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A New Class of the Weibull-Generalized Truncated Poisson Distribution for Reliability and Life Data Analysis

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Abstract

In this paper, a new class of lifetime distributions will be constructed using the ascendant order statistics. The constructed class is called Weibull-generalized truncated Poisson (WGTP) distribution. The properties of Weibull-generalized truncated Poisson (WGTP) distribution will be studied. The maximum likelihood (ML) method and the expectation maximization (EM) algorithm will be used to estimate the parameters. A comparison between the new class of distributions and other some lifetime distributions will be performed based on a real set of data.

Keywords: Lifetime distribution; Weibull distribution; Truncated Poisson; Failure rate function; Order Statistics; Maximum likelihood estimation; EM Algorithm

1. Introduction

There are several distributions have been proposed to model lifetime data by compounding some useful lifetime distributions. For example, Adamidis and Loukas (1998) introduced a new compounding distribution named the Exponential –Geometric (EG) distribution with decreasing failure rate. Also, Kus (2007) proposed the Exponential-Poisson (EP) distribution, where the baseline is the exponential distribution and the latent variable has zero-truncated Poisson. Lu and

Shi (2012) extended the (EP) distribution using the mixture of the Weibull distribution as a baseline and zero-truncated Poisson as a latent variable. Barreto-Souza and Cribari-Neto (2009) introduced another generalization of the (EP) distribution by inserting a power parameter. Rahmouni and Orabi (2017) proposed new family distributions by mixing the exponential and generalized truncated Poisson distributions called the exponential-generalized truncated Poisson (EGTP) distribution.

In this paper, we introduce a generalization of Rahmouni's and Orabi's (2017) work by compounding the Weibull distribution and the generalized truncated Poisson distribution. The new distribution called the Weibull-generalized truncated Poisson (WGTP) distribution. Kus's (2007) and Lu's and Shi's (2012) works are special cases from our work. In section 2, the new proposed model will be constructed including the probability density function (pdf) and cumulative distribution (cdf) with some special cases. In section 3, Some statistical proprieties of Weibull-generalized truncated Poisson (WGTP) distribution will be derived such that, the moment generating function, the reliability and failure rate functions, and the random number generation. The estimation of the parameters using the maximum likelihood method and the expectation maximization (EM) algorithm will be given in section 4. Finally, the application of a real data set is illustrated in section 5 to show the flexibility of the new distribution is more than some the other distributions.

2. The proposed model

Let $X=(X_1,X_2,.....,X_n)$ be random variables distributed according to Weibull distribution with parameter θ and α with probability density function (pdf) as

$$f(x)=\alpha\theta^{-\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\theta}\right)^{\alpha}} \quad ; \quad \alpha,\theta>0 \text{ and } x\geq 0$$

where θ is the scale parameter and α is the shape parameter. The cumulative distribution and the survival functions respectively are

$$F(x)=1-e^{-\left(\frac{x}{\theta}\right)^{\alpha}} \quad ; \quad x\geq 0$$

and

$$S(x)=e^{-\left(\frac{x}{\theta}\right)^{\alpha}}$$

If Z is a discrete random number following the $(k-1)$ truncated Poisson distribution with probability function $P_{Z/\lambda}(Z=z)$ given as:-

$$P_{Z/\lambda}(Z=z)=\frac{e^{-\lambda}\lambda^{z}}{z!A(k,\lambda)} \quad ; \quad \lambda\succ 0 \text{ and } z=k,k+1,k+2,...... \tag{2.1}$$

where $A(k,\lambda)=1-\sum_{n=0}^{k-1}p(N=n)$ and $p(N=n)$ is a Poisson distribution with probability mass function (pmf) defined as $P(n)=\dfrac{e^{-\lambda}\lambda^{n}}{n!}$ $n=0,\ 1,\ 2,...$. Also, the density function (pdf) of i^{th} order statistic $X_{(i)}$ is given by David and Nagaraja

(2003), Balakrishnan and Cohen (1991) Then, the pdf of i^{th}- smallest value of lifetime is given as:-

$$g_k(x/z)=C_{k,z}\left[F(x)\right]^{k-1}\left[S_{X|\{\alpha,\theta\}}(x)\right]^{z-k}f(x) \quad ; \quad \alpha,\theta>0 \text{ and } x\geq 0 \qquad (2.2)$$

where $C_{k,z}=\dfrac{z!}{(k-1)!(z-k)!}$.

In the joint probability density function we derive the ascending order $X_{(1)},X_{(2)},...,X_{(z)}$. The following equation is the result of using equations (2.1) and (2.2):-

$$g_k(x,z)=\frac{\alpha\theta^{-\alpha}x^{\alpha-1}\left(1-e^{-\left(\frac{x}{\theta}\right)^{\alpha}}\right)^{k-1}\lambda^z e^{-(z-k+1)\left(\frac{x}{\theta}\right)^{\alpha}-\lambda}}{(k-1)!(z-k)!A(k,\lambda)} \quad ; \quad k=1,2,....,z \qquad (2.3)$$

where x,z are the life time of a system and the last order statistic, respectively and $g_{k/\alpha,\theta,\lambda}(x,z)$ is the joint probability density function, this joint function is obtained by compounding a truncated at $z=k-1$ Poisson distribution and k^{th} order statistic. In order to get the probability density function for the variable x, we will sum the joint function given in equation (2.3) on the variable z. Thus, our proposed new family of lifetime distributions, named the Weibull-generalized truncated Poisson (WGTP) distribution, is the marginal density distribution of X given by

$$g_k(x)=\frac{\alpha\theta^{-\alpha}\lambda^k x^{\alpha-1}\exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)\left[1-\exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)\right]^{k-1}\exp\left[-\lambda\left(1-e^{-\left(\frac{x}{\theta}\right)^{\alpha}}\right)\right]}{(k-1)!IG(\lambda,k)} \quad ; \quad x\geq 0 \qquad (2.4)$$

where $\alpha>0$ and $\lambda>0$ are shape parameters and $\theta>0$ is scale parameter. When $\alpha=1$ the WGTP distribution reduces to Exponential-generalized truncated Poisson (EGTP) distribution.

Also, $IG(\lambda,k)$ is the lower incomplete gamma function, can be defined as:

$$IG(\lambda,k)=\int_0^{\lambda}\lambda^{k-1}e^{-\lambda}d\lambda=1-\sum\nolimits_{i=0}^{k-1}p(i)$$

According to (2.4), we have the Weibull- generalized truncated Poisson (WGTP) distribution function of $X|\{\alpha,\theta,\lambda\}$ as

$$G_k(x)=\frac{IG(\lambda v,k)}{IG(\lambda,k)} \quad ; \quad k=1,2,...,z \qquad (2.5)$$

where,

$$v=1-\exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)$$

For modeling any order statistic, the WGTP distribution would be more appropriate. As will be shown below, the minimum lifetimes are special models such as Weibull–Poisson distribution due to Lu and Shi (2012) and exponential–Poisson distribution is proposed by Kuş (2007).

In table (1), the PDF in Eq. (2.4) for some special cases at the first, second and third order statistics is presented.

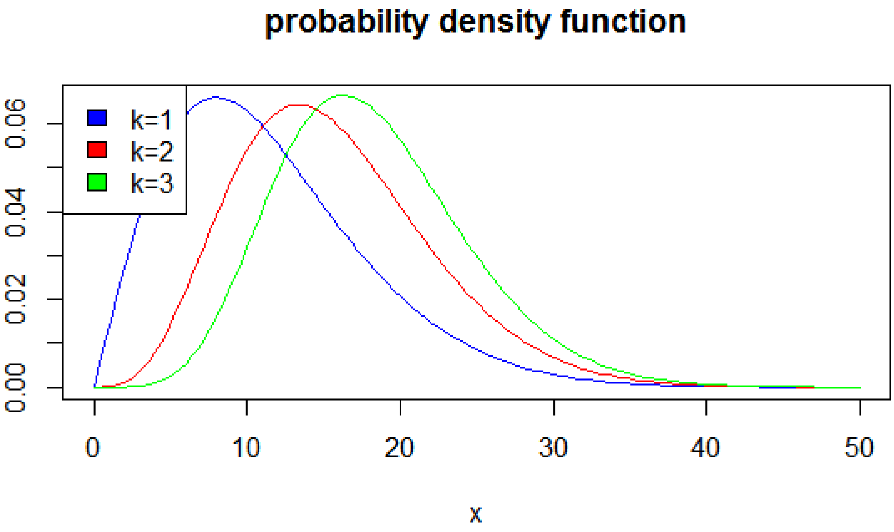
Table 1. The PDF of WGTP distribution for some special cases

Order Statistics	k	PDF
First	$k=1$	$\dfrac{\alpha\theta^{-\alpha}\lambda x^{\alpha-1}(1-v)\exp[\lambda v]}{e^{\lambda}-1}$
Second	$k=2$	$\dfrac{\alpha\theta^{-\alpha}\lambda^2 x^{\alpha-1}(1-v)v\exp[\lambda v]}{e^{\lambda}-\lambda-1}$
Third	$k=3$	$\dfrac{\alpha\theta^{-\alpha}\lambda^3 x^{\alpha-1}(1-v)v^2\exp[\lambda v]}{2e^{\lambda}-\lambda^2-2\lambda-2}$

where,

$$v=1-\exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)$$

Table (2.1) shows that the particular case of the WGTP distribution, for $k=1$ is the lifetime Weibull–Poisson distribution (WP) due to (Lu and Shi (2012)). The PDF of WP is monotone decreasing if $0<\alpha\leq 1$ and unimodal if $\alpha>1$. Note that, for $k=1$ the WGTP is monotone decreasing with a model value equal to $\lambda e^{-(\frac{x}{\theta})^{\alpha}}$ given at $x=0$. As $k=1$ and $\lambda=0$ the WGTP reduces to a two-parameter Weibull distribution. Also, $k=1$ and $\alpha=1$ the WGTP reduces to an exponential distribution. For $= 2$, $\lambda = 1$, $\theta = 15$, k={1,2,3} the probability density functions of (WGTP) are illustrated in the following graph:



3. Statistical Properties

3.1 Moment

In this subsection, the r^{th} non-central moment and moment generating function are derived.

If X has $GTPW(\alpha,\theta,\lambda)$, then the r^{th} non-central moment of x is given as following:-

$$E\left(X^r\right)=\frac{\theta^r\Gamma\left(\frac{r}{\alpha}+1\right)\exp(-\lambda)}{IG(k,\lambda)}\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}\frac{\lambda^{i+k}}{i!\,j!(k-j-1)(i+j+1)^{\frac{r}{\alpha}+1}},\ k=1,2,3,...,z$$

(3.1)

when $r=1$, the equation (3.1) becomes the expected value. From r^{th} moment of $GTPW(\alpha,\theta,\lambda)$ distribution, the first four moments function of X can be derived by using r = 1, 2, 3 and 4 as follows:-

i. $$E(x)=\frac{\theta\,\Gamma\left(\frac{1}{\alpha}+1\right)\exp(-\lambda)}{IG(k,\lambda)}\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}\frac{\lambda^{i+k}}{i!\,j!(k-j-1)(i+j+1)^{\frac{1}{\alpha}+1}},$$

ii. $$E\left(x^2\right)=\frac{\theta^2\,\Gamma\left(\frac{1}{\alpha}+1\right)\exp(-\lambda)}{IG(k,\lambda)}\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}\frac{\lambda^{i+k}}{i!\,j!(k-j-1)(i+j+1)^{\frac{2}{\alpha}+1}},$$

iii. $$E\left(x^3\right)=\frac{\theta^3\,\Gamma\left(\frac{1}{\alpha}+1\right)\exp(-\lambda)}{IG(k,\lambda)}\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}\frac{\lambda^{i+k}}{i!\,j!(k-j-1)(i+j+1)^{\frac{3}{\alpha}+1}},$$

iv. $$E\left(x^4\right)=\frac{\theta^4\,\Gamma\left(\frac{1}{\alpha}+1\right)\exp(-\lambda)}{IG(k,\lambda)}\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}\frac{\lambda^{i+k}}{i!\,j!(k-j-1)(i+j+1)^{\frac{4}{\alpha}+1}}.$$

where $IG(\lambda,k)=\int_0^{\lambda}\lambda^{k-1}e^{-\lambda}d\lambda$ is lower incomplete gamma function. Also, the variance

3.2 Random number generation

Using the CDF of X in equation (2.5), the distribution of $G_k(x)$ is right-truncated Gamma, the random variable of X can be generated by using the following steps:

Generating a random variable y belong to $[0,\lambda]$ from the truncated Gamma distribution (see, Philippe (1997)).

Solving the nonlinear equation in y :

$$U = \frac{IG(y_i, k)}{IG(\lambda, k)}$$

Where

$$y_i = \lambda v = \lambda\left[1 - \exp\left(-\left(\frac{x_i}{\theta}\right)^{\alpha}\right)\right]$$

Computing the values of X as

$$X_i = \theta\left[-\ln\left(1 - \frac{y_i}{\lambda}\right)\right]^{\frac{1}{\alpha}}$$

where X is *WGTP* random variable with parameter θ, λ and α . In particular, for $k = 1$ we obtain X directly from the following equation:

$$X_i = \theta\left[-\ln\left(1 + \frac{1}{\lambda}\ln(1 - U1)\ (1 - e^{-\lambda})\right)\right]^{\frac{1}{\alpha}} \tag{3.2}$$

where $U1$ is a random variable with standard uniform (0, 1) distribution.

3.3 Reliability functions

Let X be a random variable of the Weibull- generalized truncated Poisson distribution $WGTP\ (\theta, \lambda, \alpha)$, then the survival function can be presented as follows

$$S_{X|\{\theta,\lambda,\alpha\}}(x) = 1 - \frac{IG(\lambda v, k)}{IG(\lambda, k)}, \quad (\alpha, \theta, \lambda) \in \Re^{+} \tag{3.3}$$

Also, the hazard rate of the *WGTP* is illustrated as

$$h_{X|\{\theta,\lambda,\alpha\}}(x) = \frac{\alpha\theta^{-\alpha}\lambda^{k}x^{\alpha-1}\exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)\left[1 - \exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)\right]^{k-1}\exp\left[-\lambda\left(1 - e^{-\left(\frac{x}{\theta}\right)^{\alpha}}\right)\right]}{(k-1)\ !IG(\lambda, k) - IG(\lambda v, k)}$$

(3.4)

Analytically speaking, the hazard function and the time failure probability distribution are related to each other. This relationship leads to examining the increase or the decrease in the failure rate properties of life-length distributions. Gamma distribution is an increasing failure rate, if $H_X(x)$ increases for all X such that $G(X) < 1$.

If $v = 1 - \exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)$, the reverse rate hazard function of the *GTPW* is given as

$$R_{X|\{\theta,\lambda,\alpha\}}(x) = \frac{\alpha\theta^{-\alpha}\lambda^{k}x^{\alpha-1}(1-v)[v]^{k-1}\exp[-\lambda v]}{(k-1)!\,IG(\lambda v, k)}.$$

Furthermore, the cumulative hazard function of the *GTPW* is given as

$$H_{X|\{\theta,\lambda,\alpha\}}(x) = -\ln\left(1 - \frac{IG(\lambda v, k)}{IG(\lambda, k)}\right).$$

In table (2) the reliability function in Eq. (3.3) and the failure rate function in Eq. (3.4) for some special cases at the first, second and third order statistics are presented.

Table 2. The Survival and Failure rate functions of WGTP distribution for some special cases

Order Statistics	**K**	**Survival function**
First	$k=1$	$\frac{\Phi-1}{e^{\lambda}-1}$
Second	$k=2$	$\frac{\Phi[1+\lambda-\lambda(1-v)-\lambda-1]}{e^{\lambda}-\lambda-1}$
Third	$k=3$	$\frac{\Phi[2+2\lambda v+\lambda^2v^2-\lambda^2-2\lambda-2]}{2e^{\lambda}-\lambda^2-2\lambda-2}$
		Failure rate function
First	$k=1$	$\frac{\lambda\alpha\theta^{-\alpha}x^{\alpha-1}(1-v)\Phi}{\Phi-1}$
Second	$k=2$	$\frac{\lambda^2\alpha\theta^{-\alpha}x^{\alpha-1}(1-v)v\,\Phi}{\Phi[1+\lambda-\Phi]-\lambda-1}$
Third	$k=3$	$\frac{\lambda^3\alpha\theta^{-\alpha}x^{\alpha-1}(1-v)v^2\Phi}{\Phi[2+2\lambda v+\lambda^2v^2]-\lambda^2-2\lambda-2}$

where,

$$\Phi=\exp\left(\lambda e^{\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)}\right),$$

and

$$v=1-\exp\left(-\left(\frac{x}{\theta}\right)^{\alpha}\right)$$

4. Estimation of the Parameters

In this section, we shall derive the maximum likelihood estimates of unknown parameters α,θ and λ of *WGTP* distribution. Let $X_1, X_2, ..., X_n$ be a random sample size n from $WGTP(\alpha,\theta,\lambda)$. The maximum likelihood function of this sample is:

$$L(X_i|\alpha,\theta,\lambda)=\prod_{i=1}^{n} g_X(x_i;\lambda,\theta,\alpha) \tag{4.1}$$

Substituting from (2.4) into (4.1), we have

$$L(X_i|\alpha,\theta,\lambda)=\frac{\lambda^{nk}\theta^{-n\alpha}\alpha^{n}\left(\prod_{i=1}^{n}x_i^{\alpha-1}\right)\exp\left(-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\alpha}\right)\exp\left(-\lambda\left(n-\sum_{i=1}^{n}e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}}\right)\right)}{\left(IG(\lambda,k)\right)^{n}}.$$

The log likelihood function becomes:

$$\ln L(X_i|\alpha,\theta,\lambda)=nk\ln\lambda-n\alpha\ln\theta+n\ln\alpha+(\alpha-1)\sum_{i=1}^{n}\ln x_i-\theta^{-\alpha}\sum_{i=1}^{n}x_i-n\lambda$$
$$+\lambda\sum_{i=1}^{n}\exp\left(-\left(\frac{x_i}{\theta}\right)^{\alpha}\right)-n\ln\left(IG(\lambda,k)\right)$$

The components of the score vector for the parameters α,θ and λ are given by

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n}{\alpha}+n\ln\theta+\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\alpha}\ln\left(\frac{x_i}{\theta}\right)+(k-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{\alpha}\ln\left(\frac{x_i}{\theta}\right)e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}}}{1-e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}}}$$
$$-\lambda\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\alpha}\ln\left(\frac{x_i}{\theta}\right)e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}};$$

$$\frac{\partial\ln L}{\partial\theta}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)\ln\left(\frac{x_i}{\theta}\right)+(k-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{\alpha}\ln\left(\frac{x_i}{\theta}\right)e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}}}{1-e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}}}-\lambda\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)\ln\left(\frac{x_i}{\theta}\right)e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}};$$

and

$$\frac{\partial\ln L}{\partial\lambda}=n\left[\frac{k}{\lambda}-\frac{e^{-\lambda}\sum_{j=0}^{k-1}\frac{\lambda^{j}-j\lambda^{j-1}}{j!}}{1-\sum_{j=0}^{k-1}\frac{e^{-\lambda}\lambda^{j}}{j}}-1\right]+\sum_{i=1}^{n}\exp\left(-\left(\frac{x_i}{\theta}\right)^{\alpha}\right).$$

The maximum likelihood estimates $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\lambda}$ of the WGTP parameters can be solved using the iterative EM algorithm to treated the incomplete data problems (McLachlan and Krishnan, 1997; Dempster et al., 1977). This iterative method is composed of on frequently substitution the missing data with the new estimated ones to improve the parameter estimates. The criterion method applied to determine the MLEs is the Newton–Raphson algorithm that needed second derivatives of the log-likelihood function for all iterations. The major disadvantage of the EM algorithm is to some extent slow convergence, compared to the Newton–Raphson method, when the “missing data” contain a relatively large amount of information (Little and Rubin (1983)). Lately, various researchers have applied the EM method like Adamidis and Loukas (1998), Adamidis (1999), Ng et al. (2002), Karlis (2003) and others. Newton-Raphson is desired for the M-step of the EM algorithm.
To begin the algorithm, we ought to determine a hypothetical distribution of complete-data with PDF in equation (2.4) and then drive the conditional mass function as:

E-step:

$$E(Z|\alpha,\theta,\lambda;x) = k + \lambda e^{-\left(\frac{x_i}{\theta}\right)^{\alpha}}$$

M-step:

$$\theta^{(r+1)} = n^{-\frac{1}{\alpha^{(r+1)}}} \left\{ \sum_{i=1}^{n} - \left(\lambda^{(r)} e^{-\left(\frac{x_i}{\theta^{(r)}}\right)^{\alpha^{(r)}}} + 1 \right) x_i^{\alpha^{(r+1)}} + (k-1) \sum_{i=1}^{n} \frac{x_i^{\alpha^{(r+1)}} e^{-\left(\frac{x_i}{\theta^{(r+1)}}\right)^{\alpha^{(r+1)}}}}{1 - e^{-\left(\frac{x_i}{\theta^{(r+1)}}\right)^{\alpha^{(r+1)}}}} \right\}^{\frac{1}{\alpha^{(r+1)}}} ;$$

$$\alpha^{(r+1)} = n \left\{ n \ln \theta^{(r+1)} + \sum_{i=1}^{n} \left(\lambda^{(r)} e^{-\left(\frac{x_i}{\theta^{(r)}}\right)^{\alpha^{(r)}}} + 1 \right) \left(\frac{x_i}{\theta^{(r+1)}} \right)^{\alpha^{(r+1)}} \ln\left(\frac{x_i}{\theta^{(r+1)}} \right) - (k-1) \sum_{i=1}^{n} \frac{\left(\frac{x_i}{\theta^{(r+1)}} \right)^{\alpha^{(r+1)}} \ln\left(\frac{x_i}{\theta^{(r+1)}} \right) e^{-\left(\frac{x_i}{\theta^{(r+1)}}\right)^{\alpha^{(r+1)}}}}{1 - e^{-\left(\frac{x_i}{\theta^{(r+1)}}\right)^{\alpha^{(r+1)}}}} \right\} ;$$

$$\lambda^{(r+1)} = \frac{nk + \sum_{i=1}^{n} \lambda^{(r)} \exp\left(-\left(\frac{x_i}{\theta^{(r)}} \right)^{\alpha^{(r)}} \right)}{n \left[1 + \frac{IG'\left(\lambda^{(r+1)}, k\right)}{IG\left(\lambda^{(r+1)}, k\right)} \right]}$$

Where

$$IG' = \frac{\partial IG}{\partial \lambda}$$

5. Application Example

In the section, the fitting of WGTP distribution to a real set of data will be compared with the EGTP, EP, EG, EL, gamma and Weibull distributions. The real set of data in table (4) represents 24 observations of “time intervals between successive earthquakes” The real set of data from Kus (2007) and is analyzed by Barreto-Souza and Bakouch, (2013).

Table 3.

1163	3258	323	159	756	409	501	616
398	67	896	8592	2039	217	9	633
461	1821	4863	143	182	2117	3709	979

Table (4) shows the fitting of the GTPW, EGTP, EP, EG, EL, gamma and Weibull distributions to the real set of data in table (4). It contains estimation of parameters, calculated values of Kolmogorov–Smirnov (K-S) an p-values.

Table 4. The Goodness of Fit for real set of data

Distributions	Estimates			**K-S value**	**p-value**
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$		
WGTP:					
First order $(k = 1)$	0.879243	8.54×10^3	6.336146	0.409058	0.996152
Second order $(k = 2)$	0.220368	1.33×10^5	3.3751	1.137745	0.150137
Third order $(k = 3)$	0.463342	5.22×10^4	21.240553	0.31921	0.999957
EGTP:					
First order $(k = 1)$	-	2.77×10^3	2.6170	0.0950	0.9820
Second order $(k = 2)$	-	1.80×10^3	4.5600	0.1480	0.6680
Third order $(k = 3)$	-	1.37×10^3	6.1520	0.1830	0.3980
Fourth order $(k = 4)$	-	1.13×10^3	7.6420	0.2010	0.2880
EB	-	2.70×10^3	0.1046	0.0985	0.9738
EL	-	2.42×10^3	0.1260	0.0885	0.9885
EP	-	2.78×10^3	2.6377	0.0972	0.9772
EPL	-	3.33×10^3	0.5312	0.0712	0.9990
EG	-	3.03×10^3	0.7369	0.0964	0.9690
Gamma	-	2.00×10^3	0.7117	0.1235	0.8328
Weibull		1.23×10^3	0.7854	0.1004	0.9690

References

[1] K. Adamidis, An EM algorithm for estimating negative binomial parameters, *Australian and New Zealand journal of Statistics*, **41** (1999), no. 2, 213-221. https://doi.org/10.1111/1467-842x.00075

[2] K. Adamidis and S. Loukas, A lifetime distribution with decreasing failure rate, *Statistics and Probability Letters*, **39** (1998), no. 1, 35-42. https://doi.org/10.1016/s0167-7152(98)00012-1

[3] N. Balakrishnan and A. C. Cohen, *Order Statistics & Inference: Estimation Methods*, Academic Press, Cambridge, 1991.
https://doi.org/10.1016/c2009-0-22411-1

[4] Wagner Barreto-Souza and S. Bakouch Hassan, A new lifetime model with decreasing failure rate, *Statistics*, **47** (2013), no. 2, 465–476.
https://doi.org/10.1080/02331888.2011.595489

[5] W. Barreto-Souza and F. Cribari-Neto, A generalization of the exponential-Poisson distribution, *Statistics and Probability Letters*, **79** (2009), no. 24, 2493-2500. https://doi.org/10.1016/j.spl.2009.09.003

[6] H. A. David and H. N. Nagaraja, *Order Statistics*, 3nd edition. John Wiley and Sons, Hoboken, New Jersey, 2003.

[7] A. P. Dempster, N. M. Laird and D. B. Rubin, Maximum likelihood from incomplete data via the EM algorithm, *Journal of the Royal Statistical Society*: Series *B (Methodological)*, **39** (1977), no. 1, 1-22.
https://doi.org/10.1111/j.2517-6161.1977.tb01600.x

[8] D. Karlis, An EM algorithm for multivariate Poisson distribution and related models, *Journal of Applied Statistics*, **30** (2003), no. 1, 63-77.
https://doi.org/10.1080/0266476022000018510

[9] C. Kuş, A new lifetime distribution, *Computational Statistics and Data Analysis,* **51** (2007), no. 9, 4497–4509.

[10] R. J. Little and D. B. Rubin, On jointly estimating parameters and missing data by maximizing the complete-data likelihood, *The American Statistician*, **37** (1983), no. 3, 218-220. https://doi.org/10.1080/00031305.1983.10483106

[11] W. Lu and D. Shi, A new compounding life distribution: the Weibull–Poisson distribution, *Journal of Applied Statistics*, **39** (2012), no. 1, 21-38.
https://doi.org/10.1080/02664763.2011.575126

[12] G. J. McLachlan and T. Krishnan, *The EM Algorithm and Extensions*, Wiley, New York, 1997.

[13] H. K. T. Ng, P. S. Chan and N. Balakrishnan, Estimation of parameters from progressively censored data using EM algorithm, *Computational Statistics and Data Analysis,* **39** (2002), no. 4, 371-386.
https://doi.org/10.1016/s0167-9473(01)00091-3

[14] A. Philippe, Simulation of right and left truncated gamma distributions by mixtures, *Statistics and Computing,* **7** (1997), no. 3, 173-181.

[15] M. Rahmouni and A. Orabi, A new generalization of the exponential-Poisson distribution using order statistics, *International Journal of Applied Mathematics and Statistical Sciences*, **6** (2017), no. 6, 27–36.

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