{cases} {}^{C}_{H}D^{\gamma}_{1^+}[z(\tau) - {}_{H}J^{\gamma}_{1^+}h(\tau, z(\tau))] = g(\tau, z(\tau)), \quad \tau \in [1, e], \\ az(1) + bz'(1) = 0, \quad cz(e) + dz'(e) = k, \end{cases} \tag{1.1}$$

where $1 < \gamma \leq 2$, a, b, c, d, k are real numbers with $ace - bce + ad \neq 0$, ${}^{C}_{H}D^{\gamma}_{1^+}$, ${}_{H}J^{\gamma}_{1^+}$ is the Caputo-Hadamard fractional derivative and Hadamard fractional integral of order γ, respectively. h, g are some given continuous function on $[1, e] \times X \to X$, where X is a Banach space.

In recent years, boundary value problems of fractional differential equations have been discussed extensively since fractional calculus theory and methods have wide applications in various fields of natural sciences and social sciences. Most of the researchers have given attention to research for the fractional differential equations with Riemman-Liouville, Caputo and Hadamard derivative [1, 2, 3, 4]. Other than these fractional derivatives, another sort of fractional derivatives established in the literature is the fractional derivative because of Hadamard made known to in 1892 [5], differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains the logarithmic function of arbitrary order $(\log t - \log s)$ instead of $(t - s)$. Another important aspect of Hadamard derivative is that its expression can be regarded as the generalization operator $(t\frac{d}{dt})^n$, while the Riemann-Liouville derivative is regarded as the generalization of the classical differential operator $(\frac{d}{dt})^n$.

Recently, through Caputo correction of Hadamard derivative, another kind of derivative called Caputo-Hadamard derivative was proposed. It is obtained by changing the order of the differential and integral of the Hadamard derivative. In addition, the existence and uniqueness of solutions of Caputo-Hadamard fractional-order functional differential equations were considered. In order to study the existence and uniqueness of their solutions, many articles in this field have emerged in recent years [6, 7, 8, 9]. But there are relatively few articles of boundary value problems of fractional differential equations with Caputo-Hadamard derivative [10, 11, 12, 13]. Thus this paper is studied a class of boundary value problems for fractional differential equations with Caputo-Hadamard derivatives. Some properties and applications of the Caputo-Hadamard derivative can be found in [14, 15].

Zhang. [16] investigated the existence of positive solutions for nonlinear fractional differential equations

$$\begin{cases} {}^{C}D^{\gamma}z(\tau) = g(\tau, z(\tau)), \quad \tau \in (0, 1), \ 1 < \gamma \leq 2, \\ z(0) + z'(0) = 0, \ z(1) + z'(1) = 0, \end{cases}$$

where ${}^{C}D^{\gamma}$ is the Caputo fractional derivative, g is given continuous function on $[0,1]\times[0,+\infty]\to[0,+\infty]$.

Du et al. [17] studied the existence and uniqueness of solutions for the fractional differential equations with Caputo-Hadamard fractional derivative

$$\begin{cases} {}^{C}_{H}D^{\gamma}_{1^+}z(\tau)=g(\tau,z(\tau)), & \tau\in[1,e],\ 1<\gamma\le 2,\\ z(1)+z'(1)=0,\ Az(e)+Bz'(e)=C, \end{cases}$$

where ${}^{C}_{H}D^{\gamma}_{1^+}$ is the Caputo-Hadamard fractional derivative, $A,\ B,\ C$ is constant, g is given continuous function on $[1,e]\times X\to X$.

The main contributions are as follows:

i) We have established a differential equation (1.1) for the study of quadratic perturbation in nonlinear problems that contains in different ways the perturbations of literature [17] and as a special cases several dynamic system.

ii) According to literature [16] and [17], we obtain the boundary condition satisfying the problem (1.1) and the assumptions we propose are more general.

iii) We transform Caputo-Hadamard fractional differential equation into the form of integral equation. The existence of solutions is proved by Krasnosel'skii fixed point theorem and Banach fixed point theorem, with which we analyze the solution's continuity, equicontinuity and boundedness. Then, we use Arzela-Ascoli theory to ensure that the solution is completely continuous.

iv) The novelty of this work lies in the study of the class of Caputo-Hadamard fractional order differential equations with effects and especially in introducing the concepts of differential equation of quadratic disturbance.

Motivated by [16] and [17], we discuss the existence and uniqueness of solutions for fractional differential equations with Caputo-Hadamard fractional derivative (1.1). The next section introduces some theorems, definitions and lemmas of fractional calculus. The main conclusion are presented in Section 3, the existence and uniqueness of solutions for boundary value problem (1.1) are obtained by using the Krasnosel'skii fixed point theorem and Banach fixed point theorem. Finally, we provide two examples to verify our results.

2 Preliminaries

In the section, we present some definitions and lemmas, which are used throughout this paper.

Definition 2.1. ([*18*]) *The Hadamard fractional integral of order* $\gamma>0$ *for a function* $h:(1,\infty)\to\mathbb{R}$ *is defined as follows:*

$${}_{H}J^{\gamma}_{1^+}h(t)=\frac{1}{\Gamma(\gamma)}\int_1^t(\log t-\log s)^{\gamma-1}\frac{h(s)}{s}ds,$$

Definition 2.2. ([15]) *Let $\gamma \geq 0$, $n=[\gamma]+1$ and $\delta = t\frac{d}{dt}$. If $h(t) \in AC_\delta^n[a,b]$, where $0<a<b<\infty$ and $AC_\delta^n[a,b] = \{h:[a,b]\to C : \delta^{n-1}h(t) \in AC[a,b]\}$. The Caputo-Hadamard fractional derivative of fractional order $\gamma > 0$ for a function $h:[1,\infty)\to\mathbb{R}$ is defined as follows:*

$$ {}_H^C D_{1^+}^{\gamma} h(t) = \frac{1}{\Gamma(n-\gamma)} \int_1^t (\ln t - \ln s)^{n-\gamma-1} \delta^n \frac{h(s)}{s} ds, $$

where $[\gamma]$ denotes the integer part of the real number $\gamma > 0$.

Lemma 2.3. ([15]) *Let $\gamma \geq 0, n=[\gamma]+1$. If $z(t) \in AC_\delta^n[a,b]$, the solution of Caputo-Hadamard fractional differential equations*

$$ {}_H^C D_{1^+}^{\gamma} z(t) = 0, $$

is given by

$$ z(t) = \sum_{i=1}^{n} c_i (\ln t - \ln a)^i. $$

Then

$$ {}_H J_{1^+}^{\gamma} ({}_H^C D^{\gamma} z)(t) = z(t) - \sum_{i=0}^{n-1} c_i (\ln t - \ln a)^i, $$

where $c_i \in \mathbb{R}, i = 0,1,2,\cdots,n-1$.

Lemma 2.4. *(Banach fixed point theorem [20]) Let X be a Banach space, $D \subset X$ be closed, and $F: D \to D$ be a strict contraction, i.e., $\| Fx - Fy \| \leq k_1 \| x-y \|$ for some $k_1 \in (0,1)$ and all $x, y \in D$. Then F has a fixed point in D.*

Lemma 2.5. *(Krasnosel'skii fixed point theorem [19]) Let X be a Banach space, $D \subset X$ be closed, convex set, and operator Q, G satisfied*
(i) $Qx + Gy \in D$, for all $x, y \in D$,
(ii) Q be a completely continuous operator,
(iii) G is a contraction mapping,
then there exists point $x^* \in D$ such that $x^* = Qx^* + Gx^*$.

3 Main results

Lemma 3.1. *Let $f_1, f_2 \in C([1,e],\mathbb{R})$. Then $z \in C([1,e],\mathbb{R})$ is a solution of the fractional differential equation*

$$ \begin{cases} {}_H^C D_{1^+}^{\gamma} [z(\tau) - {}_H J_{1^+}^{\gamma} f_1(\tau)] = f_2(\tau), & \tau \in [1,e], \\ az(1) + bz'(1) = 0, \quad cz(e) + dz'(e) = k, \end{cases} \tag{3.1} $$

if and only if

$$\begin{aligned} z(\tau) &= \tfrac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\tfrac{\tau}{s})^{\gamma-1}\tfrac{f_2(s)}{s}ds+\tfrac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\tfrac{\tau}{s})^{\gamma-1}\tfrac{f_1(s)}{s}ds \\ &\quad +(b-a\ln\tau)\big[\tfrac{ce}{\lambda\Gamma(\gamma)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-1}\tfrac{f_2(s)}{s}ds+\tfrac{ce}{\lambda\Gamma(\gamma)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-1}\tfrac{f_1(s)}{s}ds \\ &\quad +\tfrac{d}{\lambda\Gamma(\gamma-1)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-2}\tfrac{f_2(s)}{s}ds+\tfrac{d}{\lambda\Gamma(\gamma-1)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-2}\tfrac{f_1(s)}{s}ds-\tfrac{ke}{\lambda}\big], \end{aligned} \tag{3.2}$$

where $\lambda = ace - bce + ad$.

Proof. Suppose z satisfies (3.1), then applying ${}_HJ^{\gamma}_{1+}$ to both sides of (3.1), By Lemma 2.3, we have

$$z(\tau) = {}_HJ^{\gamma}_{1+}f_2(\tau) + {}_HJ^{\gamma}_{1+}f_1(\tau) + c_0 + c_1\ln\tau.$$

Thus

$$z'(\tau) = \frac{1}{\tau}{}_HJ^{\gamma-1}_{1+}f_2(\tau) + \frac{1}{\tau}{}_HJ^{\gamma-1}_{1+}f_1(\tau) + \frac{c_1}{\tau}.$$

By boundary condition $az(1) + bz'(1) = 0, cz(e) + dz'(e) = k$, we obtain

$$ac_0 + bc_1 = 0,$$

$$c\big({}_HJ^{\gamma}_{1+}f_2(e) + {}_HJ^{\gamma}_{1+}f_1(e) + c_0 + c_1\big) + d\big(\frac{1}{e}{}_HJ^{\gamma-1}_{1+}f_2(e) + \frac{1}{e}{}_HJ^{\gamma-1}_{1+}f_1(e) + \frac{c_1}{e}\big) = k,$$

then

$$c_0 = \frac{bce}{\lambda}\big({}_HJ^{\gamma}_{1+}f_2(e) + {}_HJ^{\gamma}_{1+}f_1(e)\big) + \frac{bd}{\lambda}\big({}_HJ^{\gamma-1}_{1+}f_2(e) + {}_HJ^{\gamma-1}_{1+}f_1(e)\big) - \frac{bke}{\lambda},$$

$$c_1 = \frac{ake}{\lambda} - \frac{ace}{\lambda}\big({}_HJ^{\gamma}_{1+}f_2(e) + {}_HJ^{\gamma}_{1+}f_1(e)\big) - \frac{ad}{\lambda}\big({}_HJ^{\gamma-1}_{1+}f_2(e) + {}_HJ^{\gamma-1}_{1+}f_1(e)\big),$$

where $\lambda = ace - bce + ad$.
Thus

$$\begin{aligned} z(\tau) = &\frac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\frac{\tau}{s})^{\gamma-1}\frac{f_2(s)}{s}ds+\frac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\frac{\tau}{s})^{\gamma-1}\frac{f_1(s)}{s}ds \\ &+(b-a\ln\tau)\big[\frac{ce}{\lambda\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{f_2(s)}{s}ds+\frac{ce}{\lambda\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{f_1(s)}{s}ds \\ &+\frac{d}{\lambda\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{f_2(s)}{s}ds+\frac{d}{\lambda\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{f_1(s)}{s}ds-\frac{ke}{\lambda}\big]. \end{aligned}$$

This completes the proof. □

We express (3.2) as

$$z(\tau)=(Az)(\tau),$$

where the operators $A: C([1,e],\mathbb{R})\to C([1,e],\mathbb{R})$ is defined by

$$\begin{aligned}(Az)(\tau)\ &= \tfrac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\tfrac{\tau}{s})^{\gamma-1}\tfrac{g(s,z(s))}{s}ds+\tfrac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\tfrac{\tau}{s})^{\gamma-1}\tfrac{h(s,z(s))}{s}ds\\ &+(b-a\ln\tau)\big[\tfrac{ce}{\lambda\Gamma(\gamma)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-1}\tfrac{g(s,z(s))}{s}ds+\tfrac{ce}{\lambda\Gamma(\gamma)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-1}\tfrac{h(s,z(s))}{s}ds\\ &+\tfrac{d}{\lambda\Gamma(\gamma-1)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-2}\tfrac{g(s,z(s))}{s}ds+\tfrac{d}{\lambda\Gamma(\gamma-1)}\int_1^e(\ln\tfrac{e}{s})^{\gamma-2}\tfrac{h(s,z(s))}{s}ds-\tfrac{ke}{\lambda}\big].\end{aligned}\tag{3.3}$$

Let $h,g:[1,e]\times\mathbb{R}\to\mathbb{R}$ are some given continuous functions. Now, we need the following hypotheses:
$(H1)$ Assume that there exists real constants $L_1>0, L_2>0$ such that, for all $\tau\in[1,e], z_1,z_2\in\mathbb{R}$

$$\begin{aligned}\mid g(\tau,z_1(\tau))-g(\tau,z_2(\tau))\mid&\le L_1\mid z_1-z_2\mid,\\ \mid h(\tau,z_1(\tau))-h(\tau,z_2(\tau))\mid&\le L_2\mid z_1-z_2\mid.\end{aligned}$$

$(H2)$ $L\eta<1$, where $\eta:=\frac{1}{\Gamma(\gamma+1)}+\frac{|bc|e}{|\lambda|\Gamma(\gamma+1)}+\frac{|bd|}{|\lambda|\Gamma(\gamma)},\ L:=L_1+L_2$.
$(H3)$ For all $(\tau,z)\in[1,e]\times\mathbb{R}, \psi(\tau),\varphi(\tau)\in C([1,e],\mathbb{R})$,

$$\begin{aligned}\mid g(\tau,z)\mid&\le\varphi(\tau),\\ \mid h(\tau,z)\mid&\le\psi(\tau).\end{aligned}$$

Theorem 3.2. *Assume that $(H1),(H2)$ holds, then problem (1.1) has a unique solution on $[1,e]$.*

Proof. We will use the Banach fixed point theorem to prove that A has a unique fixed point. Indeed, in view of the continuities of g,h, then there exists positive constant M_1,M_2 such that $\max_{\tau\in[1,e]}\mid g(\tau,0)\mid=M_1, \max_{\tau\in[1,e]}\mid h(\tau,0)\mid=M_2$. Fixing $B_{r_1}=\{z\in C([1,e],\mathbb{R}):\parallel z\parallel\le r_1\}$, we introduce the norm $\parallel z\parallel=\max_{\tau\in[1,e]}\mid z\mid$, and choose

$$r_1\ge\frac{(M_1+M_2)\eta\mid\lambda\mid+e\mid bk\mid}{[1-(L_1+L_2)\eta]\mid\lambda\mid}.$$

First we show that $A(B_{r_1})\subset B_{r_1}$. Let $z\in B_{r_1}$, then by (3.3), we get

$$\begin{aligned}&\parallel A(z)\parallel\\ \le&\max_{\tau\in[1,e]}\Big|\frac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\frac{\tau}{s})^{\gamma-1}\frac{g(s,z(s))}{s}ds+\frac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\frac{\tau}{s})^{\gamma-1}\frac{h(s,z(s))}{s}ds\\ &+\mid(b-a\ln\tau)\mid\big[\frac{\mid c\mid e}{\mid\lambda\mid\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{g(s,z(s))}{s}ds+\frac{\mid c\mid e}{\mid\lambda\mid\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{h(s,z(s))}{s}ds\end{aligned}$$

$$+\frac{|d|}{|\lambda|\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{g(s,z(s))}{s}ds+\frac{|d|}{|\lambda|\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{h(s,z(s))}{s}ds+\frac{e\,|k|}{|\lambda|}]\Big|$$
$$\leq\frac{1}{\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{|g(s,z(s))-g(s,0)|+|g(s,0)|}{s}ds$$
$$+\frac{1}{\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{|h(s,z(s))-h(s,0)|+|h(s,0)|}{s}ds$$
$$+\frac{|bc|\,e}{|\lambda|\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{|g(s,z(s))-g(s,0)|+|g(s,0)|}{s}ds$$
$$+\frac{|bc|\,e}{|\lambda|\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{|h(s,z(s))-h(s,0)|+|h(s,0)|}{s}ds$$
$$+\frac{|bd|}{|\lambda|\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{|g(s,z(s))-g(s,0)|+|g(s,0)|}{s}ds$$
$$+\frac{|bd|}{|\lambda|\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{|h(s,z(s))-h(s,0)|+|h(s,0)|}{s}ds+\frac{e\,|bk|}{|\lambda|}$$
$$\leq\big[(L_1+L_2)r_1+M_1+M_2\big]\big[\frac{1}{\Gamma(\gamma+1)}+\frac{|bc|\,e}{|\lambda|\Gamma(\gamma+1)}+\frac{|bd|}{|\lambda|\Gamma(\gamma)}\big]+\frac{e\,|bk|}{|\lambda|}\leq r_1,$$

Hence, $A(z)\in B_{r_1}$, we have $A(B_{r_1})\subset B_{r_1}$.

Next we show that the operator A is a contraction. Let $z_1,z_2\in C([1,e],\mathbb{R})$, for any $\tau\in[1,e]$, we obtain

$$\|A(z_1)(\tau)-A(z_2)(\tau)\|$$
$$\leq\max_{\tau\in[1,e]}\Big|\frac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\frac{\tau}{s})^{\gamma-1}\frac{|g(s,z_1(s))-g(s,z_2(s))|}{s}ds+\frac{1}{\Gamma(\gamma)}\int_1^\tau(\ln\frac{\tau}{s})^{\gamma-1}\frac{|h(s,z_1(s))-h(s,z_2(s))|}{s}ds$$
$$+|(b-a\ln\tau)|\,[\frac{|c|\,e}{|\lambda|\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{|g(s,z_1(s))-g(s,z_2(s))|}{s}ds$$
$$+\frac{|c|\,e}{|\lambda|\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{|h(s,z_1(s))-h(s,z_2(s))|}{s}ds$$
$$+\frac{|d|}{|\lambda|\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{|g(s,z_1(s))-g(s,z_2(s))|}{s}ds$$
$$+\frac{|d|}{|\lambda|\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{|h(s,z_1(s))-h(s,z_2(s))|}{s}ds]\Big|$$
$$\leq(L_1+L_2)\|z_1-z_2\|\,[\frac{1}{\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{1}{s}ds+\frac{|bc|\,e}{|\lambda|\Gamma(\gamma)}\int_1^e(\ln\frac{e}{s})^{\gamma-1}\frac{1}{s}ds$$
$$+\frac{|bd|}{|\lambda|\Gamma(\gamma-1)}\int_1^e(\ln\frac{e}{s})^{\gamma-2}\frac{1}{s}ds]$$
$$\leq(L_1+L_2)\eta\,\|z_1-z_2\|\,.$$

By $(H2)$, we have $(L_1+L_2)\eta<1$. Thus the operator A is a contraction. Then by Banach fixed point theorem, we know that the operator A has a unique fixed point. Therefore problem (1.1) has a unique solution on $[1,e]$. □

Theorem 3.3. *Assume that* $(H1)$, $(H3)$ *hold, and*

$$\frac{|bc|\,e}{|\lambda|\Gamma(\gamma+1)}+\frac{|bd|}{|\lambda|\Gamma(\gamma)}<(L_1+L_2)^{-1},\tag{3.4}$$

Then problem (1.1) has at least one solution on $[1,e]$.

Proof. We introduce the norm $\| z \|= \max_{\tau\in[1,e]} | z |$, fixing $B_{r_2} = \{z \in C([1,e],\mathbb{R}) : \| z \| \leq r_2\}$, where

$$r_2 > (\| \varphi \| + \| \psi \|)\eta + \frac{e \mid bk \mid}{\mid \lambda \mid}.$$

We defined operators $Q,\ G$ on B_{r_2} , where

$$(Qz)(\tau) = \frac{1}{\Gamma(\gamma)}\int_1^\tau (\ln\frac{\tau}{s})^{\gamma-1}\frac{g(s,z(s))}{s}ds + \frac{1}{\Gamma(\gamma)}\int_1^\tau (\ln\frac{\tau}{s})^{\gamma-1}\frac{h(s,z(s))}{s}ds,$$

$$\begin{aligned}(Gz)(\tau) =& \mid (b - a\ln\tau) \mid [\frac{\mid c \mid e}{\mid \lambda \mid \Gamma(\gamma)}\int_1^e (\ln\frac{e}{s})^{\gamma-1}\frac{g(s,z(s))}{s}ds + \frac{\mid c \mid e}{\mid \lambda \mid \Gamma(\gamma)}\int_1^e (\ln\frac{e}{s})^{\gamma-1}\frac{h(s,z(s))}{s}ds \\ &+ \frac{\mid d \mid}{\mid \lambda \mid \Gamma(\gamma-1)}\int_1^e (\ln\frac{e}{s})^{\gamma-2}\frac{g(s,z(s))}{s}ds + \frac{\mid d \mid}{\mid \lambda \mid \Gamma(\gamma-1)}\int_1^e (\ln\frac{e}{s})^{\gamma-2}\frac{h(s,z(s))}{s}ds + \frac{e \mid k \mid}{\mid \lambda \mid}].\end{aligned}$$

For each $x, y \in B_{r_2}$, we get

$$\begin{aligned}\| Qx + Gy \| \leq& (\| \varphi \| + \| \psi \|)[\frac{1}{\Gamma(\gamma)}\int_1^\tau (\ln\frac{\tau}{s})^{\gamma-1}\frac{1}{s}ds + \mid (b - a\ln\tau) \mid \frac{\mid c \mid e}{\mid \lambda \mid \Gamma(\gamma)}\int_1^e (\ln\frac{e}{s})^{\gamma-1}\frac{1}{s}ds \\ &+ \mid (b - a\ln\tau) \mid \frac{\mid d \mid}{\mid \lambda \mid \Gamma(\gamma-1)}\int_1^e (\ln\frac{e}{s})^{\gamma-2}\frac{1}{s}ds] + \mid (b - a\ln\tau) \mid \frac{e \mid k \mid}{\mid \lambda \mid} \\ \leq& (\| \varphi \| + \| \psi \|)\eta + \frac{e \mid bk \mid}{\mid \lambda \mid} \leq r_2.\end{aligned}$$

Thus $Qx + Gy \in B_{r_2}$. By assumption $(H1)$ and simple calculation, we obtain

$$\| (Gz_1)(\tau) - (Gz_2)(\tau) \| \leq (L_1 + L_2) \| z_1 - z_2 \| \left(\frac{\mid bc \mid e}{\mid \lambda \mid \Gamma(\gamma+1)} + \frac{\mid bd \mid}{\mid \lambda \mid \Gamma(\gamma)}\right).$$

Then according to (3.4), we have G is a contraction mapping. In view of the continuities of g, h, it is easy to know that Q is continuous, thus Q is bounded.

Next, we show that the operator Q is equicontinuous. Let $\max_{\tau\in[1,e]\times B_{r_2}} | g(\tau, z) |= q_1$, $\max_{\tau\in[1,e]\times B_{r_2}} | h(\tau, z) |= q_2$, for any $\tau_1, \tau_2 \in [1, e]$, and $\tau_1 < \tau_2$, we have

$$\begin{aligned}\mid (Qz)(\tau_2) - (Qz)(\tau_1) \mid \leq& \frac{q_1}{\Gamma(\gamma)}\Big|\int_1^{\tau_2} (\ln\frac{\tau_2}{s})^{\gamma-1}\frac{1}{s}ds - \int_1^{\tau_1} (\ln\frac{\tau_1}{s})^{\gamma-1}\frac{1}{s}ds\Big| \\ &+ \frac{q_2}{\Gamma(\gamma)}\Big|\int_1^{\tau_2} (\ln\frac{\tau_2}{s})^{\gamma-1}\frac{1}{s}ds - \int_1^{\tau_1} (\ln\frac{\tau_1}{s})^{\gamma-1}\frac{1}{s}ds\Big| \\ \leq& \frac{q_1 + q_2}{\Gamma(\gamma+1)}\big[\ \mid (\ln\tau_2 - \ln\tau_1)^\gamma - (\ln\tau_2)^\gamma + (\ln\tau_1)^\gamma \mid + \mid (\ln\tau_2 - \ln\tau_1)^\gamma \mid\ \big].\end{aligned}$$

Then

$| (Qz)(\tau_2) - (Qz)(\tau_1) | \to 0$, as $\tau_2 \to \tau_1$.

By the Ascoli-Arzelá theorem, we know that the Q is completely continuous. According to Lemma 2.5, problem (1.1) has at least one solution on $[1, e]$. □

4 Examples

In this section, we give two examples to illustrate our main results.

Example 4.1 Consider the following equation

$$\begin{cases} {}^{C}_{H}D^{\frac{3}{2}}_{1^+}\{z(\tau) - {}_{H}I^{\frac{3}{2}}[\frac{\cos\tau}{50}(\frac{|z|^2}{|z|^2+1} + \sin z)]\} = \frac{\ln\tau}{3}(\frac{|z|^3}{|z|^3+1} + \sin z), & \tau \in [1, e], \\ 2z(1) + z'(1) = 0, \qquad 3z(e) + 2z'(e) = 5, \end{cases} \tag{4.1}$$

where $\gamma = \frac{3}{2},\ a = 2,\ b = 1,\ c = 3,\ d = 2,\ k = 5,\ \lambda = ace - bce + ad = 3e + 4,$
$\eta := \frac{1}{\Gamma(\gamma+1)} + \frac{|bc|e}{|\lambda|\Gamma(\gamma+1)} + \frac{|bd|}{|\lambda|\Gamma(\gamma)} \approx 1.443,$
and

$$| g(\tau, z_1(\tau)) - g(\tau, z_2(\tau)) | \leq \frac{1}{3} | z_1 - z_2 |,$$
$$| h(\tau, z_1(\tau)) - h(\tau, z_2(\tau)) | \leq \frac{1}{50} | z_1 - z_2 | .$$

Then

$$(L_1 + L_2)\eta = (\frac{1}{3} + \frac{1}{50}) \times 1.443 \approx 0.51 < 1.$$

We obtain that $(H1), (H2)$ hold.

Thus, the hypotheses of Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, problem (4.1) has a unique solution on $[1, e]$.

Example 4.2 Consider the following equation

$$\begin{cases} {}^{C}_{H}D^{\frac{3}{2}}_{1^+}[z(\tau) - {}_{H}I^{\frac{3}{2}}(\frac{|z|\cos\tau}{1+|z|} + 2\ln\tau)] = \frac{|z|\ln\tau}{|z|+1} + \sin\tau, & \tau \in [1, e], \\ 8z(1) - z'(1) = 0, \qquad 5z(e) + z'(e) = 5, \end{cases} \tag{4.2}$$

where $\gamma = \frac{3}{2},\ a = 8,\ b = -1,\ c = 5,\ d = 1,\ k = 5,\ \lambda = ace - bce + ad = 45e + 8,$
$\eta := \frac{1}{\Gamma(\gamma+1)} + \frac{|bc|e}{|\lambda|\Gamma(\gamma+1)} + \frac{|bd|}{|\lambda|\Gamma(\gamma)} \approx 0.839,$
and

$$| g(\tau, z_1(\tau)) - g(\tau, z_2(\tau)) | \leq | z_1 - z_2 |,$$
$$| h(\tau, z_1(\tau)) - h(\tau, z_2(\tau)) | \leq | z_1 - z_2 |,$$
$$| g(\tau, z) | = | \frac{\ln\tau\, | z |}{| z | + 1} + \sin\tau | \leq | z | + 1,$$
$$| h(\tau, z) | = | \frac{\cos\tau\, | z |}{1 + | z |} + 2\ln\tau | \leq | z | + 2.$$

Then $(L_1 + L_2)^{-1} = \frac{1}{2}$.

Thus

$$\frac{|\ bc\ |\ e}{|\ \lambda\ |\ \Gamma(\gamma+1)} + \frac{|\ bd\ |}{|\ \lambda\ |\ \Gamma(\gamma)} \approx 0.087 < (L_1 + L_2)^{-1}.$$

Thus, all conditions of Theorem 3.3 are satisfied. By Theorem 3.3, problem (4.2) has at least one solution on $[1, e]$.

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