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Some Hermite-Hadamard Type Inequalities for s-Convex Functions

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Abstract

In this paper an identity is presented in order to establish several Hermite-Hadamard type inequalities for functions whose powers of absolute values of third derivatives are s-convex. Some consequences are also presented.

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1. Introduction

The convex analysis has an important role in mathematics and in many other fields such as numerical analysis, convex programming, statistics and approximation theory. The classical inequality of Hermite-Hadamard was extended and generalized in many directions in recent years by many authors, like for example, [9, 8, 1, 12, 11, 15, 10, 5, 2, 3, 13, 4, 16, 17] and the references therein.

We begin by recalling below the classical definition for the convex functions and then for s-convex functions([5], [8],[6],[7]).

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be s-convex if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{2}$$

for each $x, y \in \mathbb{R}$ and $t \in (0,1),\ s \in (0,1]$.

The classical Hermite-Hadamard's inequality for convex functions, see [14] is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

The aim of this paper is to give several Hermite-Hadamard type inequalities for functions whose powers of absolute values of third derivatives are s-convex. For this goal an identity is presented as a main tool in the demonstrations of these results.

2. Several Hermite-Hadamard type inequalities for convex functions

Starting from a result from [2], the aim of this section is to present some Hermite-Hadamard type inequalities for functions whose powers of absolute values of third derivatives are s-convex .

Lemma 1. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R}),\ f''' \in L[a,b]$, then for all $x \in I^0$ the following inequality takes place:*

$$\frac{(x-a)^2 f'(a) - (x-b)^2 f'(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du =$$

$$= \frac{1}{6(b-a)}\int_0^1 t(1-t)^2[(x-a)^4 f'''(tx+(1-t)a) - (x-b)^4 f'''(tx+(1-t)b)]dt$$

Proof. It will be denoted $I_1 = \int_0^1 t(1-t)^2(x-a)^4 f'''(tx+(1-t)a)dt$ and $I_2 = \int_0^1 t(1-t)^2(x-b)^4 f'''(tx+(1-t)b)$. By integrating by parts three times I_1 and I_2 we get,

$$I_1 = -(x-a)^3 \int_0^1 (1-4t+3t^2) f''(tx+(1-t)a)dt =$$

$$= (x-a)^2 f'(a) + 2f(x)(x-a) + 4f(a)(x-a) - 6(x-a)\int_0^1 f(tx+(1-t)a)dt$$

and here by using $u = tx + (1-t)a$, it is obtained

$$I_1 = (x-a)^2 f'(a) + 2f(x)(x-a) + 4f(a)(x-a) - 6\int_a^x f(u)du,$$

and

$$I_2 = (x-b)^2 f'(b) + 2f(x)(x-b) + 4f(b)(x-b) + 6\int_x^b f(v)dv,$$

where $v = tx + (1-t)b$. Now substracting I_2 from I_1, we have,

$$I_1 - I_2 = (x-a)^2 f'(a) - (x-b)^2 f'(b) + 2(b-a)f(x) + 4[(x-a)f(a) - (x-b)f(b)] - 6\int_a^b f(u)du,$$

and dividing by $6(b-a)$ last equality the desired inequality is obtained.

□

Theorem 1. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f''' \in L[a,b]$, and $|f'''|$ is convex on $[a,b]$, then for all $x \in I^0$ the following inequality is satisfied:*

$$\left|\frac{(x-a)^2 f'(a) - (x-b)^2 f'(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$

$$\leq \frac{1}{60(b-a)}\left[\frac{(x-a)^4|f'''(a) + (x-b)^4|f'''(b)|}{2} + \frac{(x-a)^4 + (x-b)^4}{3}|f'''(x)|\right].$$

Proof. It will be used Lemma 1, the definition of convex functions for $|f'''|$ and the properties of the Gamma and Beta functions. We will have then

$$\left|\frac{(x-a)^2 f'(a) - (x-b)^2 f'(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2|f'''(tx+(1-t)a)|dt + \frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2|f'''(tx+(1-t)b)|dt \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2[t|f'''(x)| + (1-t)|f'''(a)|]dt +$$

$$+ \frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2[t|f'''(x)| + (1-t)|f'''(b)|]dt \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\left[|f'''(x)|\int_0^1 t^2(1-t)^2dt + |f'''(a)|\int_0^1 t(1-t)^3dt\right] +$$

$$+ \frac{(x-b)^4}{6(b-a)}\left[|f'''(x)|\int_0^1 t^2(1-t)^2dt + |f'''(b)|\int_0^1 t(1-t)^3dt\right] =$$

$$= \frac{(x-a)^4 + (x-b)^4}{6(b-a)}|f'''(x)|B(3,3) + \frac{(x-a)^4|f'''(a)| + (x-b)^4|f'''(b)|}{6(b-a)}B(2,4) =$$

$$= \frac{1}{60(b-a)}\left[\frac{(x-a)^4 + (x-b)^4}{3}|f'''(x)| + \frac{(x-a)^4|f'''(a)| + (x-b)^4|f'''(b)|}{2}\right]$$

where $B(x,y)$ is the Beta function, $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x>0,\ y>0$ and the Gamma function is $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ x>0.$

$\square$

Corollary 1. *Let $f: I^0\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^0, a,b\in I^0$ with $a<b$. If $f\in C^3(\mathbb{R}),\ f^{'''}\in L[a,b]$, and $|f^{'''}|$ is convex on $[a,b]$, then the following inequality is holds:*

$$|\frac{b-a}{24}[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du|\le$$

$$\le\frac{(b-a)^3}{960}\{\frac{1}{2}[|f^{'''}(a)|+|f^{'''}(b)|]+\frac{2}{3}|f^{'''}(\frac{a+b}{2})|].$$

Proof. We put in previous theorem $x=\frac{a+b}{2}$. $\square$

Corollary 2. *As in Corollary 1, by using the convexity of $|f^{'''}|$, we can obtain further,*

$$|\frac{b-a}{24}[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du|\le$$

$$\le\frac{(b-a)^3}{1152}\{|f^{'''}(a)|+|f^{'''}(b)|\}.$$

Theorem 2. *Let $f: I^0\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^0, with $a,b\in I^0$ and $a<b$. If $f\in C^3(\mathbb{R}),\ f^{'''}\in L[a,b]$, and $|f^{'''}|$ is s-convex on $[a,b]$, then for all $x\in I^0$ the following inequality takes place:*

$$|\frac{(x-a)^2f^{'}(a)-(x-b)^2f^{'}(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du|\le$$

$$\le\frac{1}{6(b-a)}\{(x-a)^4[2\frac{|f^{'''}(x)|}{(s+4)(s+3)(s+2)}+\frac{|f^{''}(a)|}{(s+4)(s+3)}]+$$

$$+(x-b)^4[2\frac{|f^{'''}(x)|}{(s+4)(s+3)(s+2)}+\frac{|f^{'''}(b)|}{(s+4)(s+3)}]\}.$$

Proof. The method of demonstration is analogue to Theorem 1 but it is used the definition of s-convexity instead of convexity for function $|f^{'''}|^q$. $\square$

Corollary 3. *Let $f: I^0\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^0, a,b\in I^0$ with $a<b$. If $f\in C^3(\mathbb{R}),\ f^{'''}\in L[a,b]$, and $|f^{'''}|$ is s-convex on $[a,b]$, then we get the following:*

$$|\frac{b-a}{24}[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du|\le$$

$$\leq \frac{(b-a)^3}{6}\{[2\frac{|f^{'''}(\frac{a+b}{2})|}{(s+4)(s+3)(s+2)} + \frac{|f^{''}(a)|}{(s+4)(s+3)}]+$$
$$+[2\frac{|f^{'''}(\frac{a+b}{2})|}{(s+4)(s+3)(s+2)} + \frac{|f^{'''}(b)|}{(s+4)(s+3)}]\}.$$

Theorem 3. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, and $|f^{'''}|^q$ is s-convex on $[a,b]$, $q > 1$ then for all $x \in I^0$ the following inequality holds:*

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{1}{6(b-a)}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}(x-a)^4[2\frac{|f^{'''}(x)|^q}{(s+4)(s+3)(s+2)} + \frac{|f^{''}(a)|^q}{(s+4)(s+3)}]^{\frac{1}{q}}+$$
$$+(x-b)^4[2\frac{|f^{'''}(x)|^q}{(s+4)(s+3)(s+2)} + \frac{|f^{'''}(b)|^q}{(s+4)(s+3)}]^{\frac{1}{q}}.$$

Proof. Using Lemma 1, Holder's inequality and that $|f^{'''}|^q$ is s-convex function we have succesively next inequalities,

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)a)|dt + \frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)b)|dt =$$
$$= \frac{(x-a)^4}{6(b-a)}\int_0^1 [t(1-t)^2]^{1-\frac{1}{q}}[t(1-t)^2]^{\frac{1}{q}}|f^{'''}(tx+(1-t)a)|dt+$$
$$+\frac{(x-b)^4}{6(b-a)}\int_0^1 [t(1-t)^2]^{1-\frac{1}{q}}[t(1-t)^2]^{\frac{1}{q}}|f^{'''}(tx+(1-t)b)|dt \leq$$
$$\leq \frac{(x-a)^4}{6(b-a)}\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}} +$$
$$+\frac{(x-b)^4}{6(b-a)}\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2|f^{'''}(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}} \leq$$
$$\leq \frac{(x-a)^4}{6(b-a)}[B(2,3)]^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2[t^s|f^{'''}(x)|^q + (1-t)^s|f^{'''}(a)|^q]dt\right)^{\frac{1}{q}} +$$
$$+\frac{(x-b)^4}{6(b-a)}[B(2,3)]^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2[t^s|f^{'''}(x)|^q + (1-t)^s|f^{'''}(b)|^q]dt\right)^{\frac{1}{q}} =$$
$$= \frac{\left(\frac{1}{12}\right)^{1-\frac{1}{q}}}{6(b-a)}\{(x-a)^4[|f^{''}(x)|^q B(s+2,3) + |f^{'''}(a)|^q B(2,s+3)]^{\frac{1}{q}}+$$
$$+(x-b)^4[|f^{''}(x)|^q B(s+2,3) + |f^{'''}(b)|^q B(2,s+3)]^{\frac{1}{q}}\}.$$

Now by taking into account that $B(s+2,3)=\frac{2}{(s+4)(s+3)(s+2)}$ and $B(2,s+3)=\frac{1}{(s+4)(s+3)}$ we find the inequality from conclusion.

□

Corollary 4. *Let $f: I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f''' \in L[a,b]$, and $|f'''|^q$ is s-convex on $[a,b]$, $q > 1$ then the next inequality holds:*

$$|\frac{1}{24}(b-a)[f'(a)-f'(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{(b-a)^3}{96}\frac{1}{12^{1-\frac{1}{q}}}\{[2\frac{|f'''(\frac{a+b}{2})|^q}{(s+4)(s+3)(s+2)}+\frac{|f'''(a)|^q}{(s+4)(s+3)}]^{\frac{1}{q}}+$$

$$+[2\frac{|f'''(\frac{a+b}{2})|^q}{(s+4)(s+3)(s+2)}+\frac{|f'''(b)|^q}{(s+4)(s+3)}]^{\frac{1}{q}}\}.$$

Proof. We put $x=\frac{a+b}{2}$ in Theorem 2. □

Theorem 4. *Suppose that $f: I^0 \subset \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f''' \in L[a,b]$, and $|f'''|^q$ is s-convex on $[a,b]$, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ then we have the following inequality:*

$$|\frac{(x-a)^2 f'(a)-(x-b)^2 f'(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{p^{\frac{1}{p}}\sqrt{\pi}^{\frac{1}{p}}}{b-a}\frac{2}{81}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{p}}\frac{\Gamma(p)\Gamma(p+\frac{1}{2})}{\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}\frac{1}{(s+1)^{\frac{1}{q}}}\{(x-a)^4[|f'''(x)|^q+|f'''(a)|^q]^{\frac{1}{q}}+$$

$$+(x-b)^4[|f'''(x)|^q+|f'''(b)|^q]^{\frac{1}{q}}\}.$$

Proof. Like in the proof of Theorem 1 we have,

$$|\frac{(x-a)^2 f'(a)-(x-b)^2 f'(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\int_0^1 t(1-t)^2|f'''(tx+(1-t)a)|dt+\frac{(x-b)^4}{6(b-a)}\int_0^1 t(1-t)^2|f'''(tx+(1-t)b)|dt.$$

In view of Holder's inequality and s-convexity for $|f'''|^q$ we get,

$$|\frac{(x-a)^2 f'(a)-(x-b)^2 f'(b)+2f(x)(b-a)+4[(x-a)f(a)-(x-b)f(b)]}{6(b-a)}-\frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{(x-a)^4}{6(b-a)}\left(\int_0^1 [t(1-t)^2]^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'''(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+\frac{(x-b)^4}{6(b-a)}\left(\int_0^1 [t(1-t)^2]^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'''(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{B^{\frac{1}{p}}(p+1,2p+1)}{6(b-a)}\{(x-a)^4[|f^{'''}(x)|^q\int_0^1 t^s dt+|f^{'''}(a)|^q\int_0^1(1-t)^s dt]^{\frac{1}{q}}+$$

$$+(x-b)^4[|f^{'''}(x)|^q\int_0^1 t^s dt+|f^{'''}(b)|^q\int_0^1(1-t)^s dt]^{\frac{1}{q}}\}=$$

$$=\frac{B^{\frac{1}{p}}(p+1,2p+1)}{6(b-a)}\{(x-a)^4[\frac{|f^{'''}(x)|^q+|f^{'''}(a)|^q}{s+1}]^{\frac{1}{q}}+$$

$$+(x-b)^4[\frac{|f^{'''}(x)|^q+|f^{'''}(b)|^q}{s+1}]^{\frac{1}{q}}\}.$$

On the other hand, by Gauss multiplication formula,

$$\Gamma(z)\prod_{k=1}^{n-1}\Gamma(z+\frac{k}{n})=n^{\frac{1}{2}-nz}(2\pi)^{(n-1)/2}\Gamma(nz)$$

we get

$$\Gamma(3p)=\frac{\Gamma(p)\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}{2\pi 3^{\frac{1}{2}-3p}}$$

for $n=3$ and by Legendre's duplication formula for the Gamma function,

$$\Gamma(z)\Gamma(z+\frac{1}{2})=2^{1-2z}\sqrt{\pi}\Gamma(2z),\ Rez>0$$

, we have,

$$B(p+1,2p+1)=\frac{2p\Gamma(p)\Gamma(2p)}{3(3p+1)\Gamma(3p)}=p\sqrt{\pi}\frac{\Gamma(p)\Gamma(p+\frac{1}{2})}{\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}\frac{2^{2p+1}}{3^{\frac{1}{2}+3p}}.$$

Last expression will be replaced in last inequality and the proof will be finished.

□

Corollary 5. *Under conditions of previous theorem, if we take* $x=\frac{a+b}{2}$ *the following inequality holds:*

$$|\frac{1}{24}(b-a)[f^{'}(a)-f^{'}(b)]+\frac{1}{3}[f(a)+f(b)+f(\frac{a+b}{2})]-\frac{1}{b-a}\int_a^b f(u)du|\leq$$

$$\leq p^{\frac{1}{p}}\sqrt{\pi}^{\frac{1}{p}}\frac{2}{81}\left(\frac{2}{\sqrt{3}}\right)^{\frac{1}{p}}\frac{\Gamma(p)\Gamma(p+\frac{1}{2})}{\Gamma(p+\frac{1}{3})\Gamma(p+\frac{2}{3})}\frac{(b-a)^3}{(s+1)^{\frac{1}{q}}}\{[|f^{'''}(\frac{a+b}{2})|^q+|f^{'''}(a)|^q]^{\frac{1}{q}}+$$

$$+[|f^{'''}(\frac{a+b}{2})|^q+|f^{'''}(b)|^q]^{\frac{1}{q}}\}.$$

Theorem 5. *Suppose that $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on $I^0, a, b \in I^0$ with $a < b$. If $f \in C^3(\mathbb{R})$, $f^{'''} \in L[a,b]$, and $|f^{'''}|^q$ is s-convex on $[a,b]$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ then we have the following inequality:*

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{1}{6(b-a)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\{(x-a)^4[|f^{'''}(x)|^q B(2q+1,s+1) + \frac{|f^{'''}(a)|^q}{2q+s+1}]^{\frac{1}{q}} +$$

$$+(x-b)^4[|f^{'''}(x)|^q B(2q+1,s+1) + \frac{|f^{'''}(b)|^q}{2q+s+1}]^{\frac{1}{q}}\}.$$

Proof. In view of s-convexity of $|f^{'''}|^q$ and Holder's inequality, the first inequality from the proof of Theorem becomes,

$$|\frac{(x-a)^2 f^{'}(a) - (x-b)^2 f^{'}(b) + 2f(x)(b-a) + 4[(x-a)f(a) - (x-b)f(b)]}{6(b-a)} - \frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \frac{1}{6(b-a)}\{(x-a)^4\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{2q}|f^{'''}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}} +$$

$$+(x-b)^4\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{2q}|f^{'''}(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\} \leq$$

$$\leq \frac{1}{6(b-a)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\{(x-a)^4[\int_0^1 (1-t)^{2q}\left(t^s|f^{'''}(x)|^q + (1-t)^s|f^{'''}(a)|^q\right)dt]^{\frac{1}{q}} +$$

$$+(x-b)^4[\int_0^1 (1-t)^{2q}\left(t^s|f^{'''}(x)|^q + (1-t)^s|f^{'''}(b)|^q\right)dt]^{\frac{1}{q}}\}$$

which leads to desired inequality, by taking into account the definition of Beta function $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt,\ p,q > 0$.

□

Corollary 6. *Under conditions of previous theorem, if we take $x = \frac{a+b}{2}$ we have,*

$$|\frac{1}{24}(b-a)[f^{'}(a) - f^{'}(b)] + \frac{1}{3}[f(a) + f(b) + f(\frac{a+b}{2})] - \frac{1}{b-a}\int_a^b f(u)du| \leq$$

$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{(b-a)^3}{6}\{[|f^{'''}(\frac{a+b}{2})|^q B(2q+1,s+1) + \frac{|f^{'''}(a)|^q}{2q+s+1}]^{\frac{1}{q}} +$$

$$+[|f^{'''}(\frac{a+b}{2})|^q B(2q+1,s+1) + \frac{|f^{'''}(b)|^q}{2q+s+1}]^{\frac{1}{q}}\}.$$

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