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Bifurcation Analysis of Ion-Acoustic Solitary and Periodic Waves of Superthermal Electron-Positron with a Strongly Relativistic Plasma

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Abstract

This manuscript introduces a new perspective for nonlinear ion-acoustic waves (IAWs) in an unmagnetized collisionless plasma containing superthermal electrons and positrons (ep) distribution, and strongly relativistic ions. The Korteweg-de Vries (K-dV) equation is derived by the reductive perturbations technique (RPT). By bifurcation analysis of the planar dynamical system, we predicted the types of solutions of the K-dV equation with the assistance of phase portraits. We proved the existence of IA blow-up solitary, soliton, and periodic wave solutions. These three types of exact solutions are derived by using the $\left(\frac{G'}{G}\right)$ expansion method and illustrated in 2-dimensional and 3-dimensional graphics. A novel form of the blow-up solitary wave is obtained with the effects of a strongly relativistic factor of ion, superthermality of ep particles $\kappa_{e,p}$, the ratio of electron to positron temperature, the ratio of ion to electron temperature, and the density of positron are illustrated. The derived results could be useful for understanding and explanation of numerous applications in the interstellar medium and pulsar relativistic wind which contain highly energies superthermality ep.

Key words: Bifurcation analysis , solitary and periodic waves, Korteweg-de Vries equation; strongly relativistic ions, superthermal electrons and positrons .

1 Introduction:

Nonlinear evolution equations (NLEEs) are considered as the most important branch for many phenomena in astrophysical plasmas. The attention has been attracted to study the propagation of nonlinear wave in multi components plasma electron, positron and ion (epi). In the last few decades, the epi plasma is appeared in many interplanetary spaces and astrophysical plasma where the nonlinear waves, like solitary, soliton, shock are illustrated. In the interstellar media when atoms collide with cosmic-ray nuclei, positrons are generated (Moskalenko and Strong [22]). The electrons and positrons have opposite charges but the same mass and they have high-energy charged, so the Kappa distribution will be the most appropriate to ep plasmas. Recently many researchers studied ep plasma in many applications (Gaimin et al. [11], Boubakour et al. [3], Pakzad [24], Das [5], El-Awady et al.[6], El-Tantawy and Moslem [10], Mugemana et al. [23]). The superthermal or called Kappa (κ) distribution was introduced for the first time as acceptable to particle data entrenched with the OGO satellite by Vasyliunas (Vastliunas [39]). The particles obey to kappa distribution due to it has high-energy tails straying away from a Maxwellian distribution. The Kappa distribution type gives a powerful structure for investigation of real data in auroral zone, solar wind, interstellar plasma, etc. The index κ is measuring the aberration from Maxwellian equilibrium, where $\kappa \to \infty$ the particles reach to Maxwellian distribution and in solar wind with coronal electrons will be in range $2 < \kappa < 6$ (Krimigls et al. [17], Maksimovic et al. [19], Pierrard and Lazar [25]). Many of studies with superthermal ep plasma are performed in nonrelativistic plasmas, e.g. (El-Tantawy and Moslem [9]), (Shahmansouri and Astaraki [37]) as well as (Mehdipoor [20]). But we know that when the velocity of ion accessed to the speed of light, so the relativistic effect plays a pivotal role in the behavior of IA waves. Weak relativistic plasmas are studied in many researches, (Gill et al. [12]) used the RPT to derive K-dV equation of three components plasma. They observed that the relativistic factor and other parameters consequently effect on the behavior of soliton wave, also they conclude that only compressive solitons are attained. (Saeed et al. [27]) investigated K-dV equation of plasma containing of Boltzmann electrons, positrons and weak relativistic ion, they observed that the soliton declines as relativistic factor increases. (Hafez et al. [13]) described the influence of plasma parameters with nonextensive electrons, Boltzmann positrons and weakly relativistic ions on the behavior of soliton waves. The bifurcation analysis of dynamical system was applied recently of

IAWs with kappa distributed components in each of non-perturbations plasma (Mondal et al. [21], Hafez et al. [15]) and perturbation plasma (Saha and Chatterjee [29], Saha and Chatterjee [30], El-Monier and Atteya [7]). Saha and Chatterjee [31] derived K-dV equation by RPT and studied the bifurcation analysis of electron acoustic waves. They proved the existence of two types of solutions, blow up solitary and periodic waves of plasma containing cold electron, hot superthermal electron and nonrelativistic ions. Undoubtedly, from previous researches the bifurcation theory of dynamical systems has tremendously important in study both solitary and periodic waves of many phenomena in plasma physics. To obtain the exact solutions of various NLEEs many methods were undertaken, such as the tanh-sech method, exp-function method, sine-cosine method and extended tanh method. Recently, some new methods was formulated among of them is (G'/G)-expansion method. Wang et al. [40] introduced this method to get the travelling waves solutions of NLEEs and then the attention has attracted by many researchers to use it in different NLEEs (Shahein and Seadawy [34], Shahein and El-Shehri [33], Shahein and Abdo [32]). Hafez et al. [14] derived Burger equation and investigated the IA shock waves in weakly and strongly relativistic ions with nonextensivity electrons and positrons. They proved existence of shock and periodic waves by using (G'/G)-expansion method.
Up to the best of our knowledge, there is no studies of bifurcation analysis with highly relativistic ions of plasma containing superthermal ep by using (G'/G)-expansion method. Therefore, we looking forward to this paper offers a rich source for understanding of IAWs that may develop in the interstellar medium where we will introduced a new point of view of IAWs with strongly relativistic ions. The arrangement of article as: In sec. (2) basic equations and physical situation, the derivation of K-dV equation in sec.(3), Bifurcation analysis of K-dV equation in sec. (4), in sec. (5) the exact solutions of K-dV equation with (G'/G)-expansion method, in sec. (6) results and discussion and finally conclusion in sec.(7) are given.

2 Basic equations and physical situation:

In this work, we consider a multicomponent unmagnetized collisionless plasma containing two Kappa distributed electrons and positrons in addition to relativistic ions. In equilibrium, the three components obey to quasi-neutral condition in the form $n_{e0} - n_{p0} - n_{i0} = 0$, where n_{e0}, n_{p0} and n_{i0} are the unperturbed number density of electrons, positrons and ions. Also, the phase velocity of nonlinear IAW is much less than the thermal velocities of electrons and positrons but larger than the thermal velocity of ions, so we can avoid the electrons and positrons inertia. The nonlinear dynamics of nonviscous IAW equations are governed by the normalized system of equations (Javidan and

Saadatmand [16], Hafez et al. [14]).
The continuity equation:

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i u_i)}{\partial x} = 0. \tag{1}$$

The momentum equation:

$$\frac{\partial (\mu u_i)}{\partial t} + u_i \frac{\partial (\mu u_i)}{\partial x} = -\frac{\partial \phi}{\partial x} - \frac{\sigma_i}{n_i}\frac{\partial p_i}{\partial x}. \tag{2}$$

The energy equation:

$$\frac{\partial p_i}{\partial t} + u_i \frac{\partial p_i}{\partial x} + 3p_i \frac{\partial (\mu u_i)}{\partial x} = 0. \tag{3}$$

Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} = n_e - n_p - n_i. \tag{4}$$

where μ is relativistic factor for strongly relativistic plasma reads as $\mu = (1 - \frac{u_i}{C^2})^2 \simeq 1 + \frac{u_i^2}{2C^2} + \frac{3u_i^4}{8C^4}$ (Hafez et al. [14]). The superthermal electrons and positrons distribution are (Shahmansouri and Astaraki [37], Mehdipoor [20], Hafez et al. [14]).

$$\begin{aligned} n_e &= \frac{1}{1-\rho}\left(1 - \frac{\phi}{\kappa_e - \frac{3}{2}}\right)^{-\kappa_e + \frac{1}{2}} \\ &\cong \frac{1}{1-\rho}(1 + c_1\phi + c_2\phi^2 + c_3\phi^3 + \dots), \end{aligned} \tag{5}$$

and

$$\begin{aligned} n_p &= \frac{\rho}{1-\rho}\left(1 + \frac{\sigma_p\, \phi}{\kappa_p - \frac{3}{2}}\right)^{-\kappa_p + \frac{1}{2}} \\ &\cong \frac{\rho}{1-\rho}(1 - b_1\, \sigma_p\, \phi + b_2\, \sigma_p^2\, \phi^2 - b_3\, \sigma_p^3\, \phi^3 + \dots), \end{aligned} \tag{6}$$

where $c_1 = \frac{2\kappa_e - 1}{2\kappa_e - 3}$, $c_2 = \frac{(2\kappa_e-1)(2\kappa_e+1)}{2(2\kappa_e-3)^2}$, $c_3 = \frac{(2\kappa_e-1)(2\kappa_e+1)(2\kappa_e+3)}{6(2\kappa_e-3)^3}$, $b_1 = \frac{2\kappa_p - 1}{2\kappa_p - 3}$, $b_2 = \frac{(2\kappa_p-1)(2\kappa_p+1)}{2(2\kappa_p-3)^2}$, $b_3 = \frac{(2\kappa_p-1)(2\kappa_p+1)(2\kappa_p+3)}{6(2\kappa_p-3)^3}$.
In Eqs.(1)-(4), $n_{(\alpha)}, u_i, \phi, p_i, t, x$ denote the densities ($\alpha = e, p, i$), ion velocity, electrostatic potential, ion pressure, the time and the space coordinate, respectively. These dimensionless quantities are normalized as follows: the plasma species densities by the unperturbed ion density n_{i0}, u_i by the ion acoustic speed $C_s = \sqrt{\frac{T_e}{M_i}}$ where M_i is the ion mass. Space variable x and time t by electron Debye length and the inverse of the plasma frequency $\lambda_D = \sqrt{\frac{T_e}{4\pi n_{i0} e^2}}$, $\omega_{pi} = \sqrt{\frac{4\pi n_{i0} e^2}{M_i}}$. The electrostatic potential ϕ by the thermal potential $\frac{T_e}{e}$, ion pressure p by equilibrium pressure $p_0 = n_{i0} T_i$. The parameters $\sigma_i = \frac{T_i}{T_e}$, $\rho = \frac{n_{p0}}{n_{e0}}$ and $\sigma_p = \frac{T_e}{T_p}$ are obtained due to the nondimensional process.

3 The derivation of K-dV equation:

We apply the reductive perturbation technique to derive the K-dV equation. According to this technique the independent variables x, t are stretched as (Mehdipoor [20].

$$\xi = \epsilon^{1/2}(x - v_{ps}t), \qquad \tau = \epsilon^{3/2}t, \tag{7}$$

where ϵ is a real parameter and satisfies $\epsilon \ll 1$, v_{ps} is the phase speed of the IA to be derived later. We expanded the depend variables as

$$\begin{pmatrix} n_i(x,t) \\ u_i(x,t) \\ p_i(x,t) \\ \phi(x,t) \end{pmatrix} = \begin{bmatrix} 1 \\ u_0 \\ 1 \\ 0 \end{bmatrix} + \sum_{q=0}^{\infty} \epsilon^{q+1} \begin{bmatrix} n_{q+1}(\xi,\tau) \\ u_{q+1}(\xi,\tau) \\ p_{q+1}(\xi,\tau) \\ \phi_{q+1}(\xi,\tau) \end{bmatrix}, \tag{8}$$

Substituting from Eqs.(5-8) into Eqs.(1-4) and collecting the terms with similar power of ϵ, the first order is
$n_1 = A\phi_1, \; u_1 = (v_{ps} - u_0)A\phi_1, \; p_1 = 3A\,\delta_1\phi_1,$
with phase velocity,

$$v_{ps} = u_0 \pm \sqrt{\frac{3\sigma_i\,A\,\delta_1 + 1}{A\,\delta_1}}. \tag{9}$$

For the next order of ϵ, we get

$$\begin{aligned} (u_0 - v_{ps})\frac{\partial n_2}{\partial \xi} + \frac{u_2}{\partial \xi} &= -\frac{\partial n_1}{\partial \tau} - \frac{\partial (n_1 u_1)}{\partial \xi}, \\ (u_0 - v_{ps})\delta_1\frac{\partial u_2}{\partial \xi} + \sigma_i\frac{\partial p_2}{\partial \xi} + \frac{\partial \phi_2}{\partial \xi} &= -\delta_1\frac{\partial u_1}{\partial \tau} + (v_{ps} - u_0)\delta_1\,n_1\frac{\partial u_1}{\partial \xi} \\ &\quad -(1 + \frac{9}{2}U_{ri}^2 + \frac{75}{8}U_{ri}^4 - v_{ps}\,\delta_2)u_1\frac{\partial u_1}{\partial \xi} - n_1\frac{\partial \phi_1}{\partial \xi}, \\ (u_0 - v_{ps})\frac{\partial p_2}{\partial \xi} + 3\delta_1\frac{\partial u_2}{\partial \xi} &= -\frac{\partial p_1}{\partial \tau} - 3\delta_1\,p_1\frac{\partial u_1}{\partial \xi} - u_1\frac{\partial p_1}{\partial \xi} - 3u_1\delta_2\frac{\partial u_1}{\partial \xi}, \\ n_2 - A\,\phi_2 &= B\,\phi_1^2 - \frac{\partial^2 \phi_1}{\partial \xi^2}. \end{aligned} \tag{10}$$

Where $A = \frac{c_1 + b_1\,\rho\,\sigma_p}{1-\rho}$, $B = \frac{c_2 - b_2\,\rho\,\sigma_p^2}{1-\rho}$, $\delta_1 = 1 + \frac{3}{2}U_{ri}^2 + \frac{15}{8}U_{ri}^4$ (with assuming $U_{ri} = \frac{u_0}{C}$), $\delta_2 = \frac{3U_{ri}}{C} + \frac{15U_{ri}^3}{2C}$.
Finally from Eqs. (9) and (10) we derived the nonlinear evolution K-dV equation as

$$\frac{\partial \phi_1}{\partial \tau} + q_1\,\phi_1\frac{\partial \phi_1}{\partial \xi} + q_2\,\frac{\partial^3 \phi_1}{\partial \xi^3} = 0. \tag{11}$$

In the above equation, q_1 and q_2 are the coefficients of nonlinear and dispersion terms. Which are given by

$$q_1 = \frac{[A^2(\delta_2(v_{ps}-u_0)-\delta_1)-3A^3\delta_1^2\sigma_i(1-5\delta_1)-2B\delta_1]}{2A^2\delta_1^2(v_{ps}-u_0)},$$

$$q_2 = \frac{1}{2A^2\delta_1(v_{ps}-u_0)}. \tag{12}$$

Here we will introduce a new viewpoint about the bifurcation discussion of dynamical system of K-dV equation in highly relativistic ions, therefore we can determine the kinds and stability of solutions closed the equilibrium points of these physical situations before solve it.

4 Bifurcation analysis of K-dV equation:

In this section, we transform the K-dV equation to autonomous dynamical system by replacing the independent variables ξ, τ to new variable $\chi = \xi - V\tau$, where V is a constant speed (normalized by ion acoustic speed). Integrating the obtained ordinary differential equation (ODE) with respect to ξ and the constant of integration equals zero by substitution the boundary conditions $\frac{d\phi_1}{d\chi} \to 0, \frac{d^2\phi_1}{d\chi^2} \to 0, \phi_1 \to 0$ as $\chi \to \pm\infty$. We get the second order ODE in the form:

$$-V\phi_1 + \frac{q_1}{2}\phi_1^2 + q_2\frac{d^2\phi_1}{d\chi^2} = 0. \tag{13}$$

We reduce Eq.(13) to the following an autonomous planar dynamical system :

$$\begin{cases} \phi_1' = Z \\ Z' = \frac{V}{q_2}\phi_1 - \frac{q_1}{2q_2}\phi_1^2. \end{cases} . \tag{14}$$

In the system (14), let $\phi_1' = g_1(\phi_1, Z)$ and $Z' = g_2(\phi_1, Z)$ is Hamiltonian system if such that $g_1 = \frac{\partial H}{\partial Z}$ and $g_2 = -\frac{\partial H}{\partial \phi_1}$ and the Hamiltonian function satisfies the condition $\frac{\partial g_1}{\partial \phi_1} + \frac{\partial g_2}{\partial Z} = 0$ (Chow and Hale [4], Saha and Banerjee [28], Shahein and Wahid [35]). The system (14) has Hamiltonian function as

$$H(\phi_1, Z) = \frac{Z^2}{2} + \frac{q_1}{6q_2}\phi_1^3 - \frac{V}{2q_2}\phi^2 = h. \tag{15}$$

Where h is a constant and the solutions orbits have fixed value of H along each one. The system (14) has two equilibrium points at $E_1(0,0)$ and $E_2(\frac{2V}{q_1}, 0)$. We use linearization technique by Jacobian matrix to determine the eigenvalues of two equilibrium points

$$J = \begin{pmatrix} 0 & 1 \\ \frac{V}{q_2} - \frac{q_1}{q_2}\phi_1 & 0 \end{pmatrix} .$$

Then the Jacobian matrix at E_1 and E_2 are in the form

$$J_{E_1} = \begin{pmatrix} 0 & 1 \\ \frac{V}{q_2} & 0 \end{pmatrix} ,$$

$$J_{E_2} = \begin{pmatrix} 0 & 1 \\ \frac{-V}{q_2} & 0 \end{pmatrix} .$$

Let now discus two cases.
First case for $\frac{V}{q_2} > 0$ [at $V > 0$ and $(v_{ps} - u_0)$ has positive square root in Eq. (9)]:
we have two real and distinct eigenvalues $\lambda = \pm\sqrt{\frac{V}{q_2}}$ at E_1 then we classify E_1 as an unstable saddle point, see figure 1(a). Also at E_2 the eigenvalues are $\lambda = \pm i\sqrt{\frac{V}{q_2}}$ then we classify E_2 as a stable center point, see figure 1(b). The phase portrait has one homoclinic orbit consists from saddle at E_1 and center at E_2 and a family of periodic orbits around the center point as shown in figures 1(c-e). The phase portrait of homoclinic orbits in figure 1(c-e) indicates a solitary and a periodic traveling wave solutions of K-dV equation. Substantially, the effect of increasing of U_{ri} plays an essential rule in decreasing the numbers of centers as U_{ri} increases, see figures 1(c-e). This indicates that an increase in the value of U_{ri} reduces the value of the wave amplitude. Interestingly, we noticed that there is a critical range of parameter $0 < \sigma_i < 0.1$ where as relativistic factor increases, the center point E_2 changes its place from the right of the saddle point E_1 to its left, see figures(2). This means the wave expected to reverse its direction of propagation as relativistic factor of ions increases within the critical range of ion temperature.
Second case for $\frac{V}{q_2} < 0$ [at $V > 0$ and $(v_{ps} - u_0)$ has negative square root in Eq. (9)]: then we have one homoclinic orbit with stable center at E_1 and unstable saddle at E_2 as illustrated in figures (3). We can say that the speed sign of $(v_{ps} - u_0)$ is a very important factor that can affect wave behavior.

5 The exact solutions of K-dV equation with (G'/G)-expansion method:

We assume that $\phi_1 = \phi(\chi)$ is the solution of Eq.(13) and can expressed as a polynomial of (G'/G) as (Zhang et al. [41], Liu [18], Song and Ge [38], Abazari [1])

$$\phi = \sum_{m=0}^{r} a_m \left(\frac{G'}{G}\right)^m , \tag{16}$$

where a_m are constants will be determined later and r is determined by balance between the higher order of derivatives $\frac{d^2\phi}{d\chi^2}$ and the nonlinear term ϕ^2 in Eq.(13)

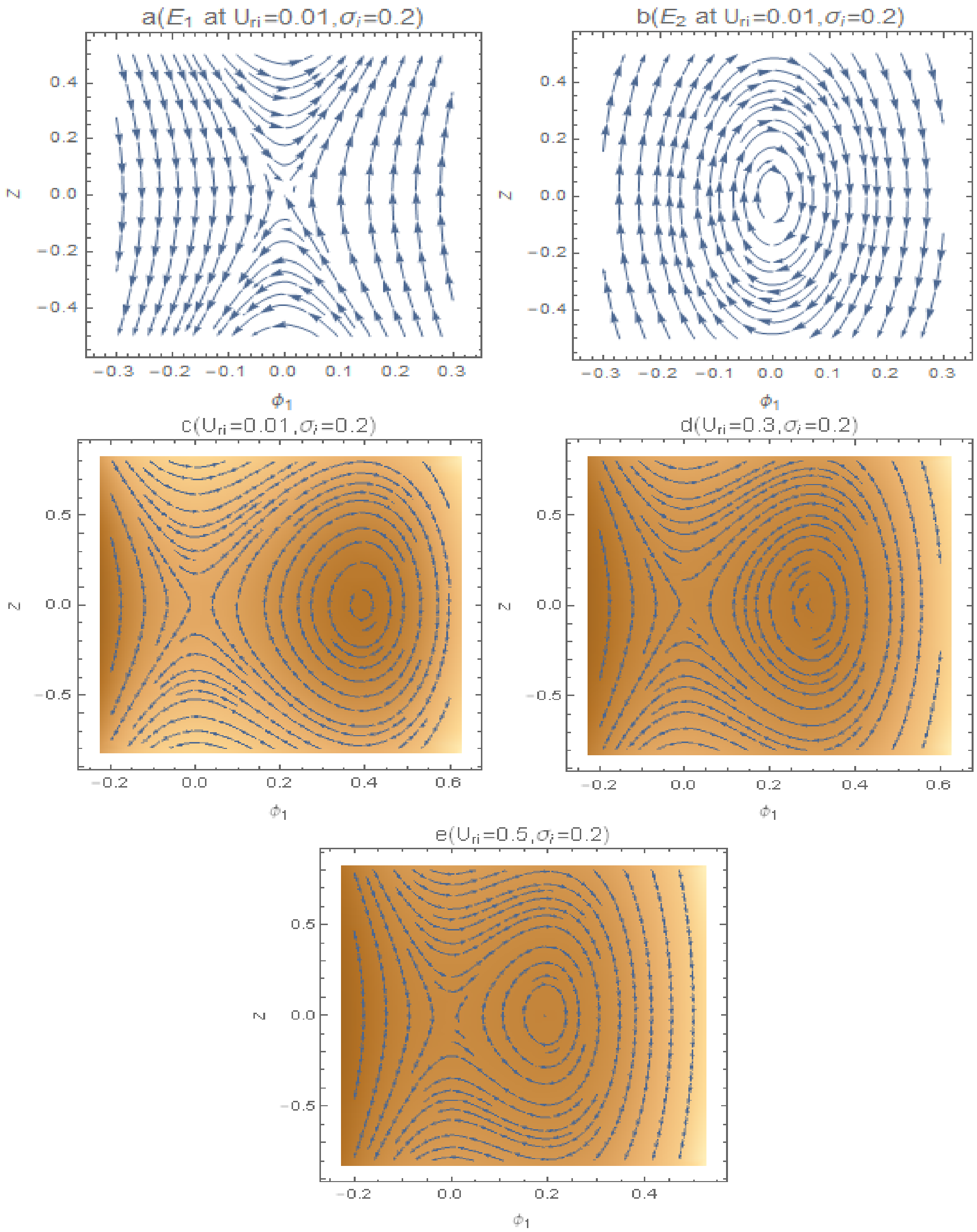

Figure 1. Phase portraits for system (14) at (a,b) and homoclinic orbits at (c-e) with positive value of phase velocity. The parameters are $\rho = 0.2$, $\sigma_p = 1.3, V = 0.59, c = 500,$ $\kappa_e = 2.5, \kappa_p = 2.8$ at different values of U_{ri} at $\sigma_i = 0.2$.

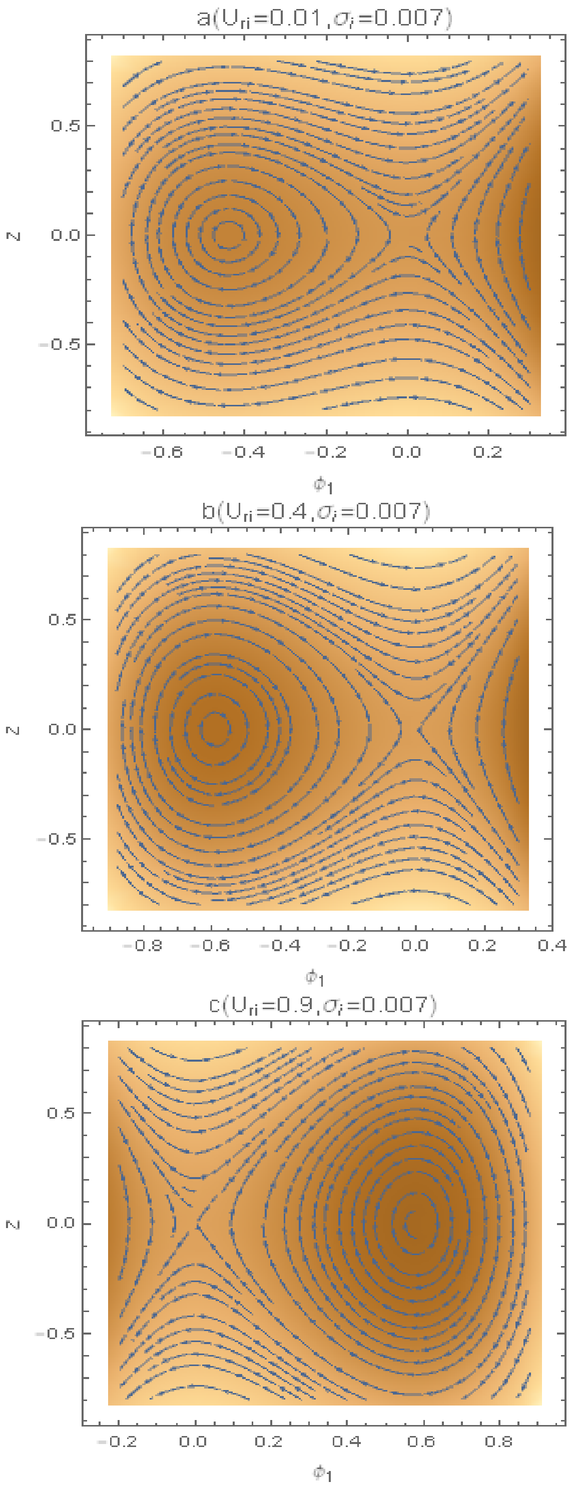

Figure 2. Phase portraits for homoclinic orbits with positive value of phase velocity. The parameters are $\sigma_i = 0.007, \rho = 0.2$, $\sigma_p = 1.3, V = 0.25, c = 500,$ $\kappa_e = 2.5, \kappa_p = 2.8$ at different values of U_{ri}.

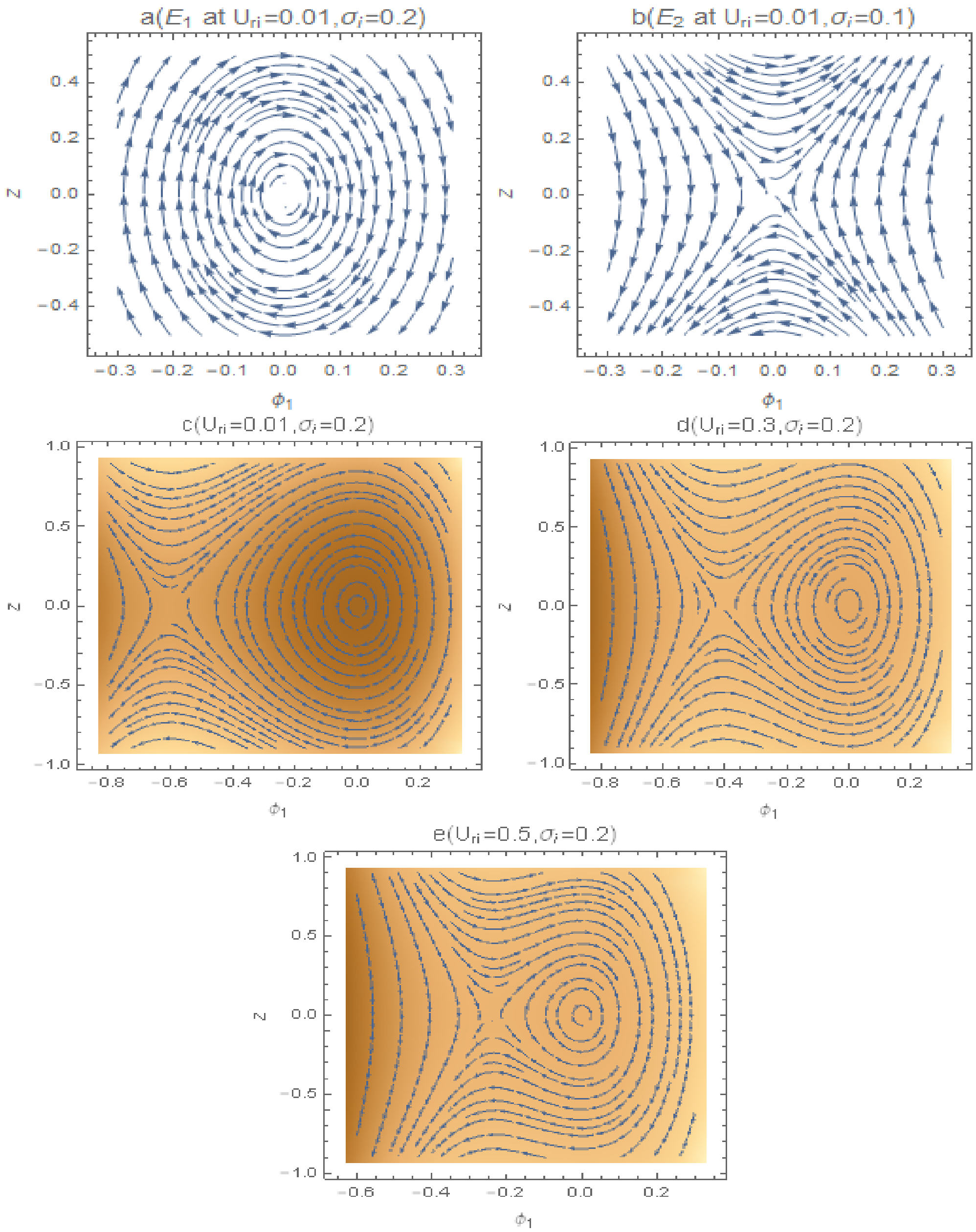

Figure 3. Phase portraits for system (14) at (a,b) and homoclinic orbits at (c-e) with negative value of phase velocity. The parameters are $\rho = 0.2$, $\sigma_p = 1.3, V = 0.4, c = 500$, $\kappa_e = 2.5, \kappa_p = 2.8$ at different values of U_{ri} at $\sigma_i = 0.2$.

we get $r+2 = 2r \Longrightarrow r = 2$ so the solution will be in the form

$$\phi = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2,\ a_2 \neq 0, \tag{17}$$

where the function $G = G(\chi)$ satisfies the second ODE equation with real constants s_1, s_2

$$G'' + s_1 G' + s_2 G = 0. \tag{18}$$

We have three solutions of Eq.(18) as

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{s_1}{2} + \frac{\sqrt{s_1^2-4s_2}}{2} \times \left(\frac{d_1 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_2 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}{d_2 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_1 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}\right), & s_1^2 - 4s_2 > 0, \\ -\frac{s_1}{2} + \frac{\sqrt{-s_1^2+4s_2}}{2} \times \left(\frac{-d_1 \sin(\frac{\chi\sqrt{-s_1^2+4s_2}}{2}) + d_2 \cos(\frac{\chi\sqrt{-s_1^2+4s_2}}{2})}{d_2 \sin(\frac{\chi\sqrt{-s_1^2+4s_2}}{2}) + d_1 \cos(\frac{\chi\sqrt{-s_1^2+4s_2}}{2})}\right), & s_1^2 - 4s_2 < 0, \\ -\frac{s_1}{2}\left(\frac{d_2}{d_2\chi + d_1}\right), \quad s_1^2 - 4s_2 = 0. \end{cases} \tag{19}$$

Substituting from Eq.(17) into Eq.(13) and collect similar terms of order (G'/G) together, then we have a set of algebraic equations for a_0, a_1, a_2 by equating the coefficients of all terms to zero. After that by solving numerically these equations, we have set of solutions as

$$a_0 = \frac{V \pm \sqrt{5(24\ q_2^2\ s_2(s_1^2 + 2s_2) + V^2)}}{\sqrt{5}\ q_1}, a_1 = \frac{-12\ q_2\ s_1}{q_1}, \quad a_2 = \frac{a_1}{s_1}. \tag{20}$$

$$a_0 = \frac{-q_2(s_1^2 + 8\ s_2) \pm V}{25\ q_1}, a_1 = \frac{-12\ q_2\ s_1}{q_1}, \quad a_2 = \frac{a_1}{s_1}. \tag{21}$$

Finally, we obtained three families of solutions for K-dV equation (13) by substituting Eqs.(20,21) and Eq.(19) into Eq.(17), then we get three solution: *Case 1 Soliton wave solution:(hyperbolic solution if* $s_1^2 - 4s_2 > 0$*)*

$$\phi_1 = \frac{V \pm \sqrt{5(24\ q_2^2\ s_2(s_1^2 + 2s_2) + V^2)}}{\sqrt{5}\ q_1} + (\frac{-12\ q_2\ s_1}{q_1})$$
$$\left[-\frac{s_1}{2} + \frac{\sqrt{s_1^2-4s_2}}{2} \times \left(\frac{d_1 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_2 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}{d_2 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_1 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}\right)\right] +$$
$$(\frac{a_1}{s_1})\left[-\frac{s_1}{2} + \frac{\sqrt{s_1^2-4s_2}}{2} \times \left(\frac{d_1 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_2 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}{d_2 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_1 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}\right)\right]^2. \tag{22}$$

Case 2 blow up solitary wave solution:(hyperbolic solution if $s_1^2 - 4s_2 > 0$)

$$\phi_2 = \frac{-q_2(s_1^2 + 8\ s_2) \pm V}{25\ q_1} + (\frac{-12\ q_2\ s_1}{q_1})$$
$$\left[-\frac{s_1}{2} + \frac{\sqrt{s_1^2 - 4s_2}}{2} \times \left(\frac{d_1 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_2 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}{d_2 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_1 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}\right)\right] +$$
$$(\frac{a_1}{s_1})\left[-\frac{s_1}{2} + \frac{\sqrt{s_1^2 - 4s_2}}{2} \times \left(\frac{d_1 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_2 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}{d_2 \sinh(\frac{\chi\sqrt{s_1^2-4s_2}}{2}) + d_1 \cosh(\frac{\chi\sqrt{s_1^2-4s_2}}{2})}\right)\right]^2. \quad (23)$$

Case 3 periodic wave solution (Trigonometric solution if $s_1^2 - 4s_2 < 0$)

$$\phi_3 = \frac{-q_2(s_1^2 + 8\ s_2) \pm V}{25\ q_1} + (\frac{-12\ q_2\ s_1}{q_1})$$
$$\left[-\frac{s_1}{2} + \frac{\sqrt{-s_1^2 + 4s_2}}{2} \times \left(\frac{-d_1 \sin(\frac{\chi\sqrt{-s_1^2+4s_2}}{2}) + d_2 \cos(\frac{\chi\sqrt{-s_1^2+4s_2}}{2})}{d_2 \sin(\frac{\chi\sqrt{-s_1^2+4s_2}}{2}) + d_1 \cos(\frac{\chi\sqrt{-s_1^2+4s_2}}{2})}\right)\right] +$$
$$(\frac{a_1}{s_1})\left[-\frac{s_1}{2} + \frac{\sqrt{-s_1^2 + 4s_2}}{2} \times \left(\frac{-d_1 \sin(\frac{\chi\sqrt{-s_1^2+4s_2}}{2}) + d_2 \cos(\frac{\chi\sqrt{-s_1^2+4s_2}}{2})}{d_2 \sin(\frac{\chi\sqrt{-s_1^2+4s_2}}{2}) + d_1 \cos(\frac{\chi\sqrt{-s_1^2+4s_2}}{2})}\right)\right]^2. \quad (24)$$

6 Results and discussion :

We introduced a new point of view for studying the types and stability of solutions for the K-dV equation by bifurcation of dynamical system. In this paper, a novel study in plasma physics have been carried out of K-dV equation of an unmagnetized plasma consists of three components superthermal electrons-positrons and strongly relativistic ions. Now we are presenting her some important effects of plasma parameters on performing the phase portraits and the behavior of solutions which are obtained. In case $(v_{ps} - u_0) > 0$, the effects of many parameters as spectral index of superthermality $(2 < \kappa_e, \kappa_p < 6)$, relativistic factor $(0 \prec U_{ri} \prec 1)$, density ratio $(0 \prec \rho \prec 1)$ of positron to electron, temperature ratio $(\sigma_p \succeq 1)$ of electron to positron as well as the temperature ratio $(0 \prec \sigma_i \preceq 1)$ of ion to electron of wave on behavior of IAW have been illustrated in 2-dimensional and 3-dimensional graphes. Figures (4,5) exhibit declining in soliton amplitude and width due to increase in each of U_{ri}, σ_p, σ_i, ρ this is congruent with the results in Ref. Javidan and Saadatmand [16]. Moreover, the increase in κ_e, κ_p lead to flourish the soliton wave which make the soliton become robust, also the wave amplitude is affected by the parameter κ_e more than κ_p as figures 4(b,c),6 this is congruent with the results of Ref.

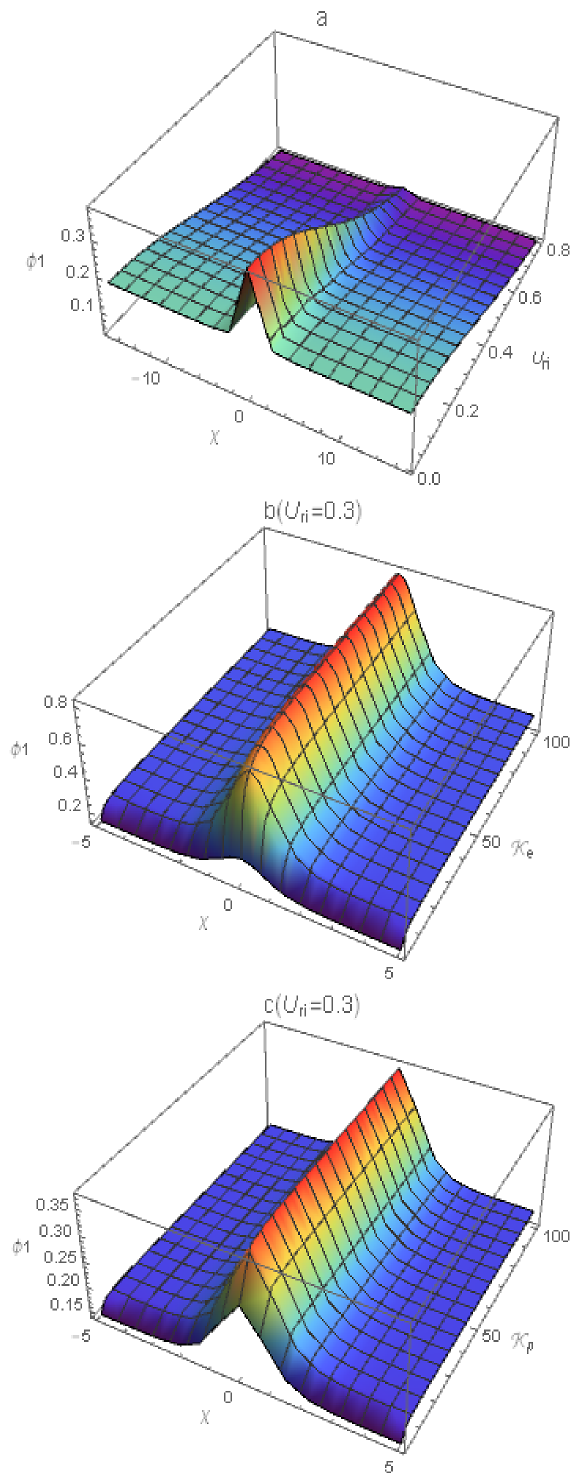

Figure 4. The soliton wave solution with relativistic factor U_{ri} at (a) for, $\kappa_e = \kappa_p = 2.5$ and with superthermality κ_e, κ_p at (b,c) for $U_{ri} = 0.3$. Other parameters $\sigma_i = 0.2, \rho = 0.2,\ \sigma_p = 1.3,\ V = 0.4, s_1 = 2,\ s_2 = 0,\ d_1 = 1,\ d_2 = 0$.

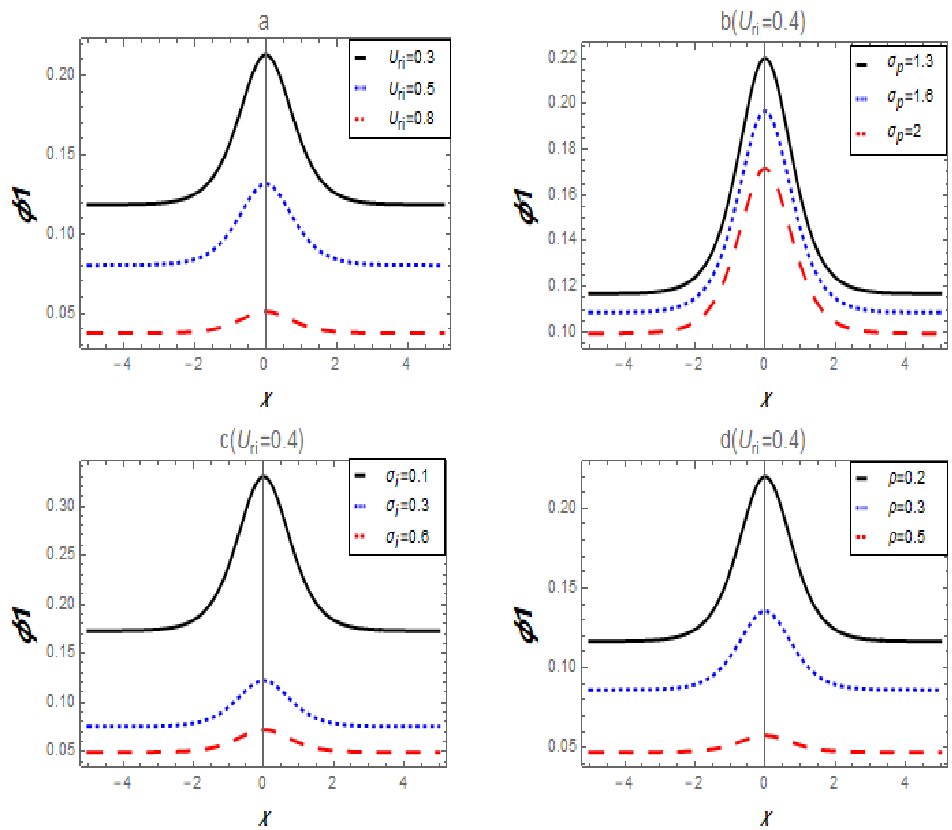

Figure 5. The soliton wave solution for parameters $\sigma_i = 0.2, \rho = 0.2,\ \sigma_p = 1.3,\ V = 0.4, s_1 = 2,\ s_2 = 0,\ d_1 = 1,\ d_2 = 0, \kappa_e = \kappa_p = 2.5$ at $U_{ri} = 0.4$ in (b-d).

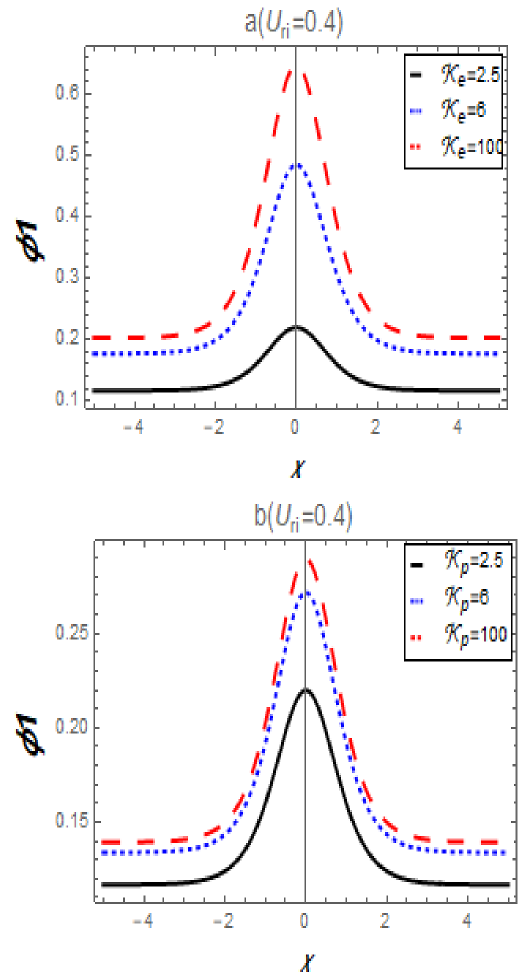

Figure 6. The soliton wave solution for parameters $\sigma_i = 0.2, \rho = 0.2,\ \sigma_p = 1.3,\ V = 0.4, s_1 = 2,\ s_2 = 0,\ d_1 = 1,\ d_2 = 0$ at $U_{ri} = 0.4$.

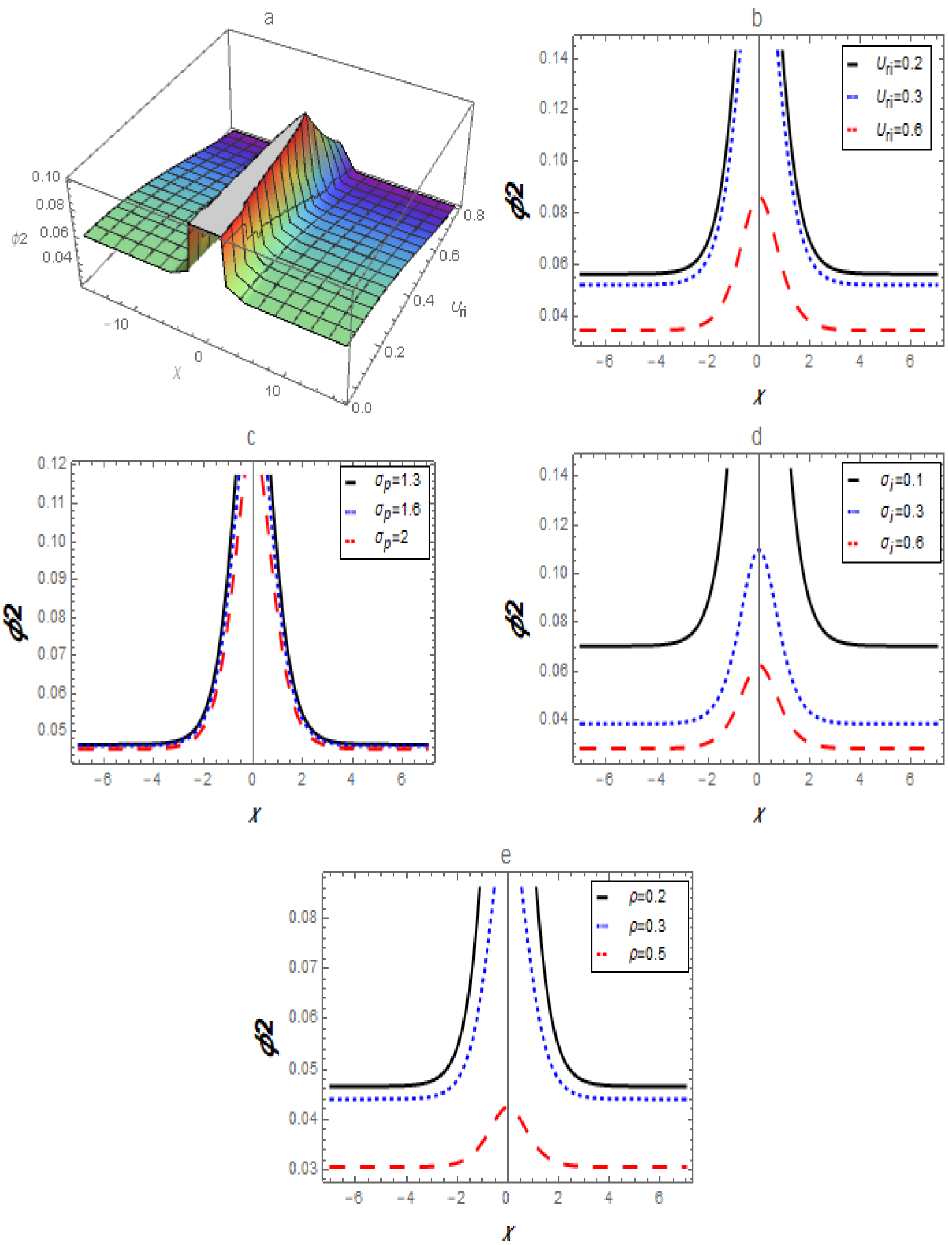

Figure 7. The blow up solitary wave solution for parameters $\sigma_i = 0.2, \rho = 0.2,\ \sigma_p = 1.3,\ V = 0.4, s_1 = 2,\ s_2 = 0,\ d_1 = 1,\ d_2 = 0$, and $U_{ri} = 0.4$.

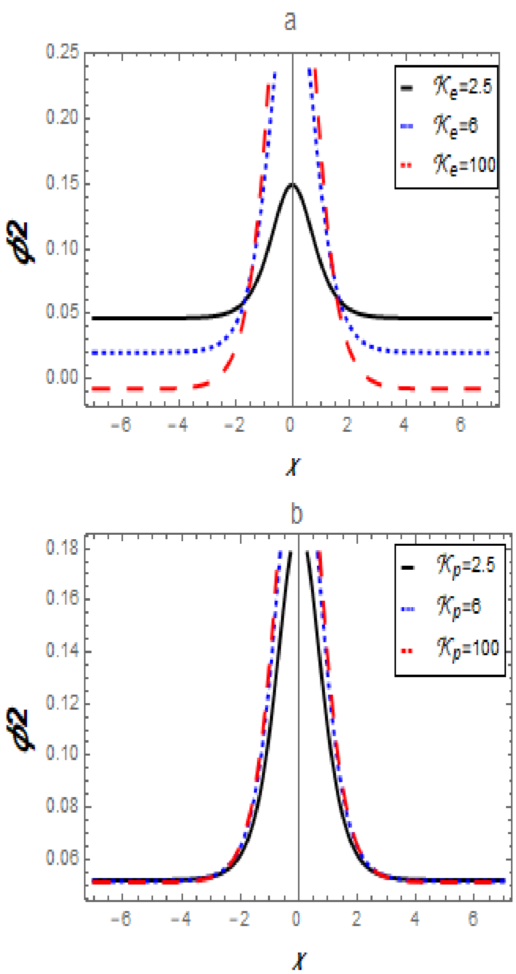

Figure 8. The blow up solitary wave solution for parameters $\sigma_i = 0.2, \rho = 0.2$, $\sigma_p = 1.3$, $V = 0.4, s_1 = 2$, $s_2 = 0$, $d_1 = 1$, $d_2 = 0$, and $U_{ri} = 0.4$ at different values of κ_e, κ_p .

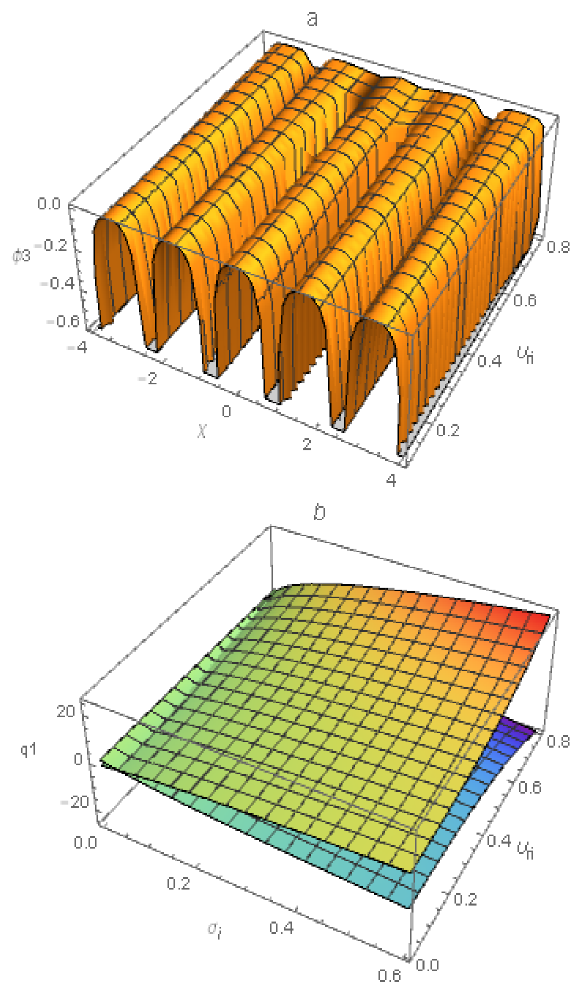

Figure 9. The periodic wave solution with the interval U_{ri} at (a) and the nonlinear coefficient term q_1 of K-dV equation with parameters $\sigma_i = 0.3, \rho = 0.5,\ \sigma_p = 1.3,\ V = 0.3, s_1 = 2,\ s_2 = 0,\ d_1 = 1,\ d_2 = 0$, and $\kappa_e = 2.5,\ \kappa_p = 2.8$.

El-Tantawy et al. [8]. The figures (7) show the new kind of solitary wave called blow up solitary wave solution where this kind appears in strongly relativistic factor up to $U_{ri} = 0.6$ as figure (7a). Also as the parameters $U_{ri},\ \sigma_p,\ \sigma_i,\ \rho$ increase, the amplitude and width of blow up solitary wave decrease and at higher values of these parameters the blow up wave changes into solitary wave (Maxwellian form) this is clear in figures 7 (b-e). Both amplitude and width of blow up solitary wave increase as superthermality index κ_e, κ_p increase as figures (8). The periodic wave solution is appeared with relativistic factor U_{ri} and independent variable χ in figure (9a). It is worth to mention here that at positive (negative) values of phase velocity in Eq.(9), we have positive (negative) values of nonlinear coefficient q_1 this leads to compressive (rarefactive) soliton for chosen parameters, see figure (9b).

7 Conclusion:

One can conclude that the derived solutions of K-dV are sensitive for relativistic ion factor and superthermality of electrons and positrons. The obtained results are helpful for basic understanding the salient features of the fully ion acoustic nonlinear waves in laboratory plasmas as well as space environments, where electron and positron with highly energies obeying superthermal distribution have been observed. In fact and without exaggeration, this new perspective gives a consolidated picture not only in the stability information of solutions but also in expecting the types of propagating wave in similar fluid system. Bifurcation analysis gives an early prediction about the wave behavior without having to obtain the solutions as illustrated in figures (1-3). Figures(1-9) and obtained results play a pivotal role in elucidating the IAW structures in epi plasma with superthermallity ep and relativistic ion that have been evidenced in auroral zone and pulsar relativistic wind.

Conflict of interest: The authors have no conflicts to disclose.

Data Availability: The data that supports the findings of this study are available within the article.

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