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Existence of Solutions of a Class of p-Laplacian Type Fractional Impulsive Differential Equation with Boundary Value Problem

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Abstract

In this paper, we study a class of p-Laplacian type fractional impulsive differential equation with boundary value problem. The existence of solutions is obtained by using the fixed point theorem. Finally, we present two examples to illustrate our main result.

Keywords: Fractional impulsive differential equation; P-Laplacian operator; Boundary value problem; Fixed point theorem

1 Introduction

In this paper, we consider the p-Laplacian fractional differential equation with boundary value problem

$$\begin{cases} {}^cD^{\beta}_{0^+}\phi_p\left({}^cD^{\alpha}_{0^+}u(t)\right) = f(t,\ u(t)), \quad t \in J' = J \setminus (t_1,\ t_2,\ \cdots,\ t_m),\ J = [0,\ 1], \\ \Delta(u(t_k)) = I_k u(t_k),\ \Delta(u'(t_k)) = J_k u(t_k),\ \Delta^{''}(u(t_k)) = Q_k u(t_k), \\ au(0) + bu(1) = 0,\ au'(0) + bu'(1) = 0, \\ au^{''}(0) + bu^{''}(1) = 0,\ {}^cD^{\alpha}_{0^+}u(0) +^c D^{\alpha}_{0^+}u(1) = 0, \end{cases} \tag{1.1}$$

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where ${}^cD^{\alpha}_{0^+}$, ${}^cD^{\beta}_{0^+}$ are Caputo fractional derivative. $\phi_p(s) = |s|^{p-2}s$ is p-Laplacian operator and satisfies $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $\phi_p^{-}(s) = \phi_q(s)$. a, b are two real constants with $a > b > 0$. $0 < \beta \leq 1$, $2 < \alpha \leq 3$, $k = 1,\ 2,\ \cdots,\ m$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. $f \in C(J \times R,\ R)$, $I_k(\cdot)$, $J_k(\cdot)$, $Q_k(\cdot) \in C(R,\ R)$. $I_k(u(t_k)) = u(t_k^+) - u(t_k^-)$, $J_k(u(t_k)) = u'(t_k^+) - u'(t_k^-)$, $Q_k(u(t_k)) = u''(t_k^+) - u''(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at the impulsive point $t = t_k (k = 1, 2, 3, \cdots, p)$, respectively, $u'(t_k^+)$, $u'(t_k^-)$, $u''(t_k^+)$, and $u''(t_k^-)$ have a similar meaning at $t = t_k (k = 1, 2, 3, \cdots, p)$.

Compared with integer order differential equations, fractional order differential equations can better describe some natural physical phenomena, such as viscoelasticity [1], fluid-dynamic traffic model [2], economics [3], etc. In the past decades, there has been a significant theoretical development and application in fractional differential equations. In this paper, we discuss the existence of solutions of fractional impulsive differential equation with p-Laplacian operator. The p-Laplacian operator is the non-standard growth operator which arises from nonlinear electrorheological fluids [4], image restoration [5], elasticity theory [6], etc. Up to now, there are many papers studied the existence of solutions of fractional differential equation with the p-Laplacian operator [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. There are some methods are used usually to study the existence of the solution for this equations, such as upper and lower solutions method [19], fixed point theory [20], coincidence degree theory [21], critical point theory [22], etc.

In [23], the author studied the fractional impulsive differential equations with boundary value conditions:

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t) = f(t,\ u(t)),\ 2 < \alpha \leq 3,\ t \in J' = J \setminus (t_1,\ t_2,\ \cdots,\ t_m),\ J = [0,\ 1], \\ \Delta(u(t_k)) = I_k(u(t_k)),\ \Delta(u'(t_k)) = J_k(u(t_k)),\ \Delta''(u(t_k)) = Q_k(u(t_k)),\quad k = 1,\ 2,\ \cdots,\ m, \\ u(0) + u(1) = 0,\ u'(0) + u'(1) = 0,\ u''(0) + u''(1) = 0, \end{cases}$$

where ${}^cD^{\alpha}_{0^+}$, ${}^cD^{\beta}_{0^+}$ are caputo fractional derivative, $f \in C(J \times R,\ R)$, $I_k(\cdot)$, $J_k(\cdot)$, $Q_k(\cdot) \in C(R,\ R)$.

In [24], the author studied the following p-Laplacian differential equations with impulsive effects:

$$\begin{cases} {}^cD^{\beta}_{0^+}\phi_p\left({}^cD^{\alpha}_{0^+}u(t)\right) = f(t,\ u(t)),\quad t \in J' = J \setminus (t_1,\ t_2,\ \cdots,\ t_m),\ J = [0,\ 1], \\ \Delta(u(t_k)) = I_k(u(t_k)),\ \Delta(u'(t_k)) = J_k(u(t_k)),\quad k = 1, 2, \cdots, m, \\ au(0) + bu(1) = 0,\ au'(0) + bu'(1) = 0,\ {}^cD^{\alpha}_{0^+}u(0) + {}^cD^{\alpha}_{0^+}u(1) = 0, \end{cases}$$

where $0 < \beta \leq 1$, $1 < \alpha \leq 2$, ${}^cD^{\alpha}_{0^+}$, ${}^cD^{\beta}_{0^+}$ are caputo fractional derivative. $\phi_p(s) = |s|^{p-2}s$ is p-Laplacian operator. $f \in C(J \times R, R)$, $I_k(\cdot)$, $J_k(\cdot) \in C(R, R)$. a, b are two real constants with $a > b > 0$.

Motivated by the works mentioned above the papers, we concentrate on the solutions for the nonlinear fractional differential equations (1.1). We obtain the existence result of the p-Laplacian type fractional impulsive differential equation with boundary value problem by using the Schauder fixed point theorem and Leray-Schauder fixed point theorem.

The main work of this paper are organized as follows: In section 2, we give some basic concepts of fractional differential equation. In section 3, we give the main result which based on the fixed point theory. Two examples are given in section 4 to illustrate our main result.

2 Preliminaries

Definition 2.1. *Let set* $J_0 = [0,\ t_1],\ J_1 = (t_1,\ t_2],\ \cdots,\ J_{m-1} = (t_{m-1},\ t_m],\ J_m = (t_m,\ 1]$ *and the spaces:*

$$PC(J,\ R) = \{u : J \to R | u \in C(J_k),\ k = 0,\ 1,\ \cdots,\ m,\ and\ u(t_k^+)\ exist,\ k = 1,\ 2,\ \cdots,\ m.\}$$

with the norm

$$\|u\| = \sup_{t \in J} |u(t)|$$

$$PC^2(J, R) = \{\ u : J \to R | u \in C^2(J_k), k = 0, 1, 2, \cdots, m,\ and\ u(t_k^+), u'(t_k^+), u''(t_k^+) exist,\\ k = 1, 2, \cdots, m.\}$$

with the norm

$$\|u\|_{PC^2} = \max(\|u\|,\ \|u')\|,\ \|u''\|),$$

obviously, $PC(J,\ R)$, $PC^2(J,\ R)$ *are Banach spaces.*

Definition 2.2. *A function* $u \in PC^2(J,\ R)$ *with the Caputo derivative of order* α *existing on* J *is a solution of (1.1) if it satisfies (1.1).*

Definition 2.3. *([25, 26]) The fractional integral of order* $\alpha(\alpha > 0)$ *of function* $f : [0,\ \infty) \to R$ *is given by*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} f(s) ds,$$

Definition 2.4. *([25, 26]) The Caputo fractional derivative of order* $\alpha(\alpha > 0)$ *of function* $f : [0,\ \infty) \to R$ *is given by*

$${}^cD_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $t > 0$, $n = [\alpha] + 1$, $\Gamma(\alpha)$ *is the Gamma fuction.*

Definition 2.5. *([27]) Let* X *and* Y *be normed linear spaces and* T *be linear operator from* X *to* Y*. If any bounded subset* M *of* X*,* TM *is relatively compact set in* Y*, then* T *is called a completely continuous operator.*

Lemma 2.6. *([25, 28]) For* $\alpha > 0$*, then*

(1) $I_{0^+}^{\alpha}\left({}^cD_{0^+}^{\alpha} u(t)\right) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},\ c_i \in \mathbb{R},\ n = [\alpha] + 1.$
(2) ${}^cD_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t).$

Lemma 2.7. *([29]) (Schauder fixed point theorem) Let* X *be Banach space and* $D \subset X$ *a convex, closed, bounded set. If* $T : D \to D$ *is a continuous operator such that* $TD \subset X$*,* TD *is relatively compact, then* T *has a fixed point* $x \in D$*.*

Lemma 2.8. *([29]) (Leray-Schauder fixed point Theorem) Let X be a Banach space, $T : X \to X$ is a completely continuous operator and $V = \{x \in X | x = \mu T x,\ 0 < \mu < 1\}$ a bounded set. Then T has a least one fixed point in X.*

Lemma 2.9. *For a given $y \in C[0,\ 1]$, a function u is a solution of the following impulsive boundary value problem*

$$\begin{cases} {}^cD^{\beta}_{0^+}\phi_p\left({}^cD^{\alpha}_{0^+}u(t)\right) = y(t), \quad 0 < \beta \le 1,\ 2 < \alpha \le 3,\ t \in J' \\ \Delta(u(t_k)) = I_k(u(t_k)),\ \Delta(u'(t_k)) = J_k(u(t_k)), \\ \Delta(u''(t_k)) = Q_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ au(0) + bu(1) = 0,\ au'(0) + bu'(1) = 0, \\ au''(0) + bu''(1) = 0,\ {}^cD^{\alpha}_{0^+}u(0) + {}^cD^{\alpha}_{0^+}u(1) = 0, \end{cases} \tag{2.1}$$

if and only if u satisfies the following integral equation.

$$u(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\phi(z(s))ds + C_1 + C_2 t + C_3 t^2, \quad t \in J_0; \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^t (t-s)^{\alpha-1}\phi(z(s))ds + \dfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\phi_q(z(s))ds \\ +\displaystyle\sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds \\ +\displaystyle\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\ +\displaystyle\sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi(z(s))ds \\ +\displaystyle\sum_{i=1}^{k-1}\frac{(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\ +\displaystyle\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi(z(s))ds \\ +\displaystyle\sum_{i=1}^{k}I_i(u(t_i)) + \sum_{i=1}^{k-1}(t_k-t_i)J_i(u(t_i)) + \sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}Q_i(u(t_i)) \\ +\displaystyle\sum_{i=1}^{k}(t-t_k)J_i(u(t_i)) + \sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)Q_i(u(t_i)) \\ +\displaystyle\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}Q_i(u(t_i)) \\ +C_1 + C_2 t + C_3 t^2, \quad t \in J_k,\ k = 1,\ 2,\ \cdots,\ m, \end{cases} \tag{2.2}$$

where

$$C_1 = \begin{cases} \dfrac{1}{a+b}\Big(\dfrac{1}{\Gamma(\alpha)}\sum\limits_{i=1}^{m+1}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\phi(z(s))ds + \sum\limits_{i=1}^{m-1}\dfrac{(t_m-t_i)}{\Gamma(\alpha-1)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds \\ +\sum\limits_{i=1}^{m-1}\dfrac{(t_m-t_i)^2}{2\Gamma(\alpha-2)}\displaystyle\int_{t_{i-1}}^{t_i}(t-s)^{\alpha-3}\phi_q(z(s))ds + \sum\limits_{i=1}^{m}\dfrac{(1-t_m)}{\Gamma(\alpha-1)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds \\ +\sum\limits_{i=1}^{m-1}\dfrac{(1-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \sum\limits_{i=1}^{m}\dfrac{(1-t_m)^2}{2\Gamma(\alpha-2)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\ +\sum\limits_{i=1}^{m}I_i(u(t_i)) + \sum\limits_{i=1}^{m-1}(t_m-t_i)J_i(u(t_i)) + \sum\limits_{i=1}^{m-1}\dfrac{(t_m-t_i)^2}{2}Q_i(u(t_i)) + \sum\limits_{i=1}^{m}(1-t_m)J_i(u(t_i)) \\ +\sum\limits_{i=1}^{m-1}(1-t_m)(t_m-t_i)Q_i(u(t_i)) + \sum\limits_{i=1}^{m}\dfrac{(1-t_m)^2}{2}Q_i(u(t_i))\Big) \\ -\dfrac{b}{(a+b)^2}\Big[\sum\limits_{i=1}^{m+1}\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s)) + \sum\limits_{i=1}^{m-1}\dfrac{(t_m-t_i)}{\Gamma(\alpha-2)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\ +\sum\limits_{i=1}^{m}\dfrac{(1-t_m)}{\Gamma(\alpha-2)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \sum\limits_{i=1}^{m}J_i(u(t_i)) + \sum\limits_{i=1}^{m-1}(t_m-t_i)Q_i(u(t_i)) \\ +\sum\limits_{i=1}^{m}(1-t_m))Q_i(u(t_i)) - \dfrac{b}{(a+b)(\Gamma(\alpha-2))}\sum\limits_{i=1}^{m+1}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds - \dfrac{b}{a+b}\sum\limits_{i=1}^{m}Q_i(u(t_i))\Big] \\ -\dfrac{b}{a+b}\left(\dfrac{1}{2(a+b)(\Gamma(\alpha-2))}\sum\limits_{i=1}^{m+1}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{(\alpha-3)}\phi_q(z(s))ds + \dfrac{1}{2(a+b)}\sum\limits_{i=1}^{p}Q_i(u(t_i))\right), \end{cases}$$

$$C_2 = \begin{cases} \dfrac{1}{a+b}\Big(\sum\limits_{i=1}^{m+1}\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds + \sum\limits_{i=1}^{m-1}\dfrac{t_m-t_i}{\Gamma(\alpha-2)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\ +\sum\limits_{i=1}^{m}\dfrac{(1-t_m)}{\Gamma(\alpha-2)}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \sum\limits_{i=1}^{m}J_i(u(t_i)) + \sum\limits_{i=1}^{m-1}(t_m-t_i)Q_i(u(t_i)) \\ +\sum\limits_{i=1}^{m}(1-t_m)Q_i(u(t_i))\Big) - \dfrac{b}{(a+b)^2\Gamma(\alpha-2)}\sum\limits_{i=1}^{m+1}\displaystyle\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds - \dfrac{b}{(a+b)}\sum\limits_{i=1}^{m}Q_i(u(t_i)), \end{cases}$$

$$C_3 = \frac{1}{2(a+b)\Gamma(\alpha-2)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \frac{1}{2(a+b)}\sum_{i=1}^{m}Q_i(u(t_i)),$$

$$z(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}y(s)ds - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}y(s)ds.$$

proof If u satisfies equation (2.1), for $t \in J_k$, applying $I^{\beta}_{0^+}$ to both sides of (2.1). One has

$$\phi_p({}^cD^{\alpha}_{0^+}u(t)) = I^{\beta}_{0^+}u(t) - c_0 = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds - c_0,$$

then

$$\phi_p({}^cD^{\alpha}_{0^+}u(0)) = -c_0,\ \phi_p({}^cD^{\alpha}_{0^+}u(1)) = \frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}u(s)ds - c_0.$$

By combing the boundary the condition ${}^cD^{\alpha}_{0^+}u(0) +^c D^{\alpha}_{0^+}u(1) = 0$, one can obtain

$$c_0 = \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}u(s)ds,$$

then

$$\phi_p({}^cD_{0+}^{\alpha}u(t)) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds - \frac{1}{\Gamma(2\beta)}\int_0^1 (1-s)^{\beta-1}u(s)ds.$$

Let

$$z(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds - \frac{1}{\Gamma(2\beta)}\int_0^1 (1-s)^{\beta-1}u(s)ds,$$

by (1) of Lemma 2.5, one has

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(z(s))ds - c_1 - c_2t - c_3t^2,\ t \in J_0,$$

then

$$u^{'}(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\phi_q(z(s))ds - c_2 - 2c_3t,\ t \in J_0,$$

$$u^{''}(t) = \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\phi_q(z(s))ds - 2c_3,\ t \in J_0,$$

where $c_1,\ c_2,\ c_3 \in R$.
If $t \in J_1$, one has

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\phi_q(z(s))ds - d_1 - d_2(t-t_1) - d_3(t-t_1)^2,\ t \in J_1,\\
u^{'}(t) &= \frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t (t-s)^{\alpha-2}\phi_q(z(s))ds - d_2 - 2d_3(t-t_1),\ t \in J_1,\\
u^{''}(t) &= \frac{1}{\Gamma(\alpha-2)}\int_{t_1}^t (t-s)^{\alpha-3}\phi_q(z(s))ds - 2d_3,\ t \in J_1,
\end{aligned}$$

where $d_1,\ d_2,\ d_3 \in R$.
Thus, one has

$$\begin{aligned}
u(t_1^-) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds - c_1 - c_2t - c_3t^2, \quad u(t_1^+) = -d_1,\\
u^{'}(t^-) &= \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1} (t-s)^{\alpha-2}\phi_q(z(s))ds - c_2 - 2c_3t_1, \quad u^{'}(t_1^+) = -d_2,\\
u^{''}(t^-) &= \frac{1}{\Gamma(\alpha-2)}\int_0^{t_1} (t-s)^{\alpha-3}\phi_q(z(s))ds - 2c_3, \quad u^{''}(t_1^+) = -2d_3.
\end{aligned}$$

In view of

$$\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1)),\ \Delta u^{'}(t_1) = u^{'}(t_1^+) - u^{'}(t_1^-) = J_1(u(t_1)),$$

and

$$\Delta u^{''}(t_1) = u^{''}(t_1^+) - u^{''}(t_1^-) = Q_1(u(t_1)),$$

one can obtain

$$
\begin{aligned}
-d_1 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds - c_1 - c_2t - c_3t^2,\\
-d_2 &= \frac{1}{\Gamma(\alpha-1)}\int_0^{t_1} (t-s)^{\alpha-2}\phi_q(z(s))ds - c_2 - 2c_3t_1,\\
-2d_3 &= \frac{1}{\Gamma(\alpha-2)}\int_0^{t_1} (t-s)^{\alpha-3}\phi_q(z(s))ds - 2c_3.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
u(t) = {}& \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}\phi(z(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t-s)^{\alpha-1}\phi(z(s))ds\\
&+ \frac{t-t_1}{\Gamma(\alpha-1)}\int_0^{t_1} (t_1-s)^{\alpha-2}\phi(z(s))ds + \frac{(t-t_1)^2}{2\Gamma(\alpha-2)}\int_0^{t_1} (t_1-s)^{\alpha-3}\phi_q(z(s))ds\\
&+ I_1(u(t_1)) + (t-t_1)J_1(u(t_1)) + \frac{1}{2}(t-{t_1}^2)Q_i(u(t_1)) - c_1 - c_2t - c_3t^2,\ t\in J_1.
\end{aligned}
$$

Similarly, one has

$$
u(t) = \begin{cases}
\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(z(s))ds + C_1 + C_2t + C_3t^2, \quad t\in J_0;\\
+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}\phi(z(s))ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\phi_q(z(s))ds\\
+\sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-2}\phi_q(z(s))ds + \sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-3}\phi_q(z(s))ds\\
+\sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-2}\phi(z(s))ds + \sum_{i=1}^{k-1}\frac{(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-3}\phi_q(z(s))ds\\
+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-3}\phi(z(s))ds + \sum_{i=1}^{k} I_i(u(t_i)) + \sum_{i=1}^{k-1}(t_k-t_i)J_i(u(t_i))\\
+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}Q_i(u(t_i)) + \sum_{i=1}^{k}(t-t_k)J_i(u(t_i)) + \sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)Q_i(u(t_i))\\
+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}Q_i(u(t_i)) + C_1 + C_2t + C_3t^2, \quad t\in J_k,\ k=1,\ 2,\ \cdots,\ m,
\end{cases}
$$

$$
\begin{aligned}
u'(t) = {}& \frac{1}{\Gamma(\alpha-1)}\int_{t_k}^t (t-s)^{\alpha-2}\phi(z(s))ds + \sum_{i=1}^{k}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-2}\phi_q(z(s))ds\\
&+ \sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-2})\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-3}\phi_q(z(s))ds + \sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-3}\phi(z(s))ds\\
&+ \sum_{i=1}^{k} J_i(u(t_i)) + \sum_{i=1}^{k-1}(t_k-t_i)Q_i(u(t_i)) + \sum_{i=1}^{k}(t-t_k)Q_i(u(t_i)) - c_2 - 2c_3t,\\
&t\in J_k, k=1,\ 2,\ \cdots,\ m,
\end{aligned}
$$

$$u^{''}(t)=\frac{1}{\Gamma(\alpha-2)}\int_{t_k}^{t}(t-s)^{\alpha-3}\phi(z(s))ds+\sum_{i=1}^{k}\frac{1}{\Gamma(\alpha-2})\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi(z(s))ds$$
$$+\sum_{i=1}^{k}Q_i(u(t_i))-2c_3,\ t\in J_k,\ k=1,\ 2,\ \cdots,\ m.$$

By the condition $au(0)+bu(1)=0$, one has

$$\begin{aligned}
&(a+b)c_1+b(c_2+c_3)\\
=&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\phi(z(s))ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\phi_q(z(s))ds\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds\\
&+\sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi(z(s))ds+\sum_{i=1}^{k-1}\frac{(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds
\end{aligned}$$

$$\begin{aligned}
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi(z(s))ds+\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k-1}(t_k-t_i)J_i(u(t_i))\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}Q_i(u(t_i))+\sum_{i=1}^{k}(t-t_k)J_i(u(t_i))+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)Q_i(u(t_i))\\
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}Q_i(u(t_i)).
\end{aligned}$$

By the condition $au^{'}(0)+bu^{'}(1)=0$, one has

$$\begin{aligned}
&(a+b)c_2+2bc_3\\
=&\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds+\sum_{i=1}^{m-1}\frac{(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds\\
&+\sum_{i=1}^{m}\frac{(1-t_k)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi(z(s))ds+\sum_{i=1}^{m}J_i(u(t_i))+\sum_{i=1}^{m-1}(t_k-t_i)Q_i(u(t_i))\\
&+\sum_{i=1}^{m}(1-t_k)Q_i(u(t_i)).
\end{aligned}$$

Combing (2.6), (2.7) and the condition $au'' + bu'' = 0$, one has

$$\begin{aligned}
c_1 = {} & \frac{1}{a+b}\Big(\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\phi(z(s))ds + \sum_{i=1}^{m-1}\frac{(t_m-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds \\
& + \sum_{i=1}^{m-1}\frac{(t_m-t_i)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t-s)^{\alpha-3}\phi_q(z(s))ds + \sum_{i=1}^{m}\frac{(1-t_m)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds \\
& + \sum_{i=1}^{m-1}\frac{(1-t_m)(t_m-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \sum_{i=1}^{m}\frac{(1-t_m)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\
& + \sum_{i=1}^{m} I_i(u(t_i)) + \sum_{i=1}^{m-1}(t_m-t_i)J_i(u(t_i)) + \sum_{i=1}^{m-1}\frac{(t_m-t_i)^2}{2}Q_i(u(t_i)) + \sum_{i=1}^{m}(1-t_m)J_i(u(t_i)) \\
& + \sum_{i=1}^{m-1}(1-t_m)(t_m-t_i)Q_i(u(t_i)) + \sum_{i=1}^{m}\frac{(1-t_m)^2}{2}Q_i(u(t_i))\Big) \\
& - \frac{b}{(a+b)^2}\Big[\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s)) + \sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\
& + \sum_{i=1}^{m}\frac{(1-t_m)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \sum_{i=1}^{m}J_i(u(t_i)) + \sum_{i=1}^{m-1}(t_m-t_i)Q_i(u(t_i)) \\
& + \sum_{i=1}^{m}(1-t_p))Q_i(u(t_i)) - \frac{b}{(a+b)(\Gamma(\alpha-2))}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds - \frac{b}{a+b}\sum_{i=1}^{m}Q_i(u(t_i))\Big] \\
& - \frac{b}{a+b}\left(\frac{1}{2(a+b)(\Gamma(\alpha-2))}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{(\alpha-3)}\phi_q(z(s))ds + \frac{1}{2(a+b)}\sum_{i=1}^{m}Q_i(u(t_i)),\right.
\end{aligned}$$

$$\begin{aligned}
c_2 = {} & \frac{1}{a+b}\Big(\sum_{i=1}^{m+1}\frac{1}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)\phi_q(z(s))ds + \sum_{i=1}^{m-1}\frac{t_m-t_i}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\
& + \sum_{i=1}^{m}\frac{(1-t_m)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \sum_{i=1}^{m}J_i(u(t_i)) + \sum_{i=1}^{m-1}(t_m-t_i)Q_i(u(t_i)) \\
& + \sum_{i=1} p(1-t_m)Q_i(u(t_i)) - 2b\Big(\frac{1}{2(a+b)\Gamma(\alpha-2)}\sum_{i=1}^{m+1}\int_{t_{i-1}} t_i(t_i-s)^{\alpha-3}\phi_q(z(s))ds \\
& + \frac{1}{2(a+b)}\sum_{i=1}^{m}Q_i(u(t_i))\Big)\Big),
\end{aligned}$$

$$c_3 = \frac{1}{2(a+b)\Gamma(\alpha-2)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds + \frac{1}{2(a+b)}\sum_{i=1}^{m}Q_i(u(t_i)).$$

Letting $C_1 = c_1$, $C_2 = c_2$, $C_3 = c_3$, we obtain the $u(t)$. Conversely, assume that $u(t)$ is the form of (2.2), then by a direct computation, it satisfies the problem (2.1). The proof is completed.

3 Main result

Theorem 3.1. *Assume that the following conditions*

$$\lim_{u\to 0}\frac{f(t,u)}{u}=0,\ \lim_{u\to 0}\frac{I_k(u)}{u}=0,\ \lim_{u\to 0}\frac{J_k(u)}{u}=0,\ \lim_{u\to 0}\frac{Q_k(u)}{u}=0, \tag{3.1}$$

hold, then the problem (1.1) *has at least one solution.*

Define an operator $T: PC(J,\ \mathbb{R})\to PC(J,\ \mathbb{R})$

$$\begin{aligned}
T(u(t)) &= \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\phi(z(s))ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\phi_q(z(s))ds\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi_q(z(s))ds+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds\\
&+\sum_{i=1}^{k}\frac{(t-t_k)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-2}\phi(z(s))ds+\sum_{i=1}^{k-1}\frac{(t-t_k)(t_k-t_i)}{\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi_q(z(s))ds\\
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2\Gamma(\alpha-2)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-3}\phi(z(s))ds+\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k-1}(t_k-t_i)J_i(u(t_i))\\
&+\sum_{i=1}^{k-1}\frac{(t_k-t_i)^2}{2}Q_i(u(t_i))+\sum_{i=1}^{k}(t-t_k)J_i(u(t_i))+\sum_{i=1}^{k-1}(t-t_k)(t_k-t_i)Q_i(u(t_i))\\
&+\sum_{i=1}^{k}\frac{(t-t_k)^2}{2}Q_i(u(t_i))+m_1+m_2t+m_3t^2,
\end{aligned}$$

where

$$m_1=C_1,\ m_2=C_2,\ m_3=C_3,$$

and

$$z(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f(s,\ u(s))ds-{}^cD_{0^+}^{\alpha}u(t)-\frac{1}{2\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f(s,\ u(s))ds.$$

proof Firstly, we prove that $T: PC(J,\ \mathbb{R})\to PC(J,\ \mathbb{R})$ is a completely continuous operator by following three steps.

Step 1 We proof that $T: PC(J,\ \mathbb{R})\to PC(J,\ \mathbb{R})$ is continuous.In view of the continuouity of functions $f,\ I_k,\ J_k,\ Q_k$, we conclude that $T: PC(J,\ \mathbb{R})\to PC(J,\ \mathbb{R})$ is continuous.

Step 2 We proof that T maps bounded sets into bounded sets. Indeed, let Ω be bounded subset on $PC(J,\ \mathbb{R})$, then there exists positive constant $L_i>0(i=1,\ 2,\ 3,\ 4)$ such that for $\forall u\in\Omega$, $|f(t,\ u)|\le L_1$, $|I_k(u)|\le L_2$, $|J_k|\le L_3$, $|Q_k|\le L_4$. By simple computations, one has

$$
\begin{aligned}
&|\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(t,u(t))dt - \frac{1}{2\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1} f(s,\ u(s)ds|\right)\\
&\le |\phi_q\left(\frac{L_1 s^\beta}{\beta\Gamma(\beta)} - \frac{L_1}{2\beta\Gamma(\beta)|}\right)\\
&\le \left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1}.
\end{aligned}
$$

$$
\begin{aligned}
|m_1| \le &\left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1}\Big[\frac{(m+1)}{(a+b)\Gamma(\alpha+1)} + \frac{1}{(a+b)\Gamma(\alpha-1)}\left(2p + \frac{b}{2(a+b)} + \frac{b^2(m+1)}{(a+b)^2}\right)\Big]\\
&+ \frac{mL_2}{a+b} + \frac{2mL_3}{a+b} + \frac{1}{a+b}\left(2m + \frac{b}{a+b}\right)L_4,
\end{aligned}
$$

$$
|m_2| \le \left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1}\left(\frac{(m+1)}{(a+b)\Gamma(\alpha)} + \frac{2m}{(a+b)\Gamma(\alpha-1)}\right) + mL_3 + 2mL_4,
$$

$$
|m_3| \le \left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1}\frac{(m+1)}{2(a+b)\Gamma(\alpha-1)} + \frac{mL_4}{2(a+b)}.
$$

Then

$$
\begin{aligned}
|T(u(t))| \le &\left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1}\Big\{\frac{1}{\Gamma(\alpha+1)}\left(m+1+\frac{m+1}{a+b}\right) + \frac{1}{\Gamma(\alpha)}\left(2m + \frac{m+1}{a+b}\right)\\
&+ \frac{1}{\Gamma(\alpha-1)}\Big[2m + \frac{1}{a+b}\left(2m + \frac{b}{2(a+b)} + \frac{b^2(m+1)}{(a+b)^2} + 2m + \frac{m+1}{2}\right)\Big]\Big\}\\
&+ \left(m + \frac{m}{a+b}\right)L_2 + \left(2m + \frac{2m}{a+b} + m\right)L_3 + (4m + \frac{1}{a+b}\left(2m + \frac{b}{a+b}\right)\\
&+ \frac{m}{2(a+b)})L_4.
\end{aligned}
$$

Let

$$
\begin{aligned}
L := &\left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1}\Big\{\frac{1}{\Gamma(\alpha+1)}\left(m+1+\frac{m+1}{a+b}\right) + \frac{1}{\Gamma(\alpha)}\left(2m + \frac{m+1}{a+b}\right)\\
&+ \frac{1}{\Gamma(\alpha-1)}\Big[2m + \frac{1}{a+b}\left(2m + \frac{b}{2(a+b)} + \frac{b^2(m+1)}{(a+b)^2} + 2m + \frac{m+1}{2}\right)\Big]\Big\}\\
&+ \left(m + \frac{m}{a+b}\right)L_2 + \left(2m + \frac{2m}{a+b} + m\right)L_3 + (4m + \frac{1}{a+b}\left(2m + \frac{b}{a+b}\right)\\
&+ \frac{m}{2(a+b)})L_4,
\end{aligned}
$$

which implies that $\|T(u(t))\|_{PC} \leq L$.
Step 3 we proof that T is equicontinuous on all the subintervals. Indeed, for any $t \in J_k,\ k = 0,\ 1,\ 2,\ \cdots,\ m$, one has

$$|(Tu)^{'}(t)| \leq \left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1} \Big[\frac{1}{\Gamma(\alpha)}\left(1+\frac{1}{a+b}\right)+\frac{1}{\Gamma(\alpha-1)}\left(2m+\frac{(3m+1)}{a+b}\right)\Big] + 2mL_3 + \left(4m+\frac{m}{a+b}\right)L_4.$$

Let

$$\overline{L} := \left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1} \Big[\frac{1}{\Gamma(\alpha)}\left(1+\frac{1}{a+b}\right)+\frac{1}{\Gamma(\alpha-1)}\left(2m+\frac{(3m+1)}{a+b}\right)\Big] + 2mL_3 + \left(4m+\frac{m}{a+b}\right)L_4.$$

Thus, for $\tau_1,\ \tau_2 \in J_k,\ k = 0,\ 1,\ 2,\ \cdots,\ m$, one has

$$|(Tu)^{'}(\tau_1) - (Tu)^{'}(\tau_2)| \leq \int_{\tau_1}^{\tau_2} |(Tu)^{'}(t)|ds \leq \overline{L}(\tau_1-\tau_2),$$

which means that T is equicontinuous on all the subintervals $t \in J_k,\ k = 1,\ 2,\ \cdots,\ m$. Thus $T\Omega$ is relativity compact. By means of the definition 2.5, we can obtain $T : PC(J,\ R) \to PC(J,\ R)$ completely continuous.

Next, we proof that for $\Omega \subset PC$ a convex, closed, bounded set, one has $T\Omega \subset \Omega$.

From the condition of (3.1), there exists $\varepsilon_i > 0 (i = 1,\ 2,\ 3)$, $r > 0$,and $|u| < r$ such that

$$|f(t,\ u)| \leq \varepsilon_1|u|,\ |I_i(u)| \leq \varepsilon_2|u|,\ |J_k(u)| \leq \varepsilon_3|u|,\ |Q_k(u)| \leq \varepsilon_4|u|,$$

and

$$\begin{aligned}&\left(\frac{\varepsilon_1}{\Gamma(\beta+1)}\right)^{q-1} \Big\{\frac{1}{\Gamma(\alpha+1)}\left(m+1+\frac{m+1}{a+b}\right)+\frac{1}{\Gamma(\alpha)}\left(2m+\frac{m+1}{a+b}\right)\\&+\frac{1}{\Gamma(\alpha-1)}\Big[2m+\frac{1}{a+b}\left(2m+\frac{b}{2(a+b)}+\frac{b^2(m+1)}{(a+b)^2}+2m+\frac{m+1}{2}\right)\Big]\Big\}\\&+\left(m+\frac{m}{a+b}\right)\varepsilon_2+\left(2m+\frac{2m}{a+b}+m\right)\varepsilon_3+\left(4m+\frac{1}{a+b}\left(2m+\frac{b}{a+b}\right)+\frac{m}{2(a+b)}\right)\varepsilon_4\\&\leq 1.\end{aligned} \tag{3.2}$$

Let $\Omega = \{u \in PC(J,\ R) \mid \|u\|_{PC} \leq r\}$. Obvious, Ω is a convex, closed and bounded set. When $u \in PC(J,\ R)$ and $u \in \partial\Omega$, one has $\|u\|_{PC} = r$. By (3.2), one has

$$\begin{aligned}&\left(\frac{\varepsilon_1}{\Gamma(\beta+1)}\right)^{q-1} \Big\{\frac{1}{\Gamma(\alpha+1)}\left(m+1+\frac{m+1}{a+b}\right)+\frac{1}{\Gamma(\alpha)}\left(2m+\frac{m+1}{a+b}\right)\\&+\frac{1}{\Gamma(\alpha-1)}\Big[2m+\frac{1}{a+b}\left(2m+\frac{b}{2(a+b)}+\frac{b^2(m+1)}{(a+b)^2}+2m+\frac{m+1}{2}\right)\Big]\Big\}\\&+\left(m+\frac{m}{a+b}\right)\varepsilon_2+\left(2m+\frac{2m}{a+b}+m\right)\varepsilon_3+\left(4m+\frac{1}{a+b}\left(2m+\frac{b}{a+b}\right)+\frac{m}{2(a+b)}\right)\varepsilon_4\\&\leq r.\end{aligned}$$

Thus

$$
\begin{aligned}
|Tu(t)| \leq & \left(\frac{\varepsilon_1}{\Gamma(\beta+1)}\right)^{q-1} \Big\{\frac{1}{\Gamma(\alpha+1)}\left(m+1+\frac{m+1}{a+b}\right)+\frac{1}{\Gamma(\alpha)}\left(2m+\frac{m+1}{a+b}\right) \\
& +\frac{1}{\Gamma(\alpha-1)}\Big[2m+\frac{1}{a+b}\left(2m+\frac{b}{2(a+b)}+\frac{b^2(m+1)}{(a+b)^2}+2m+\frac{m+1}{2}\right)\Big]\Big\} \\
& +\left(m+\frac{m}{a+b}\right)\varepsilon_2+\left(2m+\frac{2m}{a+b}+m\right)\varepsilon_3+\big(4m+\frac{1}{a+b}\left(2m+\frac{b}{a+b}\right) \\
& +\frac{m}{2(a+b)}\big)\varepsilon_4 \\
\leq & \, r.
\end{aligned}
$$

Then, $\|Tu\|_{PC} \leq r$ and $T\Omega \subset \Omega$. According to Lemma 2.7 , T has at least one fixed point in $PC(J, \mathbb{R})$. Thus, the problem (1.1) has at least one solution.The proof is completed.

Theorem 3.2. *Assume there exists positive constants $L_i > 0 (i = 1, 2, 3, 4)$ such that $f(t, u) \leq L_1$, $I_k(t, u(t)) \leq L_2$, $J_k \leq L_3$, $Q_k \leq L_4$, where $t \in J_k$, $u \in PC(J, \mathbb{R})$, then the problem (1.1) have at least one solution in $PC(J, R)$.*

proof As shown in Theorem 3.1, the operator $T : PC(J, R) \to PC(J, R)$ is completely continuous operator. We proof the set $V = \{u \in PC(J, R) | u = \mu Tu, 0 < \mu < 1\}$ is bounded. Assume $\forall u \in V$ and $\forall t \in J_k$, one has

$$
\begin{aligned}
|u(t)| = & \, |\mu T(u(t))| \\
\leq & \, \mu \left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1} \Big\{\frac{1}{\Gamma(\alpha+1)}\left(m+1+\frac{m+1}{a+b}\right)+\frac{1}{\Gamma(\alpha)}\left(2m+\frac{m+1}{a+b}\right) \\
& +\frac{1}{\Gamma(\alpha-1)}\Big[2m+\frac{1}{a+b}\left(2m+\frac{b}{2(a+b)}+\frac{b^2(m+1)}{(a+b)^2}+2m+\frac{m+1}{2}\right)\Big]\Big\} \\
& +\mu\left(m+\frac{m}{a+b}\right)L_2+\mu\left(2m+\frac{2m}{a+b}+\mu m\right)L_3 \\
& +\mu\left(4m+\frac{1}{a+b}\left(2m+\frac{b}{a+b}\right)+\frac{m}{2(a+b)}\right)L_4.
\end{aligned}
$$

Thus $\forall t \in J$, one has

$$
\begin{aligned}
\|u(t)\|_{PC} \leq & \, \mu \left(\frac{L_1}{\Gamma(\beta+1)}\right)^{q-1} \Big\{\frac{1}{\Gamma(\alpha+1)}\left(m+1+\frac{m+1}{a+b}\right)+\frac{1}{\Gamma(\alpha)}\left(2m+\frac{m+1}{a+b}\right) \\
& +\frac{1}{\Gamma(\alpha-1)}\Big[2m+\frac{1}{a+b}\left(2m+\frac{b}{2(a+b)}+\frac{b^2(m+1)}{(a+b)^2}+2m+\frac{m+1}{2}\right)\Big]\Big\} \\
& +\mu\left(m+\frac{m}{a+b}\right)L_2+\mu\left(2m+\frac{2m}{a+b}+\mu m\right)L_3 \\
& +\mu\left(4m+\frac{1}{a+b}\left(2m+\frac{b}{a+b}\right)+\frac{m}{2(a+b)}\right)L_4,
\end{aligned}
$$

which indicates that the set V is bounded.

According to Lemma 2.8, T has a fixed point $u \in PC(J, R)$. Then the problem (1.1) have at least one solution. The proof is completed.

4 Example

In this section, we give two simple examples to proof our main results.
Example 4.1 Consider the following equation:

$$\begin{cases} {}^cD^{\beta}_{0+}\phi_p\left({}^cD^{\alpha}_{0+}u(t)\right) = tu^3(t) + tu^2\sin u,\ p>1,\ 0<\beta\le 1,\ 2<\alpha\le 3, \\ \Delta(u(\frac{1}{2})) = u^2ln(1+u^2), \Delta(u^{'}(\frac{1}{2})) = \sqrt{1+u^2}-1,\ \Delta^{''}(\frac{1}{2}) = \frac{1}{u} - \frac{1}{\sin u}, \\ 3u(0) - \frac{2}{3}u(1) = 0,\ 3u'(0) - \frac{2}{3}u'(0) = 0, \\ 3u^{''}(0) - \frac{2}{3}u^{''}(1) = 0,\ {}^cD^{\alpha}_{0+}u(0) +^c D^{\alpha}_{0+}u(1) = 0, \end{cases} \tag{4.1}$$

where $f(t,u(t)) = tu^3(t)+tu^2\sin u$, $I_1(u(t)) = u^2ln(1+u^2)$, $J_1 = \sqrt{1+u^2}-1,\ Q_1 = \frac{1}{u} - \frac{1}{\sin u}$, one has

$$\lim_{u\to 0}\frac{tu^3(t)+tu^2\sin u}{u} = 0,\ \lim_{u\to 0}\frac{u^2ln(1+u^2)}{u} = 0,$$

$$\lim_{u\to 0}\frac{\sqrt{1+u^2}-1}{u} = 0,\ \lim_{u\to 0}\frac{\frac{1}{u} - \frac{1}{\sin u}}{u} = 0.$$

So, all the conditions of theorem 3.1 are satisfied. Then, the problem 4.1 has at least one solution.
Example 4.2 Consider the following equation:

$$\begin{cases} {}^cD^{\beta}_{0+}\phi_p\left({}^cD^{\alpha}_{0+}u(t)\right) = \frac{t}{1+t^2}\arctan u,\ p>1\ 0<\beta\le 1,\ 2<\alpha\le 3, \\ \Delta(u(\frac{3}{4})) = 1+3\cos^2 u,\ \Delta(u^{'}(\frac{3}{4})) = 4u^2 + \frac{1}{u^2},\ \Delta u^{''}(\frac{3}{4}) = \frac{u^2}{4+u^2}, \\ 2u(0) - \frac{1}{2}u(1) = 0,\ 2u'(0) - \frac{1}{2}u'(0) = 0, \\ 2u^{''}(0) - \frac{1}{2}u^{''}(1) = 0,\ {}^cD^{\alpha}_{0+}u(0) +^c D^{\alpha}_{0+}u(1) = 0, \end{cases} \tag{4.2}$$

where $f(t,u(t)) = \frac{t}{1+t^2}\arctan u$, $I_1(u(t)) = 1+3cos^2u$, $J_1(u(t)) = 4u^2 + \frac{1}{u^2}$, $Q_1(u(t)) = \frac{u^2}{4+u^2}$. Selection $L_1 = \frac{\pi}{2}$, $L_2 = L_3 = 4$, $L_4 = 1$, then the conditions of theorem 3.2 are easily verified. Thus the problem 4.2 has at least one solution.

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