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# Infinitely Many Solutions for a Class of $p$-Laplacian Type Fractional Dirichlet Problem with Instantaneous and Non-Instantaneous Impulses

**Weiwei Long**

College of Mathematics and Statistics, Jishou University
Jishou, Hunan 416000, P.R. China

**Guoping Chen**

School of Civil Engineering and Architecture, Jishou University
Jishou, Hunan 416000, P.R. China



**Abstract**

In this paper, the infinitely many of solutions for a class of $p$-Laplacian type fractional differential equations with both instantaneous and non-instantaneous impulses is studied. By using the applying variational methods and critical point theory, a theorem is proposed for the existence of infinitely many solutions for the impulsive problem depending on a parameters. In the end, an example is given to illustrate our theoretical results.

**Keywords** Variational method; $p$-Laplacian operator; Dirichlet problem; Instantaneous impulse; Non-instantaneous impulse

## 1 Introduction

For a long time, the theory of instantaneous impulsive differential equations has received great attention due to their extensive applications in practical situations of a wide variety of in epidemic [1], optimal control [2], mechanical and engineering [3], population ecology [4] and etc. Many authors have made attempt to use variations methods and critical point theory to discuss the existence of solutions for BVP of impulsive fractional equations [5-19]. In 2013, Hernandez and O'Regan first proposed the concept of non-instantaneous impulse differential equations [20]. Since then, many scholars extensive works have been done in the field of non-instantaneous impulses, and there has been a significant development in the theory of the existence of solutions non-instantaneous impulsive differential equations. For some recent development on this topic, one can see the interesting results of [21-24].

More recently in[6], author by the variational method discussed the existence of solutions for fractional differential equations of $p$-Laplacian with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}_tD_T{}^{\alpha}\Phi_p\left({}_0^cD_t^{\alpha}y(t)\right)+|y(t)|^{p-2}y(t)=\lambda f_j(t,y(t)), & t\in\bigcup_{j=0}^{p}(s_j,t_{j+1}],\\ \triangle\left({}_tD_T^{\alpha-1}\Phi_p\left({}_0^cD_t^{\alpha}y\right)\right)(t_j)=I_j\left(y(t_j)\right), & j=1,...,p,\\ {}_tD_T^{\alpha-1}\Phi_p\left({}_0^cD_t^{\alpha}y\right)(t)={}_tD_T^{\alpha-1}\Phi_p\left({}_0^cD_t^{\alpha}y\right)(t_j^+), & t\in\bigcup_{j=1}^{p}(t_j,s_j],\\ {}_tD_T^{\alpha-1}\Phi_p\left({}_0^cD_t^{\alpha}y\right)(s_j^-)={}_tD_T^{\alpha-1}\Phi_p\left({}_0^cD_t^{\alpha}y\right)(s_j^+), & j=1,...,p,\\ y(0)=y(T)=0, \end{cases} \tag{1.1}$$

Motivated by these works, in this paper we are interested in the infinitely many solutions for a class of $p$-Laplacian type fractional differential equations Dirichlet problem with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{\omega(t)^{p-2}}\Phi_p\big(\omega(t){}_0^cD_t^{\alpha}y(t)\big)\right)+|y(t)|^{p-2}y(t)=\lambda f_j(t,y(t)), & t\in\bigcup_{j=0}^{m}(s_j,t_{j+1}],\\ \triangle_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega{}_0^cD_t^{\alpha}y\big)\right)(t_j)=\mu I_j\left(y(t_j)\right), & j=1,...,m,\\ {}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega{}_0^cD_t^{\alpha}y\big)\right)(t)={}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega{}_0^cD_t^{\alpha}y\big)\right)(t_j^+), & t\in\bigcup_{j=1}^{m}(t_j,s_j],\\ {}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega{}_0^cD_t^{\alpha}y\big)\right)(s_j^-)={}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega{}_0^cD_t^{\alpha}y\big)\right)(s_j^+), & j=1,...,m,\\ y(0)=y(T)=0, \end{cases} \tag{1.2}$$

where $1<p<+\infty$, $\lambda$ and $\mu$ are non-negative parameters. ${}_0^cD_t^{\alpha}$ is the left Caputo fractional derivative and ${}_tD_T^{\alpha}$ denotes the right Riemann-Liouville fractional derivative of order $\alpha\in(\frac{1}{p},1]$. $0=s_0<t_1<s_1<\cdots<s_m<t_{m+1}=T$, $\omega(t)\in C^{\infty}[0,T]$ with $\omega^0=ess\ \inf_{[0,T]}\omega(t)>0$, $f_j\in C((s_j,t_{j+1}]\times\mathbb{R},\mathbb{R})$. The instantaneous impulses $I_j\in C(\mathbb{R},\mathbb{R})$ start abrupt changes at the points $t_j$, and the non-instantaneous impulses continue during the finite intervals $(t_j,s_j]$, for $j=1,2,\cdots,m$, and the $\Phi_p(t)=|t|^{p-2}t(t\neq 0)$ is the $p$-Laplacian operator and

$\Phi_p(0) = 0$.

$$\begin{aligned}\triangle {}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega_0^c D_t^{\alpha} y\big)\right)(t_j) =& {}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega_0^c D_t^{\alpha} y\big)\right)(t_j^{+})\\ &- {}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega_0^c D_t^{\alpha} y\big)\right)(t_j^{-}),\end{aligned}$$

$${}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega_0^c D_t^{\alpha} y\big)\right)(t_j^{\pm}) = \lim_{t\to t_j^{\pm}} {}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega_0^c D_t^{\alpha} y\big)\right)(t),$$

and

$${}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega_0^c D_t^{\alpha} y\big)\right)(s_j^{\pm}) = \lim_{t\to s_j^{\pm}} {}_tD_T^{\alpha-1}\left(\frac{1}{\omega^{p-2}}\Phi_p\big(\omega_0^c D_t^{\alpha} y\big)\right)(t).$$

In this paper, we mainly use the critical point theory to obtain the existence of infinitely many solutions for the problem (1.2). The novelty of this article is as follows: Firstly, when $\omega(t) = 1$ and $\mu = 1$, problem (1.2) simplifies to problem (1.1). Secondly, the existence of infinitely many solutions for perturbed nonlinear fractional $p$-Laplacian Dirichlet boundary value problem. The main aim of this paper is intended to establish infinitely many solutions for the problem (1.2) under different assumptions and our results are totally different from above problem (1.1). Therefore, our work generalizes and improved significantly some existing results in literatures.

# 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that we need in the sequel.

**Definition 2.1.** *([5]) Let $y$ be a function defined on $[0,T]$. The left and right Riemann-Liouville fractional integral of order $\alpha$ for the function $y$ denoted by ${}_0D_t^{-\alpha}$ and ${}_tD_T^{-\alpha}$, respectively, with definitions as follows*

$${}_0D_t^{-\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,\ t\in[0,T],\ \alpha>0,$$

*and*

$${}_tD_T^{-\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}y(s)ds,\ t\in[0,T],\ \alpha>0,$$

*provided the right-hand side is pointwise defined on $[0,T]$, where $\Gamma(\cdot)$ is the gamma function.*

**Definition 2.2.** *([5]) Let $0 < \alpha < 1$, $y$ be a function defined on $[0, T]$. Denote the left and right Riemann-Liouville fractional derivatives of the function $y$ by ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$, respectively, with definitions as follows*

$$ {}_0D_t^\alpha y(t) = \frac{d}{dt} {}_0D_t^{\alpha-1} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \int_0^t (t-s)^{-\alpha} y(s) ds \right), \ t \in [0, T], $$

*and*

$$ {}_tD_T^\alpha y(t) = -\frac{d}{dt} {}_tD_T^{\alpha-1} y(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \int_t^T (s-t)^{-\alpha} y(s) ds \right), \ t \in [0, T]. $$

**Definition 2.3.** *([5]) Let $0 < \alpha < 1$, $y$ be a function defined on $[0, T]$. Denote the left and right Caputo fractional integrals of the function $y$ by ${}_0^cD_t^\alpha$ and ${}_t^cD_T^\alpha$, respectively, with definitions as follows*

$$ {}_0^cD_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \left( \int_0^t (t-s)^{-\alpha} y'(s) ds \right), \ t \in [0, T], $$

$$ {}_t^cD_T^\alpha y(t) = -\frac{1}{\Gamma(1-\alpha)} \left( \int_t^T (s-t)^{-\alpha} y'(s) ds \right), \ t \in [0, T], $$

*and the following relationships hold:*

$$ {}_0^cD_t^\alpha y(t) = {}_0D_t^\alpha y(t) - \frac{y(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \ t \in [0, T], $$

$$ {}_t^cD_T^\alpha y(t) = {}_tD_T^\alpha y(t) - \frac{y(T)}{\Gamma(1-\alpha)} (T-t)^{-\alpha}, \ t \in [0, T]. $$

**Definition 2.4.** *([6]) Let $p \in (1, +\infty)$, $\alpha \in (\frac{1}{p}, 1]$. The fractional derivative space*

$$ E_0^{\alpha,p} = \{ y : [0, T] \to \mathbb{R} | y, {}_0^cD_t^\alpha y \in L^p([0, T], \mathbb{R}), y(0) = y(T) = 0, p \geq 2 \} $$

*is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$ with the norm:*

$$ \|y\|_{\alpha,p} = \left( \int_0^T |y(t)|^p dt + \int_0^T |{}_0^cD_t^\alpha y(t)|^p dt \right)^{\frac{1}{p}}. \tag{2.1} $$

**Lemma 2.5.** *([6]) Let $\alpha > 0$, $p \geq 1$, $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} \leq \alpha + 1$ ($p$, $q \neq 1$ under the case of $\frac{1}{p} + \frac{1}{q} = \alpha + 1$). If $y \in L_p(0, T)$ and $x \in L_q(0, T)$, then, we have*

$$ \int_0^T ({}_aD_t^{-\alpha} y(t)) x(t) dt = \int_0^T y(t) ({}_aD_t^{-\alpha} x(t)) dt. $$

**Lemma 2.6.** *([6]) Let $0 < \alpha \leq 1$ and $1 < p < +\infty$. For any $E_0^{\alpha,p}$, we have*

$$||y||_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} ||{}_0^c D_t^\alpha y||_{L^p}. \tag{2.2}$$

*If $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$||y||_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} ||{}_0^c D_t^\alpha y||_{L^p}. \tag{2.3}$$

for $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Since $\omega(t) \in C^\infty[0,T]$, the norm of (2.1) is equivalent to

$$\|y\| = \Big( \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} |y(t)|^p dt + \int_0^T \omega(t) |{}_0^c D_t^\alpha y(t)|^p dt \Big)^{\frac{1}{p}}. \tag{2.4}$$

Let $u \in E_0^{\alpha,p}$ and $0 < (w^0)^{\frac{1}{p}} \leq 1$, from Lemma 2.6, we obtain

$$||y||_{L^p} \leq \Lambda_1 \|y\|, \tag{2.5}$$

and

$$||y||_\infty \leq \Lambda_2 \|y\|, \tag{2.6}$$

where $\Lambda_1 = \frac{T^\alpha}{\Gamma(\alpha+1)(w^0)^{\frac{1}{p}}}$, $\Lambda_2 = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}(w^0)^{\frac{1}{p}}}$ .

*Proof.* From (2.2), for any $u \in E_0^{\alpha,p}$ and $\omega(t) \in C^\infty[0,T]$, we easy have

$$\begin{aligned}
||y||_{L^p} &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left( \int_0^T |{}_0^c D_t^\alpha y(t)|^p dt \right)^{\frac{1}{p}} \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)(w^0)^{\frac{1}{p}}} \left( \int_0^T \omega(t) |{}_0^c D_t^\alpha y(t)|^p dt \right)^{\frac{1}{p}} \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)(w^0)^{\frac{1}{p}}} \Big( \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} |y(t)|^p dt + \int_0^T \omega(t) |{}_0^c D_t^\alpha y(t)|^p dt \Big)^{\frac{1}{p}} \\
&= \Lambda_1 \|y\|.
\end{aligned}$$

The same can be proved (2.6). □

**Lemma 2.7.** *([7]) Let $\frac{1}{2} < \alpha \leq 1$, assume that the sequence $\{y_k\}$ converges weakly to $y \in E_0^{\alpha,p}$,i.e., $y_k \rightharpoonup y$. Then we have that $\{y_k\}$ converges strongly to $y \in C([0,T],R)$,i,e., $\|y_k - y\| \to 0$ as $k \to \infty$.*

**Lemma 2.8.** *([7]) The functional $y \in E_0^{\alpha,p}$ is a weak solution of (1.2) if and only $y$ is a classical solution of (1.2).*

**Definition 2.9.** *We say that $y(t) \in E_0^{\alpha,p}$ is a weak solution of problem (1.2), if the relationship defined below is satisfied:*

$$\int_0^T \frac{1}{\omega(t)^{p-2}}\Phi_p\left(\omega(t){}_0^cD_t^\alpha y(t)\right)\left({}_0^cD_t^\alpha x(t)\right)dt+\sum_{j=0}^m\int_{s_j}^{t_{j+1}}|y(t)|^{p-2}y(t)x(t)dt+\mu\sum_{j=1}^m I_j\left(y(t_j)\right)x(t_j)$$
$$=\lambda\sum_{j=0}^m\int_{s_j}^{t_{j+1}} f_j(t,y(t))x(t)dt,\ \ x(t)\in Y,$$

*where $F_j(t,y)=\int_0^y f_j(t,s)ds$.*

Consider the functional $\varphi : E_0^{\alpha,p} \to \mathbb{R}$ as follow

$$\varphi(y)=\frac{1}{p}\|y\|^p+\mu\sum_{j=1}^m\int_0^{y(t_j)} I_j(s)ds-\lambda\sum_{j=0}^m\int_{s_j}^{t_{j+1}} F_j(t,y(t))dt, \tag{2.7}$$

we esay deduce that $\varphi$ is continuous and differentiable, we have

$$\begin{aligned}\langle\varphi'(y),x\rangle=&\int_0^T\frac{1}{\omega(t)^{p-2}}\Phi_p\left(\omega(t){}_0^cD_t^\alpha y(t)\right)\left({}_0^cD_t^\alpha x(t)\right)dt+\mu\sum_{j=1}^m I_j\left(y(t_j)\right)x(t_j)\\&+\sum_{i=0}^m\int_{s_j}^{t_{j+1}}|y(t)|^{p-2}y(t)x(t)dt-\lambda\sum_{j=0}^m\int_{s_j}^{t_{j+1}} f_j(t,y(t))x(t)dt.\end{aligned}\tag{2.8}$$

Then we obviously obtain that the critical points of $\varphi$ are the weak solutions of the problem (1.2).

**Definition 2.10.** *([8]) Let $Y$ be a real Banach space and $\varphi \in C^1(Y,\mathbb{R})$ satisfy the Palais-Smale condition, i.e., every sequence $y_j \in Y$ for which $\{\varphi(y_j)\}$ is boundedand $\varphi'(y_j) \to 0$ as $j \to 0$ possesses a convergent subsequence in $Y$ and $\varphi(0)=0$.*

We denote $B_\rho$ be the open ball in $Y$ with the radius $\rho$ and centered at 0 and its boundary defined by $\partial B_\rho$.

**Theorem 2.11.** *([9]) Let $Y$ be a real Banach space, and let $\varphi \in C^1(Y,\mathbb{R})$ be even satisfying the Palais-Smale condition and $\varphi(0)=0$. If $Y=U\bigoplus V$, where $U$ is finite dimensional, and $\varphi$ satisfies that:*
*(i) There exist constants $\rho,\eta>0$ such that $\varphi|\partial B_\rho\bigcap V\geq\eta$;*
*(ii) For each finite dimensional subspace $W\subset Y$, there is $r=r(W)$ such that $\varphi(y)\leq 0$ for all $y\in W$ with $\|y\|\geq r$.*
*Then $\varphi$ possesses an unbounded sequence of critical values.*

# 3 The main existence result

To prove the existence of infinitely many classical solutions to problem (1.2), we need the following auxiliary hypotheses:
(H1) There exists $\gamma > p$, such that for all $t \in (s_j, t_{j+1}]$ and $y \in \mathbb{R}\backslash\{0\}$,

$$0 < \gamma F_j(t, y) \leq f_j(t, y)y,$$

where $F_j(t, y) = \int_0^y f_j(t, s)ds$.
(H2) $f_j(t, y)$ and $I_j(y)$ are odd about $y$ .
(H3) there exist $c_j, d_j > 0$ and $\delta_j \in [0, 1)$ such that for any $y \in R$ and $j = 1, 2, \cdots m$, we have

$$|I_j(y)| \leq c_j + d_j|y|^{\delta_j}.$$

(H4) The following inequality

$$(\frac{1}{p\Lambda_2^p} - \frac{\lambda M_1 \Lambda_1^p}{\Lambda_2^p}) - \mu \sum_{j=1}^{m}(c_j + d_j) > 0$$

hods, where $\Lambda_1$ and $\Lambda_2$ is defined in (2.5), (2.6) and $M_1 = max\{\overline{M_1}, \overline{M_2}, \cdots, \overline{M_m}\}$, where $\overline{M}_j = sup\{F_j(t, y)|t \in (s_j, t_{j+1}], j = 1, 2, \cdots, m, |y| = 1\} > 0$.
(H5) There exist constants $L_j > 0$, such that for any $u, v \in \mathbb{R}, j = 1, 2, m$, we have

$$|I_j(u) - I_j(v)| \leq L_j|u - v|,$$

where $0 < \sum_{j=1}^{m} L_j < \frac{\gamma-p}{p\Lambda_2^p(\gamma+1)}$.
(H6) The following inequality

$$(\frac{1}{p} - \lambda M_1 \Lambda_1^p)\frac{1}{\Lambda_2^p} - \sum_{j=1}^{m} \mu L_j - \mu \sum_{j=1}^{m} |I_j(0)| > 0$$

holds.
Now we are ready to prove the following result:

**Theorem 3.1.** *Assume that (H1)-(H4) hold, then the problem (1.2) has infinitely many classical solutions.*

*Proof.* It is clear to see that $\varphi \in C^1(E_0^{\alpha,p}, \mathbb{R})$ is an even functional and $\varphi(0) = 0$. Then we will apply Theorem 2.11 to show Theorem 3.1, process of proof followling there three steps.

Step 1, we need to prove that $\varphi$ satisfies the P.S. condition. Let $\{y_k\} \in E_0^{\alpha,p}$ such that $\{\varphi(y_k)\}$ be a bounded sequence and $\lim_{k\to\infty} \varphi'(y_k) = 0$. Assume that there exists a constant $C_1$ such that

$$|\varphi(y_k)| \leq C_1, \quad \| \varphi'(y_k) \| \leq C_1. \tag{3.1}$$

From the functional $\varphi$ and (H1), we have

$$\begin{aligned}\|y_k\|^p =& p\varphi(y_k) - p\mu\sum_{j=1}^{m}\int_0^{y_k(t_j)} I_j(s)ds + p\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,y_k(t))dt\\ \leq& p\varphi(y_k) - p\mu\sum_{j=1}^{m}\int_0^{y_k(t_j)} I_j(s)ds + \frac{p\lambda}{\gamma}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j(t,y_k(t))y_k(t)dt,\end{aligned}$$

together with $\langle\varphi'(y),x\rangle$ we immediately have

$$\begin{aligned}(1-\frac{p}{\gamma})\|y_k\|^p \leq& p\varphi(y_k) - p\mu\sum_{j=1}^{m}\int_0^{y_k(t_j)} I_j(s)ds + \frac{p\lambda}{\gamma}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j(t,y_k(t))y_k(t)dt\\ &- \frac{p}{\gamma}\varphi'(y_k)y_k + \frac{\mu p}{\gamma}\sum_{j=1}^{m} I_j(y_k(t_j))y_k(t_j) - \frac{p\lambda}{\gamma}\sum_{j=0}^{m}\int_{s_j}^{t_j+1} f_j(t,y_k(t))y_k(t)dt\\ =& p\varphi(y_k) - \frac{p}{\gamma}\varphi'(y_k)y_k - p\mu\sum_{j=1}^{m}\int_0^{y_k(t_j)} I_j(s)ds + \frac{\mu p}{\gamma}\sum_{j=1}^{m} I_j(y_k(t_j))y_k(t_j).\end{aligned}$$

By (2.6), (3.1) and (H3), we obtain

$$\begin{aligned}(1-\frac{p}{\gamma})\|y_k\|^p \leq& pC_1 + p\mu\|y_k\|_\infty\sum_{j=1}^{m}(c_j + d_j\|y_k\|_\infty^{\delta_j})\\ &+ \frac{p}{\gamma}C_1\|y_k\|_\infty + \frac{\mu p}{\gamma}\|y_k\|_\infty\sum_{j=1}^{m}(c_j + d_j\|y_k\|_\infty^{\delta_j})\\ \leq& pC_1 + \mu p\Lambda_2\|y_k\|\sum_{j=1}^{m}(c_j + d_j\Lambda_2^{\delta_j}\|y_k\|^{\delta_j})\\ &+ \frac{p}{\gamma}C_1\Lambda_2\|y_k\| + \frac{\mu p\Lambda_2}{\gamma}\|y_k\|\sum_{j=1}^{m}(c_j + d_j\Lambda_2^{\delta_j}\|y_k\|^{\delta_j}),\end{aligned}$$

this is implies that $\{y_k\}$ is bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is reflexive space, then we may choose a weakly convergent subsequence, we denote $\{y_k\}$ and $y_k \rightharpoonup y$ in $E_0^{\alpha,p}$, then we will prove that $y_k \to y$ in $E_0^{\alpha,p}$. By (2.8) and Schwarz inequality, we have

$$\langle\varphi'(y_k) - \varphi'(y), y_k - y\rangle \leq \|\varphi'(y_k)\|\|y_k - y\| - \langle\varphi'(y_k), -y_k - y\rangle \longrightarrow 0, \quad (3.2)$$

as $k \longrightarrow \infty$. On the other hand, by Lemma 2.7, we know $\|y_k - y\|_\infty \to 0$ as

$k \to \infty$, since

$$
\begin{aligned}
0 &\leftarrow \langle \varphi'(y_k) - \varphi'(y), y_k - y \rangle \\
&= \|y_k - y\|^p - \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} [f_j(t, y_k) - f_j(t, y)](y_k - y)dt \\
&+ \mu \sum_{j=1}^{m} [I_j(y_k(t_j)) - I_j(y(t_j))](y_k(t_j) - y(t_j)) \\
&\geq \|y_k - y\|^p - \lambda |\sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} [f_j(t, y_k) - f_j(t, y)]dt| \|y_k - y\|_\infty \\
&- \mu \sum_{j=1}^{m} [I_j(y_k(t_j)) - I_j(y(t_j))] \|y_k - y\|_\infty,
\end{aligned}
$$

by (3.2), we immediately deduce that $\|y_k - y\| \to 0$ as $k \to \infty$, this implies that $\{y_k\}$ converges strongly to $y \in E_0^{\alpha,p}$. So $\varphi$ satisfies the P.S.condition.

Step 2, we will show that the condition (i) of Theorem 2.11 holds. Assume that $U = R$ and $V = \{y \in E_0^{\alpha,p} | \sum\limits_{j=0}^{m} \int_{s_j}^{t_{j+1}} y(t)dt = 0\}$, then $E_0^{\alpha,p} = U \bigoplus V$, where $dimU = 1 < +\infty$. Suppose that $0 < \|y\|_\infty \leq 1$, if $(H1)$ and $(H2)$ hold, then the following inequalities

$$
F_j(t, y) \leq F_j(t, \frac{y}{|y|})|y|^\gamma, 0 < |y| \leq 1,
$$

$$
F_j(t, y) \geq F_j(t, \frac{y}{|y|})|y|^\gamma, |y| \geq 1,
$$

hold, which implies that $f$ is superquadratic at infinity, subquadratic at the origin. So we deduce that

$$
\sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, y(t))dt \leq \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, \frac{y}{|y|})|y|^\gamma dt \leq M_1 \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} |y|^p dt \leq M_1 \Lambda_1^p \|y\|^p. \tag{3.3}
$$

By (2.5), (2.6), (2.7), (3,3) and (H3), we have

$$
\begin{aligned}
\varphi(y) &= \frac{1}{p}\|y\|^p + \mu \sum_{j=1}^{m} \int_0^{y(t_j)} I_j(s)ds - \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, y(t))dt \\
&\geq \frac{1}{p}\|y\|^p - \mu\Lambda_2\|y\| \sum_{j=1}^{m} (c_j + d_j \Lambda_2^{\delta_j} \|y\|^{\delta_j}) - \lambda M_1 \Lambda_1^p \|y\|^p \\
&= (\frac{1}{p} - \lambda M_1 \Lambda_1^p)\|y\|^p - \mu \sum_{j=1}^{m} (c_j \Lambda_2 \|y\| + d_j \Lambda_2^{\delta_j+1} \|y\|^{\delta_j+1}).
\end{aligned}
$$

Let $\|y\| = \rho = \frac{1}{\Lambda_2}$, from (2.5), we have $0 < \|y\|_\infty \leq 1$. So

$$\varphi(y) \geq (\frac{1}{p\Lambda_2^p} - \frac{\lambda M_1 \Lambda_1^p}{\Lambda_2^p}) - \mu \sum_{j=1}^{m}(c_j + d_j),$$

let $\eta = (\frac{1}{p\Lambda_2^p} - \frac{\lambda M_1 \Lambda_1^p}{\Lambda_2^p}) - \mu \sum_{j=1}^{m}(c_j + d_j)$, then from (H4), we have $\varphi(y) \geq \eta > 0$ for any $y \in \partial B_\rho \bigcap V$.

Step 3, we will prove that the condition (ii) of Theorem 2.12 holds. By (H1), let $M_{2j} > 0$ such that for any $y \geq M_{2j} > 0$ and $t \in (s_j, t_{j+1}]$, we have

$$(\frac{F_j(t,y)}{y^\gamma})'_y = \frac{y^\gamma f_j(t,y) - \gamma y^{\gamma-1}F_j(t,y)}{y^2\gamma} = \frac{yf_j(t,y) - \gamma F_j(t,y)}{y^{\gamma+1}} \geq 0,$$

it implies that $\frac{F_j(t,y)}{y^\gamma}$ is increasing for $y$, so we deduce that

$$\frac{F_j(t,y)}{y^\gamma} \geq \frac{F_j(t, M_{2j})}{M_2^\gamma} \geq C_{2j},$$

where $C_{2j} = M_{2j}^{-\gamma} inf_{t\in(s_j,t_{j+1}]}\{F_j(t, M_{2j})\}$, this yields $F_j(t,y) \geq C_{2j}|y|^\gamma$ for any $y \geq M_{2j} > 0$ and $t \in (s_j, t_{j+1}]$. Using the same argument, we have $F_j(t,y) \geq C_{3j}|y|^\gamma$ for any $y \leq -M_{2j}$ and $t \in (s_j, t_{j+1}]$,where $C_{3j} > 0$. Owing to the continuity of $F_j(t,y)$ on $t \in (s_j, t_{j+1}] \times [-M_{2j}, M_{2j}]$, then exists $C_{5j} > 0$ such that $F_j(t,y) \geq C_{4j}|y|^\gamma - C_{5j}$ for any $(t,y) \in (s_j, t_{j+1}] \times [-M_{2j}, M_{2j}]$, where $C_{4j} = min\{C_{2j}, C_{3j}\}$, So we obtain

$$F_j(t,y) \geq C_{4j}|y|^\gamma - C_{5j}, \tag{3.4}$$

for any $(t,y) \in (s_j, t_{j+1}] \times R$. Let $E_1$ is any finite dimensional subspace in $E_0^{\alpha,p}$, then for each $\xi \in R\backslash\{0\}$ and $y \in E_1\backslash\{0\}$, combining with (2.6), (2.8), (H3) and (3.4), we obtain

$$\begin{aligned}
\varphi(\xi y) &= \frac{1}{p}\|\xi y\|^p + \mu \sum_{j=1}^{m} \int_0^{\xi y(t_j)} I_j(s)ds - \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} F_j(t, \xi y(t))dt \\
&\leq \frac{1}{p}\|\xi y\|^p + \mu\|\xi y\|\Lambda_2 \sum_{j=1}^{m}(c_j + d_j\Lambda_2^{\delta_j}\|\xi y\|^{\delta_j}) - \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} (C_{4j}|\xi y|^\gamma - C_{5j})dt \\
&= \frac{1}{p}\|\xi y\|^p + \mu\|\xi y\|\Lambda_2 \sum_{j=1}^{m}(c_j + d_j\Lambda_2^{\delta_j}\|\xi y\|^{\delta_j}) + \lambda T C_{5j} - \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} C_{4j}|\xi y|^\gamma dt.
\end{aligned}$$

Let $\tau \in E_1$ such that $\|\tau\| = 1$, since $\gamma > p$, the above inequality implies that there exists a sufficiently large $\xi$ such that $\|\xi\tau\| > \rho$ and $\varphi(\xi y) < 0$. Since $E_1$ is a finite dimensional subspace in $E_0^{\alpha,p}$, there exists $T(E_1) > 0$, such that $\varphi(y) \leq 0$ on $E_1\backslash B_{T(E_1)}$. By Theorem 2.11, $\varphi$ has infinitely many critical points that is the system (1.2) has infinitely many classical solutions. $\square$

**Theorem 3.2.** *Assume that (H1), (H2), (H5) and (H6) hold, then the problem (1.2) has infinitely many classical solutions.*

*Proof.* It is obvious to see that $\varphi \in C^1(E_0^{\alpha,p}, \mathbb{R})$ is an even functional and $\varphi(0) = 0$. Then we will apply Theorem 2.11 to show Theorem 3.2, process of proof followling there three steps.

Step 1, we need to prove that the functional $\varphi$ satisfies the P.S. condition. As in the proof of Theorem 3.1, by (2.6), (2.7), (2.8), (3.1) together with (H1) and (H5), we have

$$\begin{aligned}(1-\frac{p}{\gamma})\|y_k\|^p \le{}& p\varphi(y_k) - p\mu\sum_{j=1}^{m}\int_0^{y_k(t_j)} I_j(s)ds + \frac{p\lambda}{\gamma}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j(t,y_k(t))y_k(t)dt\\ &- \frac{p}{\gamma}\varphi'(y_k)y_k + \frac{\mu p}{\gamma}\sum_{j=1}^{m} I_j(y_k(t_j))y_k(t_j) - \frac{p\lambda}{\gamma}\sum_{j=0}^{m}\int_{s_j}^{t_j+1} f_j(t,y_k(t))y_k(t)dt\\ ={}& p\varphi(y_k) - \frac{p}{\gamma}\varphi'(y_k)y_k - p\mu\sum_{j=1}^{m}\int_0^{y_k(t_j)} I_j(s)ds + \frac{\mu p}{\gamma}\sum_{j=1}^{m} I_j(y_k(t_j))y_k(t_j)\\ \le{}& p\varphi(y_k) + \frac{p}{\gamma}\varphi'(y_k)y_k + p\mu\sum_{j=1}^{m}\int_0^{y_k(t_j)} I_j(s)ds + \frac{\mu p}{\gamma}\sum_{j=1}^{m} I_j(y_k(t_j))y_k(t_j)\\ \le{}& pC_1 + P\|u_k\|_\infty\sum_{j=0}^{m}(|I_j(0)| + L_j\|y_k\|_\infty)\\ &+ \frac{p}{\gamma}C_1\|y_k\|_\infty + \frac{p}{\gamma}\|y_k\|_\infty\sum_{j=1}^{m} I_j(y_k(t_j))y_k(t_j)\end{aligned}$$

$$\begin{aligned}&\le pC_1 + p\Lambda_2\|y_k\|\sum_{j=0}^{m}(|I_j(0)| + L_j\Lambda_2\|y_k\|)\\ &+ \frac{p}{\gamma}C_1\Lambda_2\|y_k\| + \frac{p\Lambda_2}{\gamma}\|y_k\|\sum_{j=1}^{m}(|I_j(0)| + L_j\Lambda_2\|y_k\|),\end{aligned}$$

then we have

$$[1-\frac{p}{\gamma}-p(1+\frac{1}{\gamma})\Lambda_2^2\sum_{j=0}^{m} L_1]\|y_k\|^p \le pC_1 + [\frac{pC_1\Lambda_2}{\gamma} + p\Lambda_2(1+\frac{1}{\gamma})\sum_{j=0}^{m}|I_j(0)|]\|y_k\|,$$

from (H5), we deduce that $1-\frac{p}{\gamma}-p(1+\frac{1}{\gamma})\Lambda_2^2\sum\limits_{j=0}^{m} L_1 > 0$, that implies that $\{y_k\}$ is bounded in $E_0^{\alpha,p}$. The rest of the proof of the P.S. condition is similar to that in Theorem 3.1, we omit it here.

Step 2, we will show that the condition (i) of Theorem 2.11 holds. As in the proof of Theorem 3.1, by (2.5), (2.6), (2.7), (3.3) together with (H4), we have

$$\begin{aligned}\varphi(y) &= \frac{1}{p}\|y\|^p + \mu\sum_{j=1}^{m}\int_0^{y(t_j)} I_j(s)ds - \lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,y(t))dt \\ &\geq \frac{1}{p}\|y\|^p - \mu\Lambda_2\|y\|\sum_{j=1}^{m}(|I_j(0)| + L_j\Lambda_2\|y\|) - \lambda M_1\Lambda_1^p\|y\|^p \\ &= (\frac{1}{p} - \lambda M_1\Lambda_1^p)\|y\|^p - \sum_{j=1}^{m}\mu\Lambda_2^2 L_j\|y\|^2 - \mu\Lambda_2\sum_{j=1}^{m}|I_j(0)|\|y\|.\end{aligned}$$

Let$\|y\| = \rho = \frac{1}{\Lambda_2}$, from (2.6), we have $0 < \|u\|_\infty \leq 1$, so

$$\varphi(u) \geq (\frac{1}{p} - \lambda M_1\Lambda_1^p)\frac{1}{\Lambda_2^p} - \sum_{j=1}^{m}\mu L_j - \mu\sum_{j=1}^{m}|I_j(0)|,$$

let $\eta = (\frac{1}{p} - \lambda M_1\Lambda_1^p)\frac{1}{\Lambda_2^p} - \sum\limits_{j=1}^{m}\mu L_j - \mu\sum\limits_{j=1}^{m}|I_j(0)|$, then from (H6), we have $\varphi(u) \geq \eta > 0$ for any $u \in \partial B_\rho \cap V$.

Step 3, we will prove that the condtion (ii) of Theorem 2.11 holds. As in the proof of Theorem 3.1, we assume that $E_1$ is any finite dimensional subspace in $E_0^{\alpha,p}$, then for each $\xi \in R\backslash\{0\}$ and $u \in E1\backslash\{0\}$, combining with (2.6), (2.7), (H5) and (3.4), we obtain

$$\begin{aligned}\varphi(\xi y) &= \frac{1}{p}\|\xi y\|^p + \mu\sum_{j=1}^{m}\int_0^{\xi y(t_j)} I_j(s)ds - \lambda\int_0^T F_j(t,\xi y(t))dt \\ &\leq \frac{1}{p}\|\xi y\|^p + \mu\Lambda_2\|\xi y\|\sum_{j=1}^{m}(|I_j(0)| + L_j\Lambda_2\|\xi y\|) - \lambda\int_0^T (C_{4j}|\xi y|^\gamma - C_{5j})dt, \\ &= \frac{1}{p}\|\xi y\|^p + \mu\Lambda_2\sum_{j=1}^{m}|I_j(0)|\|\xi y\| + \mu\Lambda_2^2\sum_{j=1}^{m}L_j\|\xi y\|^2 + \lambda TC_{5j} - \lambda\int_0^T C_{4j}|\xi y|^\gamma dt.\end{aligned}$$

Same as theorem 3.1 such that $\varphi(y) \leq 0$. By Theorem 2.11, $\varphi$ has infinitely many critical points that is the system (1.2) has infinitely many classical solutions. □

# 4 Example

In this part, we will give corresponding examples to illustrate the main results in our paper.

**Example 4.1** Let $\alpha = \frac{1}{2}$, $T = 1$, $p = 3$, $\lambda = \frac{1}{20}$, $\mu = 2$ and $m = 1$. consider the following fractional BVP:

$$\begin{cases} {}_tD_1^{\frac{1}{2}}\Big(\frac{1}{\omega(t)}\Phi_3\big(\omega(t){}_0^cD_t^{\frac{1}{2}}y(t)\big)\Big)+|y(t)|y(t)=\frac{1}{20}f_j(t,y(t)), \quad t\in(s_j,t_{j+1}], j=0,1\\ \triangle_tD_1^{-\frac{1}{2}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{1}{2}}y\big)\Big)(t_1)=2I_1\left(y(t_1)\right),\\ {}_tD_1^{-\frac{1}{2}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{1}{2}}y\big)\Big)(t)={}_tD_1^{-\frac{1}{2}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{1}{2}}y\big)\Big)(t_j^+), \quad t\in(t_j,s_j]j=0,1,\\ {}_tD_1^{-\frac{1}{2}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{1}{2}}y\big)\Big)(s_1^-)={}_tD_1^{-\frac{1}{2}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{1}{2}}y\big)\Big)(s_1^+),\\ y(0)=y(1)=0, \end{cases} \tag{4.1}$$

where $f_1(t,y)=y^3+ty^5$ and $I_1(y)=\frac{1}{20}|y|^{\frac{1}{2}}siny$, then we obtain that $f_1(t,y)$ and $I_1$ are odd about $y$, so $(H2)$ holds, Then we assume $\gamma=4$,$c_1=0$,$d_1=\frac{1}{20}$,$\delta_1=\frac{1}{2}$,then by verification,we obtain that $(H1)$ and $(H3)$ hold. By simple calculations, let $\omega^0=1$, we know $\Lambda_1=\Lambda_2\approx 1.27329545$, $M_1=1$ and $(\frac{1}{3\Lambda_2^3}-\frac{\lambda M_1\Lambda_1^3}{\Lambda_2^3})-2(c_1+d_1)\approx 0.01147684>0$, so $(H5)$ holds. By $Theorem 3.1$, (4.1) has infinitely many solutions.

**Example 4.2** Let $\alpha=\frac{4}{5}$, $T=1, p=3$, $\lambda=\frac{1}{10}$, $\mu=2$ and $m=1$ then consider the following fractional differential equations:

$$\begin{cases} {}_tD_1^{\frac{4}{5}}\Big(\frac{1}{\omega(t)}\Phi_3\big(\omega(t){}_0^cD_t^{\frac{4}{5}}y(t)\big)\Big)+|y(t)|y(t)=\frac{1}{10}f_j(t,y(t)), \quad t\in(s_j,t_{j+1}], j=0,1\\ \triangle_tD_1^{-\frac{1}{5}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{4}{5}}y\big)\Big)(t_1)=2I_1\left(y(t_1)\right),\\ {}_tD_1^{-\frac{1}{5}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{4}{5}}y\big)\Big)(t)={}_tD_1^{-\frac{1}{5}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{4}{5}}y\big)\Big)(t_j^+), \quad t\in(t_j,s_j]j=0,1,\\ {}_tD_1^{-\frac{1}{5}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{4}{5}}y\big)\Big)(s_1^-)={}_tD_1^{-\frac{1}{5}}\Big(\frac{1}{\omega}\Phi_3\big(\omega_0^cD_t^{\frac{4}{5}}y\big)\Big)(s_1^+),\\ y(0)=y(1)=0, \end{cases} \tag{4.2}$$

where $f_1(t,y)=y^3+ty^5$ and $I_1(y)=\frac{1}{36}|y|siny$, then we obtain that $f_1(t,y)$ and $I_1(y)$ are odd about $y$, so $(H2)$ holds. Then we assume $\gamma=4$, $L_1=\frac{1}{24}$. By a some calculation, let $\omega^0=1$, we know $\Lambda_1=\Lambda_2\approx 0.931384$, $M_1=1$, $\frac{\gamma-p}{p\Lambda_2^p(\gamma+1)}-L_1\approx 0.040846>0$ and $(\frac{1}{p}-\lambda M_1\Lambda_1^p)\frac{1}{\Lambda_2^p}-\mu L_1-\mu|I_1(0)|\approx 0.229232>0$, so we immediately have $(H1)$, $(H5)$, $(H6)$ hold. By $Theorem 3.2$, (4.2) has infinitely many solutions.

# References

[1] Z. Zhao, L. Chen, X. Song, Impulsive vaccination of SEIR epidemic model with time delay and nonlinear incidence rat, *Mathematics and Computers in Simulation*, **79** (2008), 500-510.
https://doi.org/10.1016/j.matcom.2008.02.007

[2] L. Stone, B. Shulgin, Z. Agur, Theoretical examination of the pulse vaccination policy in the SIR epidemic model, *Mathematics and Computers in Simulation*, **31** (2000), 207-215.
https://doi.org/10.1016/s0895-7177(00)00040-6

[3] E. Joelianto, H. Sutarto, Controlled switching dynamical systems using linear impulsive differential equations, *Intelligent Unmanned Systems: Theory and Applications*, **2009** (2009), 227-244.
https://doi.org/10.1007/978-3-642-00264-9_15

[4] S. Ahmad, I. Stamova, Asymptotic stability of an N-dimensional impulsive competitive system, *Nonlinear Analysis: Real World Applications*, **8** (2007), 654-663. https://doi.org/10.1016/j.nonrwa.2006.02.004

[5] Y.Zhou, JR Wang, L. Zhang, *Basic Theory of Fractional Differential Equations*, World Scientific, (2016). https://doi.org/10.1142/10238

[6] Y. Qiao, F. Chen, Y. An, Variational method for p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses, *Mathematical Methods in the Applied Sciences*, **44** (2021), 8543-8553. https://doi.org/10.1002/mma.7276

[7] D Gao, J Li, Applications of variational methods to a impulsive fractional differential equation with a parameter, *Dynamic Systems and Applications*, **27** (2018), no. 4, 973-987. https://doi.org/10.12732/dsa.v27i4.17

[8] D Gao, J Li, Infinitely many solutions for impulsive fractional differential equations through variational methods, *Quaestiones Mathematicae*, **43** (2020), no. 9, 1285-1301.
https://doi.org/10.2989/16073606.2019.1609619

[9] J Zhou, Y Li, Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, *Nonlinear Analysis: Theory, Methods Applications*, **71** (2009), no. 7-8, 2856-2865.
https://doi.org/10.1016/j.na.2009.01.140

[10] N. Nyamoradi, E. Tayyebi, Existence of Solutions for a Class of Fractional Boundary Value Equations with Impulsive Effects via Critical Point Theory, *Mediterranean Journal of Mathematics*, **15** (2018), no. 3, 79. https://doi.org/10.1007/s00009-018-1122-z

[11] F. Kamache, R. Guefaifia, S. Boulaaras, A. Alharbi, Existence of Weak Solutions for a New Class of Fractional p-Laplacian, Boundary Value Systems, *Mathematics*, **8** (2020), 475.
https://doi.org/10.3390/math8040475

[12] F. Jiao, Y. Zhou, Existence Results for Fractional Boundary Value Problem via Critical Point Theory, *International Journal of Bifurcation Chaos*, **22** (2012), no. 04, 368-224. https://doi.org/10.1142/S0218127412500861

[13] F. Jiao, Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, *Computers Mathematics with Applications*, **62** (2011), no. 3, 1181-1199. https://doi.org/10.1016/j.camwa.2011.03.086

[14] R.R. Lopez, S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, *Fractional Calculus Applied Analysis*, **17** (2014), no. 4, 1016-1038. https://doi.org/10.2478/s13540-014-0212-2

[15] G. Bonanno, R.R. Lopez, Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fractional Calculus and Applied Analysis*, **17** (2014), no. 3, 717-744. https://doi.org/10.2478/s13540-014-0196-y

[16] C. Jing, X.H. Tang, Existence and Multiplicity of Solutions for Some Fractional Boundary Value Problem via Critical Point Theory, *Abstract Applied Analysis*, **3** (2013), 185-195. https://doi.org/10.1155/2012/648635

[17] Z. Zhang, Y. Rong, An application of variational methods to Dirichlet boundary value problem with impulses, *Nonlinear Analysis Real World Applications*, **11** (2010), no. 1, 155-162. https://doi.org/10.1016/j.nonrwa.2008.10.044

[18] Y. Zhao, H. Chen, Q. Zhang, Infinitely many solutions for fractional differential system via variational method, *Journal of Applied Mathematics and Computing*, **50** (2016), no. 1, 589-609. https://doi.org/10.1007/s12190-015-0886-6

[19] B. Liang, B. Dai, Existence and multiplicity of solutions for an impulsive boundary value problem with a parameter via critical point theory, *Mathematical and Computer Modelling*, **53** (2011), no. 9-10, 1844-1855. https://doi.org/10.1016/j.mcm.2011.01.006

[20] E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, *Proceedings of the American Mathematical Society*, **141** (2013), 1641-1649. https://doi.org/10.1090/S0002-9939-2012-11613-2

[21] K. Liu, J. Wang, D. O'Regan, M. Fečkan, A New Class of $(\omega, c)$-Periodic Non-instantaneous Impulsive Differential Equations, *Mediterranean Journal of Mathematics*, **17** (2020), 155. https://doi.org/10.1007/s00009-020-01574-8

[22] R. Agarwal, S. Hristova, D. O'Regan, Non-instantaneous impulses in Caputo fractional differential equations, *Fractional Calculus and Applied Analysis*, **20** (2017), 595-622. https://doi.org/10.1515/fca-2017-0032

[23] W. Zhang, W. Liu, Variational approach to fractional Dirichlet problem with instantaneous and non-instantaneous impulses, *Applied Mathematics Letters*, **99** (2019). https://doi.org/10.1016/j.aml.2019.07.024

[24] A. Khaliq, M.U. Rehman, On variational methods to non-instantaneous impulsive fractional differential equation, *Applied Mathematics Letters*, **83** (2018). https://doi.org/10.1016/j.aml.2018.03.014

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