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Stability Analysis of a Predator-Prey System with Allee Effect

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Abstract

This paper mainly studies the stability of a predator-prey model with and without Allee effect on the prey population. Combined with numerical simulations, we found that after the introduction of Allee effect, the stability of the equilibrium points changed slightly. Specifically, the equilibrium points of the system could be changed from saddle to saddle node, or otherwise the system will take much longer time to reach the stable state even when it is stable.

Mathematics Subject Classification: 34D20, 92D25

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1 Introduction

Predation, which generally exists in nature, plays an important role in ecological development. Studying the dynamic properties of predator-prey systems is helpful to predict and estimate the population, so as to protect the ecosystem. In addition to the interaction between populations, there are many environmental factors in nature that will affect the equilibrium state of populations, such as Allee effect [1] proposed in the last century. In recent decades, more and more scholars have tended to add Allee effect to the study of population model, such as [2, 3, 4, 5]. Among them, Zhou et al. [3] considered two classical

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predator-prey models with Allee effect and obtained Allee effect is probably an unstable factor in the food web.

In this paper, we mainly discuss how Allee effect in prey population affects the stability of the following predator-prey model in [6]:

$$\begin{cases} \dfrac{dx}{dt} = x\left(r - \dfrac{r}{K}x - by\right), \\ \dfrac{dy}{dt} = y\left(\dfrac{\kappa bx}{1+bhx} - d\right), \end{cases} \tag{1}$$

where $x = x(t)$ and $y = y(t)$ denote the population densities of prey and predator at time t, r is the birth rate of prey, K represents the environmental carrying capacity of prey, b represents the predation coefficient, d is the mortality of predator, $\frac{\kappa bx(t)}{1+bhx(t)}$ represents the per capita conversion rate from prey to predator in saturated form. Furthermore, the parameters r, K, b, κ, $d > 0$ and $0 \le h < \frac{\kappa}{d}$ satisfy the constraint (K_1): $K > \frac{d+\sqrt{d^2+r\kappa d/(\kappa-hd)}}{2b(\kappa-hd)}$.

2 Stability analysis of system (1)

In order to facilitate calculation, we set $\bar{t} = rt$, $\bar{x} = \frac{1}{K}x$ and $\bar{y} = \frac{b}{r}y$, and drop the bars, then system (1) is changed to

$$\begin{cases} \dfrac{dx}{dt} = x(1-x-y), \\ \dfrac{dy}{dt} = y\left(\dfrac{\alpha x}{1+\beta x} - \gamma\right), \end{cases} \tag{2}$$

where $\alpha = \frac{\kappa bK}{r}$, $\beta = bhK$ and $\gamma = \frac{d}{r}$. From the condition (K_1), we can obtain $K > \frac{d+d}{2b(\kappa-hd)} = \frac{d}{b(\kappa-hd)}$, i.e. $\alpha > (1+\beta)\gamma$.

The equilibrium point of system (2) is the solution of the following equations:

$$\begin{cases} x(1-x-y) = 0, \\ y\left(\dfrac{\alpha x}{1+\beta x} - \gamma\right) = 0. \end{cases} \tag{3}$$

It is clear that system (2) has two boundary equilibrium points $P_1(0,0)$ and $P_2(1,0)$ and a positive equilibrium point $P_3\left(\frac{\gamma}{\alpha-\beta\gamma}, 1-\frac{\gamma}{\alpha-\beta\gamma}\right)$. Theorem 2.1 describes the stability of these three equilibrium points respectively.

Theorem 2.1. *For system* (2)*, the following conclusions are obtained. The boundary equilibrium points* $P_1(0,0)$ *and* $P_2(1,0)$ *are saddles. The positive equilibrium point* $P_3\left(\frac{\gamma}{\alpha-\beta\gamma}, 1-\frac{\gamma}{\alpha-\beta\gamma}\right)$ *is locally asymptotically stable.*

Proof. The Jacobian matrix of system (2) at any equilibrium point is

$$J(P) = \begin{pmatrix} 1-2x-y & -x \\ \frac{\alpha y}{(1+\beta x)^2} & \frac{\alpha x}{1+\beta x}-\gamma \end{pmatrix}. \tag{4}$$

The Jacobian matrix of system (2) at P_1 is given by $J(P_1) = \begin{pmatrix} 1 & 0 \\ 0 & -\gamma \end{pmatrix}$, and the two eigenvalues of $J(P_1)$ are $\lambda_1(P_1) = 1 > 0$ and $\lambda_2(P_1) = -\gamma < 0$, so the equilibrium point P_1 is a saddle. Using similar methods mentioned above, we can conclude that the equilibrium point P_2 is also a saddle.

Let $x_3 = \frac{\gamma}{\alpha-\beta\gamma} > 0$ and $y_3 = 1 - \frac{\gamma}{\alpha-\beta\gamma} > 0$ which satisfy $\frac{\alpha x_3}{1+\beta x_3} - \gamma = 0$. Then the Jacobian matrix of system (2) at the positive equilibrium point P_3 is $J(P_3) = \begin{pmatrix} 1-2x_3-y_3 & -x_3 \\ \frac{\alpha y_3}{(1+\beta x_3)^2} & 0 \end{pmatrix}$, and the characteristic equation corresponding to $J(P_3)$ can be written as:

$$\lambda^2 - (1-2x_3-y_3)\lambda + \frac{\alpha x_3 y_3}{(1+\beta x_3)^2} = 0. \tag{5}$$

We can get the determinant and trace of Jacobian matrix $J(P_3)$:

$$Det[J(P_3)] = \lambda_1(P_3)\cdot\lambda_2(P_3) = \frac{\alpha x_3 y_3}{(1+\beta x_3)^2}, \tag{6}$$

and

$$Tr[J(P_3)] = \lambda_1(P_3) + \lambda_2(P_3) = 1-2x_3-y_3 = -x_3. \tag{7}$$

It is easy to get $Det[J(P_3)] > 0$ and $Tr[J(P_3)] < 0$, so P_3 is locally asymptotically stable. This completes the proof of Theorem 2.1.

3 Allee effect on prey population

Based on the method of introducing Allee effect in [4, 5], we consider system (2) with Allee effect for the prey population as follows:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x^2}{x+\xi}(1-x-y), \\ \dfrac{dy}{dt} = y\left(\dfrac{\alpha x}{1+\beta x} - \gamma\right), \end{cases} \tag{8}$$

where ξ is Allee effect coefficient with $\xi > 0$. The bigger ξ is, the stronger Allee effect of the prey population will be.

The equilibrium points of system (8) satisfy the following equations:

$$\begin{cases} \dfrac{x^2}{x+\xi}(1-x-y)=0, \\ y\left(\dfrac{\alpha x}{1+\beta x}-\gamma\right)=0. \end{cases} \tag{9}$$

Through simple calculation, we find that system (8) also has three equilibrium points: $P_1(0,0)$, $P_2(1,0)$ and $P_3\left(\frac{\gamma}{\alpha-\beta\gamma}, 1-\frac{\gamma}{\alpha-\beta\gamma}\right)$. For these points, we have the following results.

Theorem 3.1. *For system* (8)*, the following conclusions are obtained. The boundary equilibrium point* $P_1(0,0)$ *is a saddle node. The boundary equilibrium point* $P_2(1,0)$ *is a saddle. The positive equilibrium point* $P_3\left(\frac{\gamma}{\alpha-\beta\gamma}, 1-\frac{\gamma}{\alpha-\beta\gamma}\right)$ *is locally asymptotically stable.*

Proof. The Jacobian matrix of system (8) at any equilibrium point is

$$J(P)=\begin{pmatrix} H(x,y) & -\frac{x^2}{x+\xi} \\ \frac{\alpha y}{(1+\beta x)^2} & \frac{\alpha x}{1+\beta x}-\gamma \end{pmatrix}, \tag{10}$$

where $H(x,y)=\frac{\left(2x-3x^2-2xy\right)(x+\xi)-x^2(1-x-y)}{(x+\xi)^2}$.

The Jacobian matrix of system (8) at the boundary equilibrium point P_1 is $J(P_1)=\begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix}$, and two eigenvalues are $\lambda_1(P_1)=0$ and $\lambda_2(P_1)=-\gamma<0$. Obviously, the equilibrium point P_1 is non-hyperbolic, so it is difficult to directly judge the stability and type of the equilibrium point. Next, we use Theorem 7.1 in Chapter 2 in [7] to overcome this difficulty.

System (8) needs to be transformed into a standard form. Because the equilibrium point P_1 is the origin, we directly expand system (8) in series up to the fourth order around the origin as follows:

$$\begin{cases} \dfrac{dx}{dt}=\dfrac{1}{\xi}x^2-\dfrac{1+\xi}{\xi^2}x^3-\dfrac{1}{\xi}x^2y+\dfrac{1+\xi}{\xi^3}x^4+\dfrac{1}{\xi^2}x^3y+R_0(x,y), \\ \dfrac{dy}{dt}=-\gamma y+\alpha xy-\alpha\beta x^2y+\alpha\beta^2x^3y+R_1(x,y), \end{cases} \tag{11}$$

where $R_0(x,y)$ and $R_1(x,y)$ represent series with terms x^iy^j $(i+j\geq 5)$.

When a new time variable $\tau=-\gamma t$ is introduced, system (11) becomes

$$\begin{cases} \dfrac{dx}{d\tau}=-\dfrac{1}{\gamma\xi}x^2+\dfrac{1+\xi}{\gamma\xi^2}x^3+\dfrac{1}{\gamma\xi}x^2y-\dfrac{1+\xi}{\gamma\xi^3}x^4-\dfrac{1}{\gamma\xi^2}x^3y-\dfrac{1}{\gamma}R_0(x,y), \\ \dfrac{dy}{d\tau}=y-\dfrac{\alpha}{\gamma}xy+\dfrac{\alpha\beta}{\gamma}x^2y-\dfrac{\alpha\beta^2}{\gamma}x^3y-\dfrac{1}{\gamma}R_1(x,y)\triangleq y+Q(x,y). \end{cases} \tag{12}$$

Based on the implicit function theorem, it can be deduced from $\frac{dy}{d\tau}$=0 that there is a unique function $y = \phi(x) = 0$ satisfying $\phi(0) = \phi'(0) = 0$ and $\phi(x) + Q(x, \phi(x)) = 0$. Then substituting it into the first equation of system (12), we get

$$\frac{dx}{d\tau} = -\frac{1}{\gamma\xi}x^2 + \frac{1+\xi}{\gamma\xi^2}x^3 - \frac{1+\xi}{\gamma\xi^3}x^4 + R_2(x), \tag{13}$$

where $R_2(x)$ represents a series with terms x^i $(i \geq 5)$. We have the coefficient of x^2 is $-\frac{1}{\gamma\xi} < 0$ from equation (13). Note that $m = 2$ and $a_m < 0$, so, according to theorem 7.1 in [7], P_1 is a saddle node.

The Jacobian matrix of system (8) at the boundary equilibrium point P_2 is $J(P_2) = \begin{pmatrix} -\frac{1}{1+\xi} & -\frac{1}{1+\xi} \\ 0 & \frac{\alpha}{1+\beta} - \gamma \end{pmatrix}$, and two eigenvalues are $\lambda_1(P_2) = -\frac{1}{1+\xi} < 0$ and $\lambda_2(P_2) = \frac{\alpha}{1+\beta} - \gamma > 0$. Hence, the equilibrium point P_2 is a saddle.

The Jacobian matrix of system (8) at the positive equilibrium point P_3 is $J(P_3) = \begin{pmatrix} H(x_3, y_3) & -\frac{{x_3}^2}{x_3+\xi} \\ \frac{\alpha y_3}{(1+\beta x_3)^2} & 0 \end{pmatrix}$. Then the characteristic equation corresponding to Jacobian matrix $J(P_3)$ is

$$\lambda^2 - H(x_3, y_3)\lambda + \frac{\alpha {x_3}^2 y_3}{(x_3 + \xi)(1 + \beta x_3)^2} = 0. \tag{14}$$

The determinant and trace of Jacobian matrix $J(P_3)$ can be written as:

$$Det[J(P_3)] = \lambda_1(P_3) \cdot \lambda_2(P_3) = \frac{\alpha {x_3}^2 y_3}{(x_3 + \xi)(1 + \beta x_3)^2}, \tag{15}$$

and

$$Tr[J(P_3)] = \lambda_1(P_3) + \lambda_2(P_3) = H(x_3, y_3) = \frac{-{x_3}^2}{x_3 + \xi}. \tag{16}$$

We can judge $Det[J(P_3)] > 0$ and $Tr[J(P_3)] < 0$, so P_3 is locally asymptotically stable. The proof of Theorem 3.1 is finished.

4 Numerical simulations

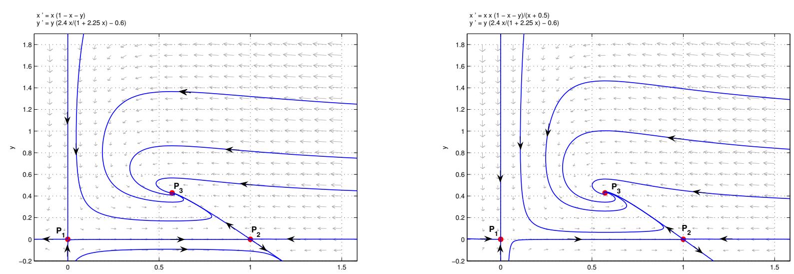
In this section, we show the numerical simulation results of the predator-prey model with and without Allee effect, to further illustrate the previous theoretical analysis. We take parameters: $\alpha = 2.4$, $\beta = 2.25$, $\gamma = 0.6$ and $\xi = 0.5$. Consider the following specific differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x - y), \\ \dfrac{dy}{dt} = y\left(\dfrac{2.4x}{1 + 2.25x} - 0.6\right), \end{cases} \tag{17}$$

and

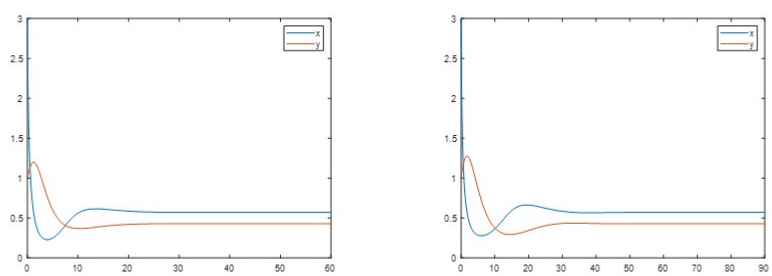
$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x^2}{x+0.5}\left(1-x-y\right), \\ \dfrac{dy}{dt} = y\left(\dfrac{2.4x}{1+2.25x} - 0.6\right). \end{cases} \tag{18}$$

Phase portraits of systems (17) and (18) are shown in Figure 1. Obviously, the boundary equilibrium point $P_1\,(0,0)$ changes from saddle to saddle node and the type of other equilibrium points remain unchanged under the influence of Allee effect. The trajectories of prey population and predator population are displayed in Figure 2. Observing the trend of curves in Figure 2, we see that the positive equilibrium point $P_3(0.57, 0.43)$ of system (18) is still asymptotically stable, but system (18) needs longer time to reach the stable state than system (17). Hence, we can obtain that Allee effect may be a destabilizing force in predator-prey systems.



(a) (b)

Figure 1: Phase portraits of systems (17) and (18).



(a) (b)

Figure 2: (a)-(b) The trajectories of prey and predator population of system (17) and (18) with the initial condition $(x(0), y(0)) = (3, 1)$, respectively.

5 Conclusion

The stability of the equilibrium points of a predator-prey model with and without Allee effect was investigated in this paper. Combined with numerical simulation, we found that Allee effect may affect the stability of the equilibrium points of predator-prey systems. When the prey population subject to Allee effect, the position of the equilibrium points doesn't change, but the stability of the equilibrium points changes. The changes of the stability may be directly reflected in the type of the equilibrium points, such as an equilibrium point could be changed from saddle to saddle node. Furthermore, even the equilibrium point is still stable, the time for the system to reach the stable state will be longer.

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