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Uncertain Portfolio Selection Model with Stock Index and Put Option

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Abstract

In this paper, we discuss a portfolio selection model with two stock indices and put options under the framework of uncertainty theory. We use VaR and CVaR to measure risk and treat the expiration price of stock indices as uncertain variables. Based on this, we propose a model with risk aversion coefficient, which is different from most portfolio selection models. The numerical experimental results show that different risk aversion coefficients lead to different investment ratios and optimum values. The paper concludes by verifying that the model including put options is superior to the model without options.

Mathematics Subject Classification: 91G10

Keywords: Portfolio selection; Uncertainty theory; Risk aversion coefficient; Stock index and put option

1 Introduction

The core of portfolio selection, as a hot issue of research in the field of finance, lies in the efficient allocation of assets and the rational distribution of assets among different financial products.

In 1952, Markowitz creatively proposed the mean-variance model [8], marking the birth of portfolio theory. Jorion introduced the concept of VaR [4]. Subsequently, to address the problem of measuring tail risk, CVaR was proposed [1], which measures the expected value of loss over VaR.

In previous studies, scholars have used probability theory to estimate future prices of securities. However, financial markets are complex and uncertain, for example, historical data is not effective in predicting the future when there are large changes in economic policies or other unexpected events. Therefore, probability theory loses its usefulness when historical data is invalid or missing. Based on this, Liu proposed uncertainty theory [6]. Experts in the relevant field make their own estimates and judgments about future prices based on their professional experience [7].

In the field of portfolio selection, Huang combined uncertainty theory and portfolio theory in 2010 and proposed uncertain portfolio selection theory [2]. Wang and Huang considered a single-stage portfolio model that includes risk-free assets, a stock index, and call options, treating the price of the stock index as an uncertain variable[3]. Zhu studied an optimal control problem and applied it to portfolio selection[9].

According to Khodamoradi et al. [5] in 2020, this paper distinguishes itself from previous portfolio selection models by using risk aversion coefficient to consider risk and return together in the objective function.

In this paper, we consider a model with two stock indices and put options. The purpose of choosing put option is that put option is a kind of protective option. When the price falls, an investor is able to sell the stock at the exercise price, and when the price rises, an investor does not execute, simply losing the option premium.

The rest of this paper is structured as follows. Section 2 briefly introduces the uncertainty theory. Section 3 presents a portfolio selection model with stock indices and put options. Section 4 is an empirical analysis that validates the results of the model. Finally, we conclude the paper in Section 5.

2 Preliminary Notes

Liu [6] proposed uncertainty theory which has been widely used in the field of finance. This section introduces the basic definitions of uncertainty theory and the theorems used in this paper. In uncertainty theory, Liu uses uncertain distribution to describe an uncertain variable.

Definition 2.1 *([6]) The uncertainty distribution $\Phi : \mathcal{R} \longrightarrow [0,1]$ of an uncertain variable ξ is define by*

$$\Phi(x) = \mathcal{M}\{\xi \leq X\}. \tag{1}$$

For example, a normal uncertain variable has the following uncertainty distribution

$$\Phi(x) = \left(1 + exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}. \tag{2}$$

Definition 2.2 *([6]) let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ, where $\alpha \in (0,1)$.*

For example, the inverse uncertain distribution of normal uncertain variable $\mathcal{N}(\mu, \sigma)$ is

$$\Phi^{-1}(\alpha) = \mu + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{3}$$

Theorem 2.3 *([6]) Let $\xi_1, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \cdots, \Phi_n$, respectively. If $f(\xi_1, \cdots, \xi_n)$ is continuous and strictly increasing with respect to $\xi_1, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \cdots, \xi_n$. Then $\xi = f(\xi_1, \cdots, \xi_n)$ has an expected value*

$$\mathrm{E}[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)\right) d\alpha. \tag{4}$$

3 Uncertain portfolio selection model with stock indices and put options

In this paper, we assume an investment in the 50ETF index and the 300ETF index, and the corresponding put options, respectively. We consider the price of indices as uncertain variables, denoted by S_1 and S_2. The exercise price of 50ETF index options is denoted by K and the exercise price of 300ETF index options is denoted by X. In this paper, n European put options on the 50ETF index and m European put options on the 300ETF index are invested separately. Other variable symbols to be used in this paper are described as follows.

x_i: proportion of investment in 50ETF index put option i, $i = 1, 2, \ldots n$;

w_j: proportion of investment in 300ETF index put option j, $j = 1, 2, \ldots m$;

y_1: proportion of investment in 50ETF index;

y_2: proportion of investment in 300ETF index;

S_1: the price of 50ETF index;

S_2: the price of 300ETF index;

K_i: exercise price of the i-th put option of 50ETF index, $i = 1, 2, \ldots n$, $K_1 \leq K_2 \leq \ldots \leq K_n$;

X_j: exercise price of the j-th put option of 300ETF index, $j = 1, 2, \dots m$, $X_1 \le X_2 \le \dots \le X_m$.

We assume that 50ETF index price S_1 and 300ETF index price S_2 are uncertain variables and independent of each other. The expected values of stock indices at maturity are μ_1 and μ_2.

The value function V of the portfolio may be expressed in the following form

$$V = y_1\mu_1 + \sum_{i=1}^{n} x_i(K_i - S_1)^+ + y_2\mu_2 + \sum_{j=1}^{m} w_j(X_j - S_2)^+. \tag{5}$$

Theorem 3.1 *Suppose 50ETF index S_1 has a regular uncertain distribution function $\Phi_1(x)$, the expected value of $y_1\mu_1 + \sum_{i=1}^{n} x_i(K_i - S_1)^+$ is*

$$E[V_1] = y_1\mu_1 + \sum_{i=1}^{n} x_i \int_{1-\Phi_1(K_i)}^{1} (K_i - \Phi_1^{-1}(1-\alpha))d\alpha. \tag{6}$$

Proof: When $0 < S_1 < K_1$, all options are executed; when $K_l < S_1 < K_{l+1}$, the $(l+1)$-th to n-th options are executed; and when $S_1 > K_n$, all options are not executed. By Theorem 2.3, we have

$$\begin{aligned} E[V_1] &= y_1\mu_1 + \int_0^1 \sum_{i=1}^{n} x_i(K_i - \Phi_1^{-1}(1-\alpha))^+ d\alpha \\ &= y_1\mu_1 + \int_{1-\Phi_1(K_1)}^{1} \sum_{i=1}^{n} x_i(K_i - \Phi_1^{-1}(1-\alpha))d\alpha \\ &\quad + \sum_{l=1}^{n-1} \int_{1-\Phi_1(K_{l+1})}^{1-\Phi_1(K_l)} \sum_{i=l+1}^{n} x_i(K_i - \Phi_1^{-1}(1-\alpha))d\alpha \\ &= y_1\mu_1 + \sum_{i=1}^{n} x_i \int_{1-\Phi_1(K_i)}^{1} (K_i - \Phi_1^{-1}(1-\alpha))d\alpha. \end{aligned} \tag{7}$$

Theorem 3.1 is proved.

Similarly, we can get the expected value of 300ETF index price $E[V_2]$ by

$$E[V_2] = y_2\mu_2 + \sum_{j=1}^{m} w_j \int_{1-\Phi_2(X_j)}^{1} (X_j - \Phi_2^{-1}(1-\alpha))d\alpha. \tag{8}$$

In this paper, we use VaR and CVaR to measure risk. VaR measures the maximum loss at a certain confidence level and CVaR is defined as the average loss in excess of VaR. Suppose the risk metrics for 50ETF index and 300ETF index are denoted as VaR_1, $CVaR_1$, VaR_2 and $CVaR_2$, respectively. 50ETF index price S_1 and 300ETF index price S_2 are independent of each other. Uncertain portfolio selection model can be expressed in the following form

$$\begin{cases} \max\ (1-\lambda)E(V) - \lambda \left(y_1 VaR_1 + \sum\limits_{i=1}^{n} x_i VaR_1 + y_2 VaR_2 + \sum\limits_{j=1}^{m} w_j VaR_2 \right) \\ \text{subject to} \\ \quad y_1 CVaR_1 + \sum\limits_{i=1}^{n} x_i CVaR_1 + y_2 CVaR_2 + \sum\limits_{j=1}^{m} w_j CVaR_2 \le \delta \\ \quad y_1 + \sum\limits_{i=1}^{n} x_i + y_2 + \sum\limits_{j=1}^{m} w_j = 1 \\ \quad L \le y_1, y_2, x_i, w_j \le U, \end{cases} \tag{9}$$

where λ is a risk aversion coefficient, L and U are constants. When $\lambda = 0$, the model represents to find the maximum value of the return function; when $\lambda = 1$, the model represents to find the minimum value of the risk function; when λ is between 0 and 1, it represents to find the equilibrium between risk and value.

Theorem 3.2 *Suppose the 50ETF index S_1 and 300ETF index S_2 are uncertain normal variables and independent of each other, $S_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $S_2 \sim \mathcal{N}(\mu_2, \sigma_2)$. S_1 has a regular uncertain distribution function $\Phi_1(x)$ and S_2 has a regular uncertain distribution function $\Phi_2(x)$. Uncertain portfolio selection model (9) can be converted into the following equivalent form*

$$\begin{cases} \max\ (1-\lambda) \left[y_1\mu_1 + y_2\mu_2 + \sum\limits_{i=1}^{n} x_i (K_i - \mu_1)\Phi_1(K_i) + \sum\limits_{j=1}^{m} w_j (X_j - \mu_2)\Phi_2(X_j) \right] \\ \quad -(1-\lambda) \sum\limits_{i=1}^{n} x_i \left[\frac{\sqrt{3}\sigma_1}{\pi}(1-\Phi_1(K_i))\ln(1-\Phi_1(K_i)) + \frac{\sqrt{3}\sigma_1}{\pi}\Phi_1(K_i)\ln\Phi_1(K_i) \right] \\ \quad -(1-\lambda) \sum\limits_{j=1}^{m} w_j \left[\frac{\sqrt{3}\sigma_2}{\pi}(1-\Phi_2(X_j))\ln(1-\Phi_2(X_j)) + \frac{\sqrt{3}\sigma_2}{\pi}\Phi_2(X_j)\ln\Phi_2(X_j) \right] \\ \quad -\lambda \left[(y_1 + \sum\limits_{i=1}^{n} x_i)(\mu_1 - \frac{\sqrt{3}\sigma_1}{\pi}\ln\frac{\alpha}{1-\alpha}) + (y_2 + \sum\limits_{j=1}^{m} w_j)(\mu_2 - \frac{\sqrt{3}\sigma_2}{\pi}\ln\frac{\alpha}{1-\alpha}) \right] \\ \textit{subject to} \\ \quad \frac{1}{\alpha}(y_1 + \sum\limits_{i=1}^{n} x_i) \left\{ \mu_1\alpha - \frac{\sqrt{3}\sigma_1}{\pi}\left[\alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)\right] \right\} \\ \qquad + \frac{1}{\alpha}(y_2 + \sum\limits_{j=1}^{m} w_j) \left\{ \mu_2\alpha - \frac{\sqrt{3}\sigma_2}{\pi}\left[\alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)\right] \right\} \le \delta, \\ \quad y_1 + \sum\limits_{i=1}^{n} x_i + y_2 + \sum\limits_{j=1}^{m} w_j = 1, \\ \quad L \le y_1, y_2, x_i, w_j \le U. \end{cases} \tag{10}$$

Proof: According to the definitions of VaR and CVaR, we can get

$$VaR(\alpha) = \Phi^{-1}(1-\alpha) = \mu - \frac{\sqrt{3}\sigma}{\pi}\ln\frac{\alpha}{1-\alpha} \tag{11}$$

and

$$\begin{aligned} CVaR(\alpha) &= \frac{1}{\alpha}\int_0^{\alpha}\Phi^{-1}(1-\beta)d\beta \\ &= \frac{1}{\alpha}\left\{\mu\alpha - \frac{\sqrt{3}\sigma}{\pi}\left[\alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)\right]\right\}. \end{aligned} \tag{12}$$

Substituting (11) into (7) and (8), we can get

$$\begin{aligned} E[V] &= y_1\mu_1 + \sum_{i=1}^{n} x_i \int_{1-\Phi_1(K_i)}^{1} [K_i - (\mu_1 - \frac{\sqrt{3}\sigma_1}{\pi}\ln\frac{\alpha}{1-\alpha})]d\alpha \\ &\quad + y_2\mu_2 + \sum_{j=1}^{m} w_j \int_{1-\Phi_2(X_j)}^{1} [X_j - (\mu_2 - \frac{\sqrt{3}\sigma_2}{\pi}\ln\frac{\alpha}{1-\alpha})]d\alpha. \end{aligned} \tag{13}$$

Since the two integrals in (13) are of the same form, we solve the above integrals using one of them as an example by

$$\begin{aligned} &\int_{1-\Phi_1(K_i)}^{1} [K_i - (\mu_1 - \frac{\sqrt{3}\sigma_1}{\pi}\ln\frac{\alpha}{1-\alpha})]d\alpha \\ = \; &\Phi_1(K_i)(K_i-\mu_1) + \frac{\sqrt{3}\sigma_1}{\pi}\int_{1-\Phi_1(K_i)}^{1}\ln\alpha d\alpha - \frac{\sqrt{3}\sigma_1}{\pi}\int_{1-\Phi_1(K_i)}^{1}\ln(1-\alpha)d\alpha \\ = \; &\Phi_1(K_i)(K_i-\mu_1) - \left[\frac{\sqrt{3}\sigma_1}{\pi}(1-\Phi_1(K_i))\ln(1-\Phi_1(K_i)) \right. \\ &\left. + \frac{\sqrt{3}\sigma_1}{\pi}\Phi_1(K_i)\ln\Phi_1(K_i)\right]. \end{aligned} \tag{14}$$

Theorem 3.2 is proved.

4 Numerial example

To address the programming problem, we use LINGO to solve the optimization problem. We select put options traded on the Shanghai Stock Exchange that are exercised on December 22, 2021. Six data sets are selected for each option in our portfolio. The exercise price data sets of the options are presented in Table 1.

Table 1: Data for 50ETF index and 300ETF index put options

50ETF option	Exercise price K_i	300ETF option	Exercise price X_j
Option K_1	3.4	Option X_1	5
Option K_2	3.5	Option X_2	5.25
Option K_3	3.6	Option X_3	5.5
Option K_4	3.7	Option X_4	5.75
Option K_5	3.8	Option X_5	6
Option K_6	3.9	Option X_6	6.25

We take S_1 and S_2 as uncertain variables, and we assume that they obey uncertain normal distribution, $S_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $S_2 \sim \mathcal{N}(\mu_2, \sigma_2)$. We may invite expert analysts to give their forecast values based on their experience and obtain the distributions by $S_1 \sim \mathcal{N}(3.582, 0.062)$ and $S_2 \sim \mathcal{N}(5.261, 0.096)$.

In our model, λ is a very important parameter and we want to analyze the effects of different values of λ on the model results. First, let $\delta = 5$, $L = 0$ and $U = 0.3$. To better match the real financial market, we set the investment ratios as integers in this paper.

Table 2: Proportion of each asset and the optimum value when $\lambda = 0$

50ETF	300ETF	K_i	X_1	X_2	X_3	X_4	X_5	X_6	Optimum value
0.3	0.3	0	0	0	0	0	0.1	0.3	2.862

Table 3: Proportion of each asset and the optimum value when $\lambda = 1$

50ETF	300ETF	K_1	K_2	K_3	K_4	K_5	K_6	X_j	Optimum value
0.3	0	0	0	0	0.1	0.3	0.3	0	-4.976

When $\lambda = 0$, we are seeking to maximize the value function. Based on the results in Table 2, we find that stock ETF indices and 300ETF options are invested. When $\lambda = 1$, we are seeking to minimize the risk function. Based on the results in Table 3, we find that 50ETF index and options are invested.

At last, we compare the effect of including options and not including options in the model on the results.

Table 4: Optimum values with and without put option

	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
Model with option	2.862	-1.448	-4.976
Model without option	2.161	-1.752	-4.996

From Tabel 4, we find that the model including options has greater gain and smaller loss, suggesting that introducing options into portfolio is meaningful.

5 Conclusion

In this paper, we discuss a problem of uncertain portfolio with stock indices and put options. We use VaR and CVaR to measure risk. In portfolio selection model, we innovatively use λ to connect return and risk and discuss the preference for return-risk by taking different values of λ. Finally, we find that the model including put options generates larger gains and smaller losses than the model without options.

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