

Meshless Method for the Numerical Solution of Coupled Burgers Equation

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Abstract

The development and interest of numerical techniques for obtaining approximate solutions of partial differential equations has increased very much in last decades. Among there are meshless methods. Recently radial base functions have been used in meshless methods applied to numerical solutions of partial differential equations, pioneers works being those of Kansa, Fasshauer, Wendland and Bohamid among others. In this paper, we employ the method, using two RBFs, TPS and MQ, to obtain numerical solution of coupled Burgers equation.

Mathematics Subject Classification: 35A40, 65D05, 65M12, 65M15

Keywords: Burgers equation, TPS, RBF, Meshless Method

1 Introduction

In this paper, we give a numerical solution for coupled and homogeneous Burgers equation with a meshless method and using radial base functions (RBF). The equation is defined in an open set Ω , with smooth boundary and connected in \mathbb{R}^2 , for a time interval $[t_0, T]$ y $t_0 \geq 0$:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) - \nu \Delta u(x, t) + (u(x, t) \cdot \nabla)u(x, t) = 0, & \text{if } x \in \Omega, t \in [t_0; T] \\ u(x, t) = g(x, t), & \text{if } x \in \partial\Omega, t \in [t_0; T] \\ u(x, 0) = u_0(x), & \text{if } x \in \Omega \end{cases} \quad (1)$$

where $u = (u(x, t), v(x, t)) : \bar{\Omega} \times [t_0, T] \rightarrow \mathbb{R}^2$, is a solution of (1), $g = (g_1(x, t), g_2(x, t)) : \partial\Omega \times [t_0, T] \rightarrow \mathbb{R}^2$ are given functions and $x = (x, y)$.

Burgers equation comes out in turbulence studies in fluid mechanics [14, 15], and presents nonlinear terms associated to convective phenomena $(u(x, t) \cdot \nabla)u(x, t)$ and the term $\Delta u(x, t)$ of diffusive phenomenon. In the study of turbulence, [18] discovered that small perturbations grow up in a big domain of smooth flow, joined by a vortex layer on which the strong turbulence is concentrated. Recently, all this became a matter of interest for researchers, due mainly to applications of Burgers model in statistical physics and fluid dynamics.

In order to decide whether the proposed approximation method is adequate, it is necessary to have an exact solution of the equation with initial and boundary conditions. In [19] some approximate solutions and exact solutions to Burgers equation, with the aid of Hopf-Cole transformation are given. In [2] 35 different analytic solutions with various initial conditions are proposed. [22] considered and applies a procedure to extend analytic solution to n -dimensional problems by using coset sets. In [16] there is given a method to find the exact solution of unhomogeneous Burgers equation using Hopf-Cole and Darboux transformations.

From the numerical viewpoint, [1] proposes a finite difference scheme totally implicit in that the nonlinear system is solved by Newton method. [10] uses finite differences of three points at two levels, fourth order in space and second order in time. Von-Neumann stability analysis showed that the method is unconditionally stable. [12] uses a standard Euler scheme with a constant discretization and standard scheme of finite differences upwind in spatial direction and using the process of quasi-linearity to come up non-linearity. It was proved that the solution sequence of the linear equations obtained after applying cuasi-linearization converges quadratically to the solution of the original nonlinear problem.

Afterwards, in [13] this same author proposed a numerical method based on Crank-Nicolson scheme. In [11], there is given the modified local method of Crank-Nicolson (MLCN) for one and 2-dimensional Burgers equations. MLCN is a scheme of finite explicit differences with a simple calculus and unconditionally stable. In [20], it is proposed an implicit exponential scheme of finite difference to solve he coupled Burgers equation with adequate initial and boundary conditions. From the numerical perspective of meshless methods, in [3] it was introduced a new method where Burgers n -dimensional equation is taken and its solution is approximated using thin plate splines (TPS) as well as RBF for the case of stable state. With this methodology the coupled PDE is put into a system of non-linear ODE's, which are solved using Newton method.

The advantage of approximating the solution using this interpolation is that

it becomes easier to simulate the behavior of the solution on parameterized and amorphous domains (such as Lipschitz) showing advantages as compared with other methods as FEM, FDM. Afterwards [4] used the same method already mentioned but including the time variable, giving rise to a complete development for approximation of Burgers n -dimensional in transition state by using TPS for a spatial discretizing problem and an implicit scheme of s -stages Runge-Kutta in time. This work is developed based on these two articles, implementing another RBF and using different polynomial bases. For more detailed revision of Burgers equation, see for example [23].

2 Meshless approximation method

In this section we develop a method to calculate the approximate solution of the coupled Burgers equation using RBFs of order m in \mathbb{R}^2 . Let the following two finite subsets be with n (interior points) and n' (boundary location points) in Ω and $\partial\Omega$ respectively, $\mathbf{A}_n = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \Omega$, $\mathbf{A}'_{n'} = \{\mathbf{x}'_1, \dots, \mathbf{x}'_{n'}\} \subset \partial\Omega$.

We assume that $\mathbf{A}'_{n'}$ contains a subset of points $\Pi_{m-1}(\mathbb{R}^2) - solvent$, where $\Pi_{m-1}(\mathbb{R}^2)$ is the space of polynomials of degree $(m-1)$ in \mathbb{R}^2 . Let $N = n + n'$ total number of points in $\bar{\Omega}$ and $\mathbf{A}_N = \mathbf{A}_n \cup \mathbf{A}'_{n'}$. Moreover, h is defined as the Hausdorff distance from \mathbf{A}_n to $\bar{\Omega}$ ($h = h_{\mathbf{A}_n, \bar{\Omega}}$), $h_{\mathbf{A}_n, \bar{\Omega}} = \sup_{\mathbf{x} \in \bar{\Omega}} \min_{\mathbf{x}_i \in \mathbf{A}_n} \|\mathbf{x} - \mathbf{x}_i\|_2$.

If Ω has a smooth boundary $\partial\Omega$ and contains N location points, then Burgers equation can be approximated by an interpolating procedure represented by a system of algebraic equations [4]. The idea is that such an approximation $u_h(\mathbf{x}, t) \in \mathbb{R}^2$ interpolates to each point in \mathbf{A}_n by using the function

$$u_h(\mathbf{x}) = \sum_{i=1}^n \alpha_i \phi_m(\|\mathbf{x} - \mathbf{x}_i\|) + \sum_{j=1}^{d_m} \beta_j q_j(\mathbf{x}), \quad (2)$$

subject to orthogonality conditions $\sum_{i=1}^n \alpha_i q_j(\mathbf{x}_i) = 0, j = 1, \dots, d_m$, where d_m is the dimension of space $\Pi_{m-1}(\mathbb{R}^2)$ [8, 14, 15, 17]. If the interpolation is defined with MQ as RBF and $m = 1$, it is not necessary to include the second term on the right side of equation (2) and the orthogonal conditions. The approximation (2) can be written as a system of equations $(n + d_m) \times (n + d_m)$ nonsingular with the form:

$$\begin{pmatrix} K & Q \\ Q^T & O \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix},$$

where α and β are coefficients (vectors) to be determined, z is the vector of solutions known at each interior point $z = (u(\mathbf{x}_1), \dots, u(\mathbf{x}_n))^T$ 0 is null matrix $d_m \times d_m$, $\Phi_m(\mathbf{x}) = \phi_m(\|\mathbf{x}\|)$ is the RBF K and Q given by:

$$K = [\Phi_m(x_i - x_j)]_{1 \leq i, j \leq n}, \quad Q = [q_i(x_j)]_{\substack{1 \leq j \leq n \\ 1 \leq i \leq d_m}}.$$

Let $u = (u, v) : \bar{\Omega} \times [t_0, T] \rightarrow \mathbb{R}^2$ be exact solution of (1) and X , unknown matrix function $n \times 2$

$$X(t) = \begin{bmatrix} u(x_1, t) & v(x_1, t) \\ \vdots & \vdots \\ u(x_n, t) & v(x_n, t) \end{bmatrix}, \quad \text{for } t \in [t_0, T]. \quad (3)$$

According to [8], for any $t \in [t_0, T]$, there is an unique approximation $u_h(x, t)$ interpolating $u_h(x, t)$ on the point set A_N . After some simple calculi and in accord to Burgers equation structure, we obtain a matrix-valued function $F_m : \mathbb{R}^2 \times \mathbb{R}^{n \times 2} \times [t_0, T] \rightarrow \mathbb{R}^2$:

$$F_m(x, X(t), t) = \nu \left([\Delta a_m(x)]^T X(t) + [\Delta b_m(x)]^T \tilde{G}(t) \right) - \left(a_m(x)^T X(t) + b_m(x)^T \tilde{G}(t) \right) \left([\nabla a_m(x)]^T X(t) + [\nabla b_m(x)]^T \tilde{G}(t) \right),$$

and a function $\mathcal{F}_m : \mathbb{R}^{n \times 2} \times [t_0, T] \rightarrow \mathbb{R}^{n \times 2}$ given by

$$\mathcal{F}_m(X(t), t) = \begin{bmatrix} F_m(x_1, X(t), t) \\ \vdots \\ F_m(x_n, X(t), t) \end{bmatrix}.$$

For more details in this development and definitions of coefficients $a_m(x)$, $b_m(x)$ y $\tilde{G}(t)$, see for example [4]. The expression u_h defined in (2) satisfied the approximation scheme:

$$\begin{cases} \frac{\partial u_h}{\partial t}(x, t) = \nu \Delta u_h(x, t) - (u_h(x) \cdot \nabla) u_h(x, t), & \forall (x, t) \in A_n \times [t_0, T] \\ u_h(x, t) = g(x, t), & \forall (x, t) \in A'_{n'} \times [t_0, T]. \end{cases} \quad (4)$$

if and only if $X(t)$ defined by (3) satisfies the matrix differential system [4]

$$\begin{cases} X'(t) = \mathcal{F}_m(X(t), t), & \text{for } t \in [t_0, T] \\ X(t_0) = X_0, \end{cases} \quad (5)$$

where $X_0 = [u_0(x_1)^t, \dots, u_0(x_n)^t]$. In this way, we transform partial differential equation into a system of nonlinear ordinary differential equations to be solved by any known method. In this case, we used the explicit Runge-Kutta method of four stages [5].

3 Numerical experiments

In this section, we give numerical results for the implementation developed just before. We analyzed two types of RBF, TPS and MQ. For TPS, we used

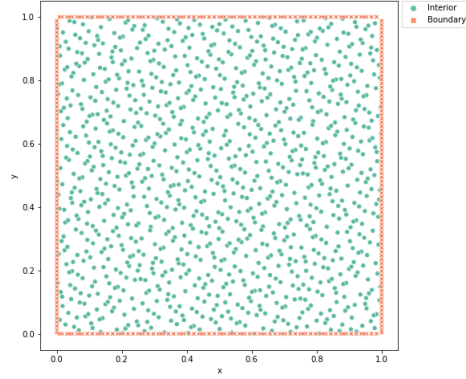


Figure 1: Square domain of $[0, 1] \times [0, 1]$, with 1089 interior points and 324 boundary points.

Hermite polynomials as a basis for the space $\Pi_{m-1}(\mathbb{R}^2)$. We took order of RBF $m = 2$ for TPS and $m = 1$ for MQ, hence $d_m = 3$ [8].

The implementating was developed in Python 3.9 and the square domain just shown in Figure 1. To compare the approximate solution u_h with the analytic one, we used the relative error given by norms 2 and infinity:

$$E_r = \frac{\sum_{x \in \Omega} \|u(x, t) - u_h(x, t)\|_p}{\sum_{x \in \Omega} \|u(x, t)\|_p}, \quad \text{for } p = 2, \infty. \quad (6)$$

Analytic solution of (1) is derived directly from the proposed solution in [9].

3.1 Thin plate splines (TPS)

In this case $\Phi_m(x)$ is defined as [8, 14, 15, 17]

$$\Phi_2(x) = \|x\|^2 \log \|x\|. \quad (7)$$

Therefore, the Burgers equation using TPS and polynomial basis with dimension $d_m = 3$ is given by

$$u_h(x) = \sum_{i=1}^n \alpha_i \|x - x_i\|^2 \log (\|x - x_i\|) + \beta_1 q_1(x) + \beta_2 q_2(x) + \beta_3 q_3(x), \quad (8)$$

subject to orthogonality conditions

$$\sum_{i=1}^n \alpha_i q_1(x_i) = 0, \quad \sum_{i=1}^n \alpha_i q_2(x_i) = 0, \quad \sum_{i=1}^n \alpha_i q_3(x_i) = 0. \quad (9)$$

$\{q_1, q_2, q_3\}$ is a basis for the space $\Pi_1(\mathbb{R}^2)$. The 2D Hermite polynomials take the form [24]

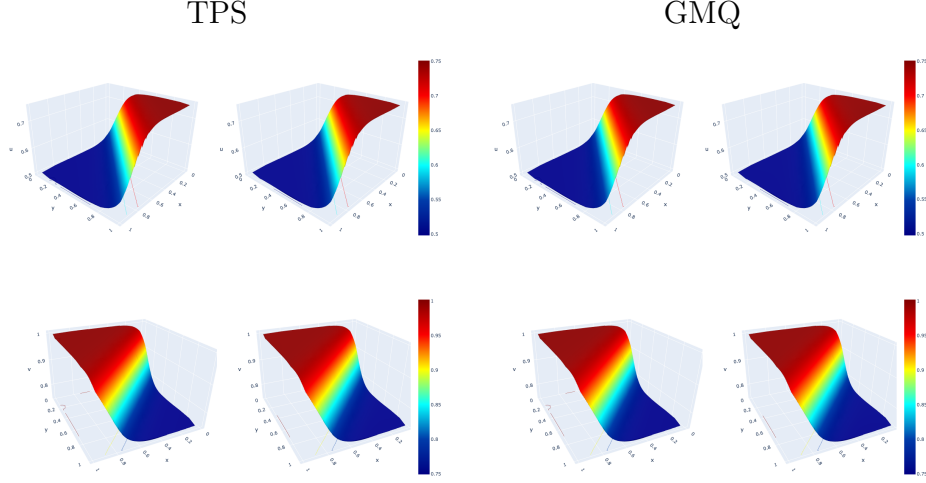


Figure 2: Analytic solution (left) and approximate (right) for $t = 0.5$, with 500 interior points and $\text{Re} = 100$. u -component (up), v -component (down).

n	L_2			L_∞		
	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$
25	2.2321×10^{-3}	8.4215×10^{-2}	9.7583×10^{-1}	3.0882×10^{-2}	1.4845×10^{-1}	1.2948
80	1.0856×10^{-3}	9.1466×10^{-3}	4.5582×10^{-1}	6.9662×10^{-3}	1.7405×10^{-2}	8.1577×10^{-1}
500	5.8435×10^{-4}	7.7409×10^{-4}	1.7702×10^{-1}	1.0587×10^{-3}	1.5403×10^{-3}	4.6921×10^{-1}

Table 1: Relative error for $t = 0.5$ of component u using Hermite polynomials.

$$H_{m,n}(x, y) = \left(2x - \frac{\partial}{\partial x}\right)^m \left(2y - \frac{\partial}{\partial y}\right)^n, \quad (10)$$

therefore,

$$q_1(x) = 1, \quad q_2(x) = 2x, \quad q_3(x) = 2y. \quad (11)$$

Results are shown in tables 1, 2 and Figure 2.

n	L_2			L_∞		
	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$
25	1.2108×10^{-3}	5.5166×10^{-2}	8.9118×10^{-1}	2.0833×10^{-3}	1.1274×10^{-1}	1.0095
80	9.1856×10^{-4}	6.0869×10^{-3}	3.9302×10^{-1}	1.5661×10^{-3}	1.4004×10^{-2}	7.1451×10^{-1}
500	3.9360×10^{-4}	5.3524×10^{-4}	1.2171×10^{-1}	8.9174×10^{-4}	1.3292×10^{-3}	3.3440×10^{-1}

Table 2: Relative error for $t = 0.5$ of component v using Hermite polynomials.

n	L_2			L_∞		
	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$
25	4.6101×10^{-3}	1.8761×10^{-1}	9.9101×10^{-1}	6.3053×10^{-2}	4.0744×10^{-1}	1.3662
80	2.1571×10^{-3}	9.1466×10^{-2}	5.3042×10^{-1}	1.2250×10^{-2}	1.7405×10^{-1}	9.2023×10^{-1}
500	6.7752×10^{-4}	1.1519×10^{-3}	1.8463×10^{-1}	1.2274×10^{-3}	2.2920×10^{-3}	4.8938×10^{-1}

Table 3: Relative error for $t = 0.5$ of component u using MQ.

n	L_2			L_∞		
	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.02$	$\nu = 0.01$	$\nu = 0.001$
25	1.8122×10^{-3}	1.2148×10^{-1}	1.0230	4.0401×10^{-2}	3.0617×10^{-1}	1.1873
80	1.0091×10^{-3}	6.0869×10^{-2}	4.9811×10^{-1}	9.1033×10^{-3}	1.4004×10^{-1}	8.6100×10^{-1}
500	4.5635×10^{-4}	7.9644×10^{-4}	1.2694×10^{-1}	1.0339×10^{-3}	1.9778×10^{-3}	3.4878×10^{-1}

Table 4: Relative error for $t = 0.5$ of component v using MQ.

3.2 Generalized Multiquadratic (GMQ)

Taking $\beta = 0.5$, functions GMQ are defined by

$$\Phi_1(\mathbf{x}) = (1 + \|\mathbf{x}\|^2)^{1/2}, \quad \mathbf{x} = (x, y). \quad (12)$$

Therefore, the approximation using MQ is:

$$u_h(\mathbf{x}) = \sum_{i=1}^n \alpha_i (1 + \|\mathbf{x} - \mathbf{x}_i\|^2)^{1/2}. \quad (13)$$

Results are shown in tables 3, 4 and Figure 2.

Experiments shown that the best approximation was obtained using TPS as RBF and Hermite polynomials. Also we used Laguerre polynomials, but the results are not given in this paper. The results are finally shown in Figure 3. Big Reynolds (> 1000) shown less variation among all other methods, and the approximation error increases when value Re does. Whereas, taking Reynolds number in laminar flow (50, 100) is plain that Hermite polynomials show better approximation. We conclude that Reynolds number has direct effect on the exactness of u_h and it is an open research field, as mentioned in [4], where numerical experiments correspond to small Reynolds numbers (less than 50).

Acknowledgements. We wish to thank the Mathematics department and its MSc program at Universidad EAFIT, professor Cristhian Montoya for his comments and suggestions leading to improvements of this paper.

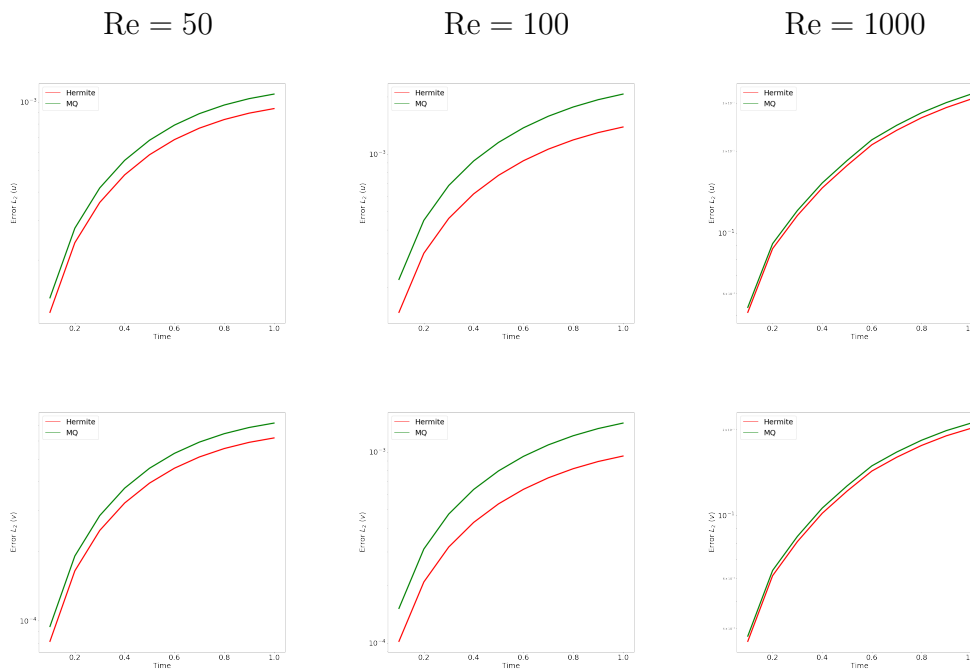


Figure 3: Comparative of L_2 error for 500 points and various Reynolds numbers for u (up) and v (down).

References

- [1] A.R. Bahadir, A fully implicit finite-difference scheme for two-dimensional Burgers' equations, *Applied Mathematics and Computation*, **137** (2003), 131-137. [https://doi.org/10.1016/s0096-3003\(02\)00091-7](https://doi.org/10.1016/s0096-3003(02)00091-7)
- [2] E.R. Benton, G.W. Platzman, A table of solutions of the one-dimensional Burgers equation, *Quarterly of Applied Mathematics*, **30** (1972), 195-212. <https://doi.org/10.1090/qam/306736>
- [3] A. Bouhamidi, M. Hached, K. Jbilou, A meshless method for the numerical computation of the solution of steady Burgers-type equations, *Applied Numerical Mathematics*, **74** (2013), 95-110. <https://doi.org/10.1016/j.apnum.2013.07.004>
- [4] A. Bouhamidi, M. Hached, K. Jbilou, A meshless RBF method for computing a numerical solution of unsteady Burgers'-type equations, *Computers & Mathematics with Applications*, **68** (2014), 238-256. <https://doi.org/10.1016/j.camwa.2014.05.022>
- [5] J.C. Butcher, *Numerical methods for ordinary differential equations*, John Wiley & Sons, 2016. <https://doi.org/10.1002/0470868279>

- [6] J. Caldwell, P. Wanless, A.E. Cook, A finite element approach to Burgers' equation, *Applied Mathematical Modelling*, **5** (1981), 189-193.
[https://doi.org/10.1016/0307-904x\(81\)90043-3](https://doi.org/10.1016/0307-904x(81)90043-3)
- [7] L. Debnath, *Nonlinear Partial Differential Equations: for Scientists and Engineers*, Second Edition, Birkhäuser, Boston, 2005.
<https://doi.org/10.1007/b138648>
- [8] G.E. Fasshauer, *Meshfree approximation methods with MATLAB*, World Scientific, 2007. <https://doi.org/10.1142/6437>
- [9] C.A Fletcher, Generating exact solutions of the two-dimensional Burgers' equations, *International Journal for Numerical Methods in Fluids*, **3** (1983), 213-216. <https://doi.org/10.1002/flid.1650030302>
- [10] I.A. Hassanien, A.A. Salama, H.A. Hosham, Fourth-order finite difference method for solving Burgers' equation, *Applied Mathematics and Computation*, **170** (2005), 781-800. <https://doi.org/10.1016/j.amc.2004.12.052>
- [11] P. Huang, A. Abduwali, The modified local Crank–Nicolson method for one-and two-dimensional Burgers' equations, *Computers & Mathematics with Applications*, **59** (2010), 2452-2463.
<https://doi.org/10.1016/j.camwa.2009.08.069>
- [12] M.K. Kadalbajoo, K.K Sharma, A. Awasthi, A parameter-uniform implicit difference scheme for solving time-dependent Burgers' equations, *Applied Mathematics and Computation*, **170** (2005), 1365-1393.
<https://doi.org/10.1016/j.amc.2005.01.032>
- [13] M.K. Kadalbajoo, A. Awasthi, A numerical method based on Crank–Nicolson scheme for Burgers' equation, *Applied Mathematics and Computation*, **182** (2006), 1430-1442. <https://doi.org/10.1016/j.amc.2006.05.030>
- [14] E. J. Kansa, Multiquadrics—a scattered data approximation scheme with applications to computational fluid dynamics— I. Surface approximations and partial derivative estimates, *Computers Math. Applic.*, **19** (8/9) (1990) 127-145. [https://doi.org/10.1016/0898-1221\(90\)90270-t](https://doi.org/10.1016/0898-1221(90)90270-t)
- [15] E.J. Kansa, Multiquatric - A scattered data approximation scheme with applications to computational fluid dynamics II, *Comp. Math. Appl.*, **19** (8/9) (1990) 147-161. [https://doi.org/10.1016/0898-1221\(90\)90271-k](https://doi.org/10.1016/0898-1221(90)90271-k)
- [16] A.G. Kudryavtsev, O.A. Sapozhnikov, Determination of the exact solutions to the inhomogeneous Burgers equation with the use of the darboux transformation, *Acoustical Physics*, **57** (2011), 311-319.
<https://doi.org/10.1134/s1063771011030080>

- [17] J. Li, Y.T. Chen, *Computational partial differential equations using MATLAB*, CRC press, 2019. <https://doi.org/10.1201/9780429266027>
- [18] J.D. Murray, On Burgers' model equations for turbulence, *Journal of Fluid Mechanics*, **59** (1973), 263-279.
<https://doi.org/10.1017/s0022112073001564>
- [19] E.Y. Rodin, On some approximate and exact solutions of boundary value problems for Burgers' equation, *Journal of Mathematical Analysis and Applications*, **30** (1970), 401-414. [https://doi.org/10.1016/0022-247x\(70\)90171-x](https://doi.org/10.1016/0022-247x(70)90171-x)
- [20] V.K. Srivastava, M.K. Awasthi, S. Singh, An implicit logarithmic finite-difference technique for two dimensional coupled viscous Burgers' equation, *AIP Advances*, **3** (2013), 122105. <https://doi.org/10.1063/1.4842595>
- [21] E. Varoğlu, W.D. Liam Finn, Space-time finite elements incorporating characteristics for the Burgers' equation, *International Journal for Numerical Methods in Engineering*, **16** (1980), 171-184.
<https://doi.org/10.1002/nme.1620160112>
- [22] K.B. Wolf, L. Hlavatý, S. Steinberg, Nonlinear differential equations as invariants under group action on coset bundles: Burgers and Korteweg-de Vries equation families, *Journal of Mathematical Analysis and Applications*, **2** (1986), 340-359. [https://doi.org/10.1016/0022-247x\(86\)90088-0](https://doi.org/10.1016/0022-247x(86)90088-0)
- [23] W.A. Woyczynski, *Burgers-KPZ turbulence*, Göttingen lectures, Springer, 2006.
- [24] A. Wünsche, Hermite and Laguerre 2D polynomials, *Journal of Computational and Applied Mathematics*, **133** (2001), 665-678.
[https://doi.org/10.1016/s0377-0427\(00\)00681-6](https://doi.org/10.1016/s0377-0427(00)00681-6)

Received: May 21, 2022; Published: June 12, 2022