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Existence Results for a Coupled System Involving right Caputo and Left Riemann-Liouville Fractional Derivatives with Nonlocal Integral Boundary Conditions[1]

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Abstract

In this paper, we investigate the existence of solutions for a nonlocal integral boundary value problem of involving right Caputo and left Riemann-Liouville fractional derivatives coupled system. Existence and uniqueness results for the given problem are derived with the aid of Leray-Schauder's alternative and Banach's contraction principle. The existence and uniqueness result is elaborated with the aid of an example.

Keywords: Fractional differential systems; Fractional derivative; Integral boundary conditions

1 Introduction

Fractional differential equations have gained importance due to their numerous applications in many fifields of science and engineering, diffusive transport akin to diffusion, rheology, probability, electrical networks, etc$[1-6]$. The study of coupled systems of fractional-order differential equations is found to be of

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great value and interest in view of the occurrence of such systems in a variety of problems of applied nature[7 − 13].

In [14], the authors disscuss the existence of coupled fractional differential system involving right Caputo and left Riemann-Liouville fractional derivatives,

$$\begin{cases} {}^cD_{1-}^{\alpha}D_{0+}^{\beta}x(t) = f(t, x(t), y(t)), \quad t \in J := [0,1], \\ {}^cD_{1-}^{p}D_{0+}^{q}y(t) = g(t, x(t), y(t)), \quad t \in J := [0,1], \\ x(0) = x'(0) = 0, \quad x(1) = \gamma y(\eta), \quad 0 < \eta < 1, \\ y(0) = y'(0) = 0, \quad y(1) = \delta x(\theta), \quad 0 < \theta < 1, \end{cases}$$

where ${}^cD_{1-}^{\alpha}$,${}^cD_{1-}^{p}$ denote the right Caputo fractional derivatives of order $\alpha, p \in (1,2]$ and D_{0+}^{β},D_{0+}^{q} denote the left Riemann-Liouville fractional derivatives of order $\beta, q \in (0,1]$, $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\gamma, \delta \in \mathbb{R}$ are appropriate constants.

This motivates us to consider the following nonlinear fractional differential equations with nonlocal integral boundary conditions:

$$\begin{cases} {}^cD_{1-}^{\alpha}D_{0+}^{\beta}x(t) = f(t, x(t), y(t)), \quad t \in [0,1], \\ {}^cD_{1-}^{p}D_{0+}^{q}y(t) = g(t, x(t), y(t)), \quad t \in [0,1], \\ x(0) = x'(0) = 0, \quad x(\xi) = a\int_0^{\eta} \frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)}x(s)\,ds, \quad \varphi > 0, \\ y(0) = y'(0) = 0, \quad y(\delta) = b\int_0^{\theta} \frac{(\theta-s)^{\tau-1}}{\Gamma(\tau)}y(s)\,ds, \quad \tau > 0, \end{cases} \tag{1.1}$$

where ${}^cD_{1-}^{\alpha}$,${}^cD_{1-}^{p}$ denote the right Caputo fractional derivatives of order $\alpha, p \in (1,2]$ and D_{0+}^{β},D_{0+}^{q} denote the left Riemann-Liouville fractional derivatives of order $\beta, q \in (0,1]$, $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, a, b are real constants, and $0 < \eta < \xi < 1, 0 < \theta < \delta < 1$.

The rest of the contents of the paper is organized as follows. In Section 2, we give some definitions and lemmas. In Section 3, we give the main results, the first result based on Leray-Schauder's alternative, the second result based on Banach's contraction principle. Finally, we provide an example to illustrate our results.

2 Preliminaries

Before presenting an auxiliary lemma, we recall some basic definitions of fractional calculus.

Definition 2.1. ([15, 16]) *We define the left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ respectively as*

$$I_{0+}^{\alpha}g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)\,ds, \tag{2.1}$$

$$I_{1-}^{\alpha} g\left(t\right) = \int_t^1 \frac{\left(s-t\right)^{\alpha -1}}{\Gamma\left(\alpha\right)} g\left(s\right) ds, \tag{2.2}$$

provided the right-hand sides are point-wise defined on $(0,\infty)$, *where* Γ *is the gamma function.*

Definition 2.2. ([15, 16]) *The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* $g:(0,\infty)\to\mathbb{R}$ *such that* $g\in C^n\left((0,\infty),\mathbb{R}\right)$ *are respectively given by*

$$D_{0+}^{\alpha} g\left(t\right) = \frac{d^n}{dt^n}\left(I_{0+}^{n-\alpha} g\right)\left(t\right),$$

$$^{c}D_{1-}^{\alpha} g\left(t\right) = \left(-1\right)^n I_{1-}^{n-\alpha} g^{(n)}\left(t\right),$$

where $n-1<\alpha<n$.

To define the solution for the problem (1.1), we use the following lemma. For the reader's convenience, we outline it's proof.

Lemma 2.3. *Let* $h,k\in C\left([0,1],\mathbb{R}\right)$,*the solution of the linear fractional differential system supplemented*

$$\begin{cases} ^{c}D_{1-}^{\alpha} D_{0+}^{\beta} x\left(t\right) = h\left(t\right), & t\in[0,1], \\ ^{c}D_{1-}^{p} D_{0+}^{q} y\left(t\right) = k\left(t\right), & t\in[0,1], \\ x\left(0\right) = x'\left(0\right) = 0, \quad x(\xi) = a\int_0^{\eta} \frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)} x\left(s\right) ds, & \varphi>0, \\ y\left(0\right) = y'\left(0\right) = 0, \quad y(\delta) = b\int_0^{\theta} \frac{(\theta-s)^{\tau-1}}{\Gamma(\tau)} y\left(s\right) ds, & \tau>0, \end{cases} \tag{2.3}$$

is equivalent to a system of integral equations given by

$$x\left(t\right) = I_{0+}^{\beta} I_{1-}^{\alpha} h\left(t\right) + \frac{\left(\beta+2\right) t^{\beta+1}}{A}\left[I_{0+}^{\beta} I_{1-}^{\alpha} h\left(\xi\right) - a\int_0^{\eta} \frac{\left(\eta-s\right)^{\varphi-1}}{\Gamma\left(\varphi\right)} I_{0+}^{\beta} I_{1-}^{\alpha} h\left(s\right) ds\right], \tag{2.4}$$

$$y\left(t\right) = I_{0+}^{q} I_{1-}^{p} k\left(t\right) + \frac{\left(q+2\right) t^{q+1}}{B}\left[I_{0+}^{q} I_{1-}^{p} k\left(\delta\right) - b\int_0^{\theta} \frac{\left(\theta-s\right)^{\tau-1}}{\Gamma\left(\tau\right)} I_{0+}^{q} I_{1-}^{p} k\left(s\right) ds\right], \tag{2.5}$$

where $A = a\eta^{\beta+2} - \left(\beta+2\right)\xi^{\beta+1}$ *and* $B = b\theta^{q+2} - \left(q+2\right)\delta^{q+1}$.

Proof. We first apply the right fractional integrals I_{1-}^{α},I_{1-}^{p} to the fractional differential equations in (2.3) and then the left fractional integrals I_{0+}^{β},I_{0+}^{q} to the resulting equations, and using the properties of Caputo and Riemann-Liouville fractional derivatives, we get

$$\begin{aligned} x(t) &= I_{0+}^{\beta}\left(I_{1-}^{\alpha}h(t)+c_0+c_1t\right)+c_2t^{\beta-1} \\ &= I_{0+}^{\beta}I_{1-}^{\alpha}h(t)+c_0\frac{t^{\beta}}{\Gamma(\beta+1)}+c_1\frac{t^{\beta+1}}{\Gamma(\beta+2)}+c_2t^{\beta-1}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} y(t) &= I_{0+}^{q}\left(I_{1-}^{p}k(t)+d_0+d_1t\right)+d_2t^{q-1} \\ &= I_{0+}^{q}I_{1-}^{p}k(t)+d_0\frac{t^{q}}{\Gamma(q+1)}+d_1\frac{t^{q+1}}{\Gamma(q+2)}+d_2t^{q-1}. \end{aligned} \tag{2.7}$$

Using the conditions $x(0)=0$, $x^{'}(0)=0$, $y(0)=0$, $y^{'}(0)=0$ in (2.6) and (2.7) yields $c_0=0$, $d_0=0$, $c_2=0$, $d_2=0$. In consequence, the system of equations (2.6) and (2.7) reduces to the form:

$$x(t)=I_{0+}^{\beta}I_{1-}^{\alpha}h(t)+c_1\frac{t^{\beta+1}}{\Gamma(\beta+2)}, \tag{2.8}$$

$$y(t)=I_{0+}^{q}I_{1-}^{p}k(t)+d_1\frac{t^{q+1}}{\Gamma(q+2)}. \tag{2.9}$$

Making use of the conditions $x(\xi)=a\int_0^{\eta}\frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)}x(s)\,ds$, $y(\delta)=b\int_0^{\theta}\frac{(\theta-s)^{\tau-1}}{\Gamma(\tau)}y(s)\,ds$ in (2.8) and (2.9) and solving the resulting equations for c_1 and d_1, we find that

$$c_1=\frac{I_{0+}^{\beta}I_{1-}^{\alpha}h(\xi)-a\int_0^{\eta}\frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)}I_{0+}^{\beta}I_{1-}^{\alpha}h(s)\,ds}{a\int_0^{\eta}\frac{s^{\beta+1}}{\Gamma(\beta+2)}ds-\frac{\xi^{\beta+1}}{\Gamma(\beta+2)}},$$

$$d_1=\frac{I_{0+}^{q}I_{1-}^{p}k(\delta)-b\int_0^{\theta}\frac{(\theta-s)^{\tau-1}}{\Gamma(\tau)}I_{0+}^{q}I_{1-}^{p}k(s)\,ds}{b\int_0^{\theta}\frac{s^{q+1}}{\Gamma(q+2)}ds-\frac{\delta^{q+1}}{\Gamma(q+2)}},$$

which, on substituting in (2.8) and (2.9), leads to the solution system (2.4)-(2.5). The converse follows by direct computation. The proof is completed. □

Lemma 2.4. *(Leray-Schauder alternative)* ([17]). *Let $F:E\to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\varepsilon(F)=\{x\in E:x=\lambda F(x)\,for\ some\ 0<\lambda<1\}.$$

Then either the set $\varepsilon(F)$ is unbounded, or F has at least one fixed point.

3 Main results

Let us introduce the space $X = \{x(t) \,|x(t) \in C([0,1], \mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$ and note that $(X, \|\cdot\|)$ is a Banach space. Then the product space $(X \times X, \|(x,y)\|)$ is also a Banach space equipped with the norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 2.3, we define an operator $T : X \times X \to X \times X$ by

$$T(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned} T_1(x,y)(t) = & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u), y(u))\, duds \\ & + \frac{(\beta+2)\, t^{\beta+1}}{A} \left[\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u), y(u))\, duds \right. \\ & \left. - a \int_0^\eta \frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)} \int_0^s \frac{(s-\nu)^{\beta-1}}{\Gamma(\beta)} \int_\nu^1 \frac{(u-\nu)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u), y(u))\, du d\nu ds \right], \end{aligned}$$

and

$$\begin{aligned} T_2(x,y)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} g(u, x(u), y(u))\, duds \\ & + \frac{(q+2)\, t^{q+1}}{B} \left[\int_0^\delta \frac{(\delta-s)^{q-1}}{\Gamma(q)} \int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} g(u, x(u), y(u))\, duds \right. \\ & \left. - b \int_0^\theta \frac{(\theta-s)^{\tau-1}}{\Gamma(\tau)} \int_0^s \frac{(s-\nu)^{q-1}}{\Gamma(q)} \int_\nu^1 \frac{(u-\nu)^{p-1}}{\Gamma(p)} g(u, x(u), y(u))\, du d\nu ds \right]. \end{aligned}$$

In order to get the result of our result, we introduce the following hypotheses.

(H_1) Assume that there exist real constants $k_i, \lambda_i \geq 0$ $(i = 1,2)$ and $k_0 > 0, \lambda_0 > 0$ such that $\forall x_i \in \mathbb{R}, i = 1,2$, we have

$$|f(t, x_1, x_2)| \leq k_0 + k_1 |x_1| + k_2 |x_2|,$$

$$|g(t, x_1, x_2)| \leq \lambda_0 + \lambda_1 |x_1| + \lambda_2 |x_2|.$$

(H_2) Assume that $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist constants $m_i, n_i, i = 1,2$ such that for all $t \in [0,1]$ and $x_i, y_i \in \mathbb{R}, i = 1,2,$

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq m_1 |x_1 - x_2| + m_2 |y_1 - y_2|,$$

and

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq n_1 |x_1 - x_2| + n_2 |y_1 - y_2|.$$

For computational convenience, we set

$$q_0 = \sup_{t\in[0,1]} \left| \frac{(\beta+2)\, t^{\beta+1}}{A} \right|, \tag{3.1}$$

$$p_0 = \sup_{t\in[0,1]} \left| \frac{(q+2)\, t^{q+1}}{B} \right|, \tag{3.2}$$

$$M_1 = \frac{1}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} + q_0 \left[\frac{(1-\xi)^{\alpha}\,\xi^{\beta}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} + |a| \frac{(1-\eta)^{\alpha}\,\eta^{\beta+\varphi}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\varphi+1)} \right], \tag{3.3}$$

$$M_2 = \frac{1}{\Gamma(p+1)\,\Gamma(q+1)} + p_0 \left[\frac{(1-\delta)^{p}\,\delta^{q}}{\Gamma(p+1)\,\Gamma(q+1)} + |b| \frac{(1-\theta)^{p}\,\theta^{q+\tau}}{\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(\tau+1)} \right]. \tag{3.4}$$

The first result is based on Leray-Schauder alternative.

Theorem 3.1. *Assume that* (H_1) *holds. In addition it is assumed that*

$$M_1 k_1 + M_2 \lambda_1 < 1 \text{ and } M_1 k_2 + M_2 \lambda_2 < 1,$$

where M_1 *and* M_2 *are given by* (3.3) *and* (3.4) *respectively. Then the boundary value problem* (1.1) *has at least one solution.*

Proof. First we show that the operator $T : X \times X \to X \times X$ is completely continuous. By continuity of functions f and g, the operator T is continuous. Let $\Omega \subset X \times X$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, x(t), y(t))| \le L_1, \qquad |g(t, x(t), y(t))| \le L_2, \qquad \forall (x, y) \in \Omega.$$

Then for any $(x, y) \in \Omega$, we have

$$\begin{aligned} |T_1(x,y)(t)| \le \Bigg| & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u), y(u))\, du ds \\ & + \frac{(\beta+2)\, t^{\beta+1}}{A} \Bigg[\int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u), y(u))\, du ds \\ & - a \int_0^{\eta} \frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)} \int_0^{s} \frac{(s-\nu)^{\beta-1}}{\Gamma(\beta)} \int_{\nu}^1 \frac{(u-\nu)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u), y(u))\, du d\nu ds \Bigg] \Bigg| \\ \le L_1 & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big(\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du \Big) ds + q_0 L_1 \Bigg[\int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \Big(\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du \Big) ds \end{aligned}$$

$$+|a|\int_0^{\eta}\frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)}(\int_0^{s}\frac{(s-\nu)^{\beta-1}}{\Gamma(\beta)}(\int_{\nu}^{1}\frac{(u-\nu)^{\alpha-1}}{\Gamma(\alpha)}du)d\nu)ds\Bigg]$$
$$\le L_1\left\{\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}+q_0\left[\frac{(1-\xi)^{\alpha}\xi^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right.\right.$$
$$\left.\left.+|a|\frac{(1-\eta)^{\alpha}\eta^{\beta+\varphi}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\varphi+1)}\right]\right\},$$

which implies that

$$\|T_1(x,y)(t)\|\le L_1\left\{\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}+q_0\left[\frac{(1-\xi)^{\alpha}\xi^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right.\right.$$
$$\left.\left.+|a|\frac{(1-\eta)^{\alpha}\eta^{\beta+\varphi}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\varphi+1)}\right]\right\}$$
$$=L_1M_1.$$

Similarly, we get

$$\|T_2(x,y)(t)\|\le L_2\left\{\frac{1}{\Gamma(p+1)\Gamma(q+1)}+p_0\left[\frac{(1-\delta)^{p}\xi^{q}}{\Gamma(p+1)\Gamma(q+1)}\right.\right.$$
$$\left.\left.+|b|\frac{(1-\theta)^{p}\theta^{q+\tau}}{\Gamma(p+1)\Gamma(q+1)\Gamma(\tau+1)}\right]\right\}$$
$$=L_2M_2.$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$. Then we have

$$|T_1(x(t_2),y(t_2))-T_1(x(t_1),y(t_1))|$$
$$\le L_1\left|\int_0^{t_1}\frac{(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}du\right)ds\right.$$
$$\left.+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}du\right)ds\right|$$
$$+L_1\left|\frac{(\beta+2)\left(t_2^{\beta+1}-t_1^{\beta+1}\right)}{A}\times\left[\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left(\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}du\right)ds\right.\right.$$
$$\left.\left.-a\int_0^{\eta}\frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)}\left(\int_0^{s}\frac{(s-\nu)^{\beta-1}}{\Gamma(\beta)}\left(\int_{\nu}^{1}\frac{(u-\nu)^{\alpha-1}}{\Gamma(\alpha)}du\right)d\nu\right)ds\right]\right|.$$

Analogously, we can obtain

$$|T_2(x(t_2),y(t_2))-T_2(x(t_1),y(t_1))|$$

$$
\begin{aligned}
&\leq L_2 \left| \int_0^{t_1} \frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}}{\Gamma(q)} \left(\int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} du \right) ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} \left(\int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} du \right) ds \right| \\
&\quad + L_2 \left| \frac{(q+2)\left(t_2^{q+1}-t_1^{q+1}\right)}{B} \times \left[\int_0^{\delta} \frac{(\delta-s)^{q-1}}{\Gamma(q)} \left(\int_s^1 \frac{(u-s)^{p-1}}{\Gamma(p)} du \right) ds \right. \right. \\
&\quad \left. \left. -b \int_0^{\theta} \frac{(\theta-s)^{\tau-1}}{\Gamma(\tau)} \left(\int_0^s \frac{(s-\nu)^{q-1}}{\Gamma(q)} \left(\int_{\nu}^1 \frac{(u-\nu)^{p-1}}{\Gamma(p)} du \right) d\nu \right) ds \right] \right| .
\end{aligned}
$$

Therefore, the operator $T(x,y)$ is equicontinuous, and thus the operator $T(x,y)$ is completely continuous.

Finally, it will be verified that the set
$\varepsilon = \{(x,y) \in X \times X \,|(x,y) = \lambda T(x,y), 0 \leq \lambda \leq 1\}$
is bounded. Let $(x,y) \in \varepsilon$, then $(x,y) = \lambda T(x,y)$. For any $t \in [0,1]$, we have

$$x(t) = \lambda T_1(x,y)(t), \qquad y(t) = \lambda T_2(x,y)(t).$$

Then

$$
\begin{aligned}
|x(t)| \leq L_1 &\left\{ \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + q_0 \left[\frac{(1-\xi)^{\alpha}\xi^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |a| \frac{(1-\eta)^{\alpha}\eta^{\beta+\varphi}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\varphi+1)} \right] \right\} \\
&\times (k_0 + k_1\|x\| + k_2\|y\|),
\end{aligned}
$$

and

$$
\begin{aligned}
|y(t)| \leq L_2 &\left\{ \frac{1}{\Gamma(p+1)\Gamma(q+1)} + p_0 \left[\frac{(1-\delta)^{p}\xi^{q}}{\Gamma(p+1)\Gamma(q+1)} + |b| \frac{(1-\theta)^{p}\eta^{q+\tau}}{\Gamma(p+1)\Gamma(q+1)\Gamma(\tau+1)} \right] \right\} \\
&\times (\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|).
\end{aligned}
$$

Hence we have

$$\|x\| \leq M_1(k_0 + k_1\|x\| + k_2\|y\|),$$

and

$$\|y\| \leq M_2(\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|).$$

Which imply that

$$\|x\| + \|y\| = (M_1k_0 + M_2\lambda_0) + (M_1k_1 + M_2\lambda_1)\|x\| + (M_1k_2 + M_2\lambda_2)\|y\|.$$

Consequently,

$$\|(x,y)\| \leq \frac{M_1k_0 + M_2\lambda_0}{M_0},$$

for any $t \in [0,1]$, where $M_0 = \min\{1 - (M_1 k_1 + M_2 \lambda_1), 1 - (M_1 k_2 + M_2 \lambda_2)\}$, which proves that $\varepsilon(T)$ is bounded. Thus, by Lemma 2.4, the operator T has at least one fixed point. Hence the boundary value problem (1.1) has at least one solution. The proof is complete. □

In the second result, we prove existence and uniqueness of solutions of the boundary value problem (1.1) via Banach's contraction principle.

Theorem 3.2. *Assume that* (H_2) *holds. In addition, assume that*

$$M_1(m_1 + m_2) + M_2(n_1 + n_2) < 1,$$

where M_1 *and* M_2 *are given by* (3.3) *and* (3.4) *respectively. Then the boundary value problem* (1.1) *has a unique solution.*

Proof. Define $\sup_{t\in[0,1]} f(t,0,0) = N_1 < \infty$ and $\sup_{t\in[0,1]} g(t,0,0) = N_2 < \infty$ such that

$$r \geq \frac{N_1 M_1 + N_2 M_2}{1 - M_1(m_1 + m_2) - M_2(n_1 + n_2)}.$$

We show that $TB_r \subset B_r$, where $B_r = \{(x,y) \in X \times X : \|(x,y)\| \leq r\}$.
For $(x,y) \in B_r$, we have

$$\begin{aligned}
&|T_1(x,y)(t)| \\
&\leq \max_{t\in[0,1]} \Bigg| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |f(u,x(u),y(u)) - f(s,0,0)| + |f(s,0,0)|\, du \right) ds \\
&\quad + \frac{(\beta+2)t^{\beta+1}}{A} \Bigg[\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \left(\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |f(u,x(u),y(u)) - f(s,0,0)| + |f(s,0,0)|\, du \right) ds \\
&\quad - a \int_0^\eta \frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)} \left(\int_0^s \frac{(s-\nu)^{\beta-1}}{\Gamma(\beta)} \left(\int_\nu^1 \frac{(u-\nu)^{\alpha-1}}{\Gamma(\alpha)} |f(u,x(u),y(u)) - f(s,0,0)| \right.\right. \\
&\quad \left.\left. + |f(s,0,0)|\, du \right) d\nu \right) ds \Bigg] \Bigg| \\
&\leq \left\{ \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + q_0 \left[\frac{(1-\xi)^\alpha \xi^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |a| \frac{(1-\eta)^\alpha \eta^{\beta+\varphi}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\varphi+1)} \right] \right\} \\
&\quad \times (m_1\|x\| + m_2\|y\| + N_1) \\
&\leq M_1 [(m_1 + m_2) r + N_1].
\end{aligned}$$

Hence

$$\|T_1(x,y)\| \leq M_1 [(m_1 + m_2) r + N_1].$$

In the same way, we can obtain that

$$\|T_2(x,y)\| \leq M_2 [(n_1 + n_2) r + N_2].$$

Consequently, $\|T(x,y)\| \leq r$. Now for $(x_2,y_2),(x_1,y_1) \in X \times X$, and for any $t \in [0,1]$, we get

$$\begin{aligned}
&|T_1(x_2,y_2)(t) - T_1(x_1,y_1)(t)| \\
&\leq \Bigg| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |f(u,x_2(u),y_2(u)) - f(u,x_1(u),y_1(u))| \, du \right) ds \\
&\quad + \frac{(\beta+2)\,t^{\beta+1}}{A} \Bigg[\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \left(\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |f(u,x_2(u),y_2(u)) - f(u,x_1(u),y_1(u))| \, du \right) ds \\
&\quad - a \int_0^\eta \frac{(\eta-s)^{\varphi-1}}{\Gamma(\varphi)} \left(\int_0^s \frac{(s-\nu)^{\beta-1}}{\Gamma(\beta)} \left(\int_\nu^1 \frac{(u-\nu)^{\alpha-1}}{\Gamma(\alpha)} |f(u,x(u),y(u)) \right.\right. \\
&\quad \left.\left. - \; f(u,x(u),y(u))| \, du \right) d\nu \right) ds \Bigg] \Bigg| \\
&\leq M_1 \left(m_1 \|x_2 - x_1\| + m_2 \|y_2 - y_1\| \right) \\
&\leq M_1 (m_1 + m_2) \left(\|x_2 - x_1\| + \|y_2 - y_1\| \right),
\end{aligned}$$

and consequently we obtain

$$\|T_1(x_2,y_2)(t) - T_1(x_1,y_1)(t)\| \leq M_1(m_1+m_2)\left(\|x_2-x_1\| + \|y_2-y_1\|\right).$$

Similarly,

$$\|T_2(x_2,y_2)(t) - T_2(x_1,y_1)(t)\| \leq M_2(n_1+n_2)\left(\|x_2-x_1\| + \|y_2-y_1\|\right).$$

It follows from (3.6) and (3.7) that

$$\|T(x_2,y_2) - T(x_1,y_1)\| \leq [M_1(m_1+m_2) + M_2(n_1+n_2)]\left(\|x_2-x_1\| + \|y_2-y_1\|\right).$$

Since $M_1(m_1+m_2) + M_2(n_1+n_2) < 1$, therefore, T is a contraction operator. So, by Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1.1) . This completes the proof. □

4 Applications

In this section, we will give an example to illustrate our main results.

Example 4.1 Consider the following equation

$$\begin{cases}
{}^cD_{1-}^{\frac{3}{2}} D_{0+}^{\frac{1}{2}} x(t) = \frac{1}{8(t+2)^2} \frac{|x|}{1+|x|} + 1 + \frac{1}{36} \sin^2 y, t \in [0,1], \\
{}^cD_{1-}^{\frac{3}{2}} D_{0+}^{\frac{1}{2}} y(t) = \frac{1}{32\pi} \sin(2\pi x) + \frac{|y|}{16(1+|y|)} + \frac{1}{2}, t \in [0,1], \\
x(0) = 0, x'(0) = 0, x\left(\frac{1}{3}\right) = \int_0^{\frac{1}{4}} x(s)\,ds, \\
y(0) = 0, y'(0) = 0, y\left(\frac{1}{4}\right) = \int_0^{\frac{1}{5}} y(s)\,ds.
\end{cases} \tag{4.1}$$

Here $\alpha = \frac{3}{2},\ \beta = \frac{1}{2},\ p = \frac{3}{2},\ q = \frac{1}{2},\ \xi = \frac{1}{3},\ a = 1,\ \eta = \frac{1}{4},\ \varphi = 1,\ \delta = \frac{1}{4},\ b = 1,\ \theta = \frac{1}{5},\ \tau = 1,$
and

$$q_0 = \sup_{t \in [0,1]} \left| \frac{(\beta+2)\,t^{\beta+1}}{A} \right| \approx 5.557099,$$

$$p_0 = \sup_{t\in[0,1]} \left| \frac{(q+2)\, t^{q+1}}{B} \right| \approx 8.485766,$$

$$M_1 = \frac{1}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} + q_0 \left[\frac{(1-\xi)^{\alpha}\, \xi^{\beta}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)} + |a| \frac{(1-\eta)^{\alpha}\, \eta^{\beta+\varphi}}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\varphi+1)} \right] \approx 2.714217,$$

$$M_2 = \frac{1}{\Gamma(p+1)\,\Gamma(q+1)} + p_0 \left[\frac{(1-\delta)^{p}\, \delta^{q}}{\Gamma(p+1)\,\Gamma(q+1)} + |b| \frac{(1-\theta)^{p}\, \theta^{q+\tau}}{\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(\tau+1)} \right] \approx 3.649045.$$

Also, $f(t,x(t),y(t)) = \frac{1}{8(t+2)^2}\frac{|x|}{1+|x|} + 1 + \frac{1}{36}\sin^2 y$, $g(t,x(t),y(t)) = \frac{1}{32\pi}\sin(2\pi x) + \frac{|y|}{16(1+|y|)} + \frac{1}{2}$.
Note that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \frac{1}{32}|x_1 - x_2| + \frac{1}{32}|y_1 - y_2|,$$

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le \frac{1}{16}|x_1 - x_2| + \frac{1}{16}|y_1 - y_2|,$$

and

$$M_1(m_1 + m_2) + M_2(n_1 + n_2) \approx 0.625769 < 1.$$

Thus all the conditions of Theorem 3.3 are satisfied and consequently, its conclusion applies to the problem (4.1).

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