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Numerical Methods for Pricing American Maximum Options

Rui Hou [1], **Yongqing Xu** [2], **Jinhua Fan** [1] **and Yuanguo Zhu** [1]

[1] School of Science
Nanjing University of Science and Technology
Nanjing 210094, China

[2] College of Big Data and Internet
Shenzhen Technology University
Shenzhen 518118, China

Abstract

Numerical methods are used to price American maximum options on the maximum of a risk asset and a fixed income. Based on Black-Scholes model, the payoff function, and the smooth matching condition, the partial differential equation which the option price satisfied can be obtained. Choosing specific interval and equivalent substitution, finite difference method for solving this equation can be derived. Least-squares Monte Carlo simulation method is also given to solve the equation. Two different numerical methods are used to compute the option price with given parameters. Graphs of the option price and the optimal exercise boundaries can be plotted using the computed result. The graphs show that the American maximum option has two optimal exercise boundaries. By comparing the results getting from two methods, the validity of finite difference method is verified. Finite difference method is more efficient in practical use because it does not need a simulation of asset price.

Mathematics Subject Classification: 91G20

Keywords: American maximum options; Finite difference method; Least-squares Monte Carlo simulation; Boundary curves

1 Introduction

The valuation of options is a classical problem of mathematical finance. In 1973, Black and Scholes[1] derived a closed-form solution for European call options in their seminal paper. They solved the problem of pricing European options. Roll[8], Geske[3], and Whaley[10] have derived the analytic solutions for American options on assets with discrete dividends but no analytic solution exists for American options if the underlying assets pay continuous dividends.

Merton[7] recognized that the valuation of American options is a free boundary problem. The mathematical difficulty stems from the early exercise right and the optimal stopping time of the option. The unknown exercise boundary should be part of the solution.

Kim[6], Jacka[4], and Carr et al.[2] divided American option value into the corresponding European option value and an early exercise right premium. The premium can be written as an integral over the optimal exercise boundary. Because of the unknown exercise boundary, numerical methods including finite difference method, least-squares Monte Carlo simulation methods and binomial tree method are used to price American options.

Based on the review of existing research results, we notice that much of the research has focused only on standard American options on single asset such as simple call or put options. Few studies have explored more complex American-style options deeply. However, options traded in modern financial markets are highly diversified. So there is an urgent need for pricing these options. Lishang Jiang[5] analyzed American options on the maximum(minimum) of two risk assets. We have special interest in American options on the maximum of two assets. One of them is risk asset like stock, the other is risk free asset like treasury bond. We call them American maximum options. This type of options is extremely common in the market. Some investment funds have their returns linked to share price but also promise a minimum payoff. Convertible bonds give holders the right to exchange fixed income for floating income. They are essentially a special form of American maximum options.

This paper is structured as follows. Section 2 gives the model for pricing American maximum options. Section 3 derives a finite difference method for pricing American maximum options. Section 4 shows a least-squares Monte Carlo simulation method for pricing American maximum options. Section 5 is a numerical example. Two different numerical methods are used to compute the option price with given parameters. Graphs of the option price and the optimal exercise boundaries are plotted with computed results. We conclude the paper in Section 6.

2 Pricing Model

We investigate American maximum option which has the right to get the maximum of an underlying asset and a fixed income. The minimum payoff equals to a positive constant L. The payoff function is $\max(S, L)$. S is the underlying asset price following geometric Brownian motion

$$\mathrm{d}S_t = S_t[(r-D)\mathrm{d}t + \sigma \mathrm{d}W_t], \tag{1}$$

where r, D, and σ are the risk-free rate, dividend yield, and volatility of the price, respectively. r, D, and σ are assumed constant. W_t is the standard Brownian motion. According to Black-Scholes theory[1], the option value $V(S,t)$ satisfies the following equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0,\ 0 < t < T_F,\ S_a(t) < S < S_b(t), \tag{2}$$

where T_F is the time of expiry. It changes to a double boundaries problem. When the underlying asset price drops below the lower boundary $S_a(t)$ or raises beyond the upper boundary $S_b(t)$, the option needs to be exercised. We get the smooth matching conditions across double optimal exercise boundaries

$$\frac{\partial V}{\partial S} = 0,\ V = L,\ S = S_a(t), \tag{3}$$

$$\frac{\partial V}{\partial S} = 1,\ V = S,\ S = S_b(t). \tag{4}$$

At time T_F, the final value of the option is the payoff

$$V(S, T_F) = \max(S, L). \tag{5}$$

Other assumptions for this model include market completeness and efficiency, continuous trading, no transaction costs and taxes, no short-sale restrictions, assets paying continuous dividends, and trading of assets and options in any divisible amounts.

3 Finite difference method

In order to simplify the calculations, we define new variables and parameters

$$S = L\mathrm{e}^x, \qquad V = L \cdot Y, \qquad \tau = T_F - t, \tag{6}$$

$$S_a(t) = L\mathrm{e}^{a(\tau)}, \qquad S_b(t) = L\mathrm{e}^{b(\tau)}. \tag{7}$$

Then (2) and (5) can be rewritten in the new form

$$\frac{\partial Y}{\partial \tau}-\frac{\sigma^2}{2}\frac{\partial^2 Y}{\partial x^2}-(r-D-\frac{\sigma^2}{2})\frac{\partial Y}{\partial x}+rY=0,\ 0<\tau<T_F,\ a(\tau)<x<b(\tau). \tag{8}$$

$$Y(x,0)=\max(\mathrm{e}^x,1). \tag{9}$$

Given Δx,$\Delta\tau>0$, we divide intervals $x\in[-\infty,+\infty]$ and $\tau\in[0,T_F]$. $Y_j^n=Y(j\Delta x,n\Delta\tau)$ represents the option value at $(j\Delta x,n\Delta\tau)$. Using difference schemes

$$\frac{\partial Y}{\partial \tau}=\frac{Y_j^n-Y_j^{n-1}}{\Delta\tau}, \tag{10}$$

$$\frac{\partial^2 Y}{\partial x^2}=\frac{Y_{j+1}^{n-1}-2Y_j^{n-1}+Y_{j-1}^{n-1}}{\Delta x^2}, \tag{11}$$

$$\frac{\partial Y}{\partial x}=\frac{Y_{j+1}^{n-1}-Y_{j-1}^{n-1}}{2\Delta x}, \tag{12}$$

equations (8) and (9) change to difference equation

$$\begin{cases} min\{\dfrac{Y_j^n-Y_j^{n-1}}{\Delta\tau}-\dfrac{\sigma^2}{2}\cdot\dfrac{Y_{j+1}^{n-1}-2Y_j^{n-1}+Y_{j-1}^{n-1}}{\Delta x^2}-(r-D-\dfrac{\sigma^2}{2})\dfrac{Y_{j+1}^{n-1}-Y_{j-1}^{n-1}}{2\Delta x} \\ \qquad +rY_j^n, Y_j^n-\max\{\mathrm{e}^x,1\}\}=0, \\ Y_j^0=\max\{\mathrm{e}^{j\Delta x},1\}. \end{cases} \tag{13}$$

To get the recurrence relation of Y_n^j, we write the deformation of (13)

$$\begin{cases} Y_j^n=&\max\{\dfrac{1}{1+r\cdot\Delta\tau}[(1-\dfrac{\sigma^2\cdot\Delta\tau}{\Delta x^2})\cdot Y_j^{n-1}+\dfrac{\sigma^2+(r-D-\frac{\sigma^2}{2})\cdot\Delta x}{2\cdot\Delta x^2}\cdot\Delta\tau \\ &\cdot Y_{j+1}^{n-1}+\dfrac{\sigma^2-(r-D-\frac{\sigma^2}{2})\cdot\Delta x}{2\cdot\Delta x^2}\cdot\Delta\tau\cdot Y_{j-1}^{n-1}],1,\mathrm{e}^{j\Delta x}\}, \\ Y_j^0=&\max\{\mathrm{e}^{j\Delta x},1\}. \end{cases} \tag{14}$$

Letting $\frac{\sigma^2\cdot\Delta\tau}{\Delta x^2}=1$, we get

$$\begin{cases} Y_j^n=\max\{\dfrac{1}{1+r\cdot\Delta\tau}[(\dfrac{1}{2}+\dfrac{(r-D-\frac{\sigma^2}{2})}{2\sigma}\cdot\sqrt{\Delta\tau})\cdot Y_{j+1}^{n-1} \\ \qquad +(\dfrac{1}{2}-\dfrac{(r-D-\frac{\sigma^2}{2})}{2\sigma}\cdot\sqrt{\Delta\tau})\cdot Y_{j-1}^{n-1}],1,\mathrm{e}^{j\Delta x}\}, \\ Y_j^0=\max\{\mathrm{e}^{j\Delta x},1\}. \end{cases} \tag{15}$$

Then we set $u=\mathrm{e}^{\sigma\sqrt{\Delta\tau}}$, $d=u^{-1}$, $\rho=\mathrm{e}^{r\cdot\Delta\tau}$, $p=(\mathrm{e}^{(r-q)\cdot\Delta\tau}-d)\cdot(u-d)^{-1}$. According to Taylor formula, when $\Delta\tau\to 0^+$ we get

$$\frac{1}{2}+\frac{(r-D-\frac{\sigma^2}{2})}{2\sigma}\cdot\sqrt{\Delta\tau}=p+o(\sqrt{\Delta\tau}),\qquad \frac{1}{1+r\cdot\Delta\tau}=\frac{1}{\rho}+O(\Delta\tau^2). \tag{16}$$

Keeping the leading order to $\sqrt{\Delta\tau}$ we obtain

$$\begin{cases} Y_j^n = \max\{\frac{1}{\rho}[pY_{j+1}^{n-1} + (1-p)Y_{j-1}^{n-1}], 1, \mathrm{e}^{j\Delta x}\}, \\ Y_j^0 = \max\{\mathrm{e}^{j\Delta x}, 1\}. \end{cases} \tag{17}$$

At the time of expire, $Y_j^0 = \max\{\mathrm{e}^{j\Delta x}, 1\}$. We can get $Y_j^0 (j = 0, \pm 1, \pm 2, \pm 3, \cdots)$ for different underlying asset price. Then using the recurrence relation $Y_j^n = \max\{\frac{1}{\rho}[pY_{j+1}^{n-1} + (1-p)Y_{j-1}^{n-1}], 1, \mathrm{e}^{j\Delta x}\}$, we obtain the option price for $n = 1, 2, \cdots, N (N\Delta\tau = T_F)$ in turn. Finally using variable substitution in (6) we get $V(S,t)$ from $Y(x,\tau)$.

4 Least-squares Monte Carlo method

Least-squares Monte Carlo simulation is another way to price American maximum option. This method is based on Francis A. Longstaff and Eduardo S. Schwartz's work[9].

First we simulate M paths of underlying asset price using Monte Carlo simulation. Then we divide the interval $[0, T]$ into N segments. In each segment we have $\Delta t = T/N$. S_m^n is the option price on the m path at time $n\Delta t$. The underlying asset price follows geometric Brownian motion. We get

$$S_m^{n+1} = S_m^n \cdot \mathrm{e}^{(r-D-\sigma^2/2)\cdot\Delta t + \sigma\cdot\sqrt{\Delta t}\cdot N(0,1)}, \tag{18}$$

where $N(0,1)$ is the standard normal distribution. Given S_m^0 we can simulate the option price.

The expected return of holding the option is calculated by least-squares regression to determine when the American maximum option should be exercised on each asset price path. At time N, the payment of the option equals to $\max\{S_m^N, L\}$. Whether to exercise the option at time $N-1$ depends on whether the return of exercising is greater than the expected return of holding the option. Let X denote the asset price S_m^{N-1} and Y denote the corresponding discounted cash flows reveived at time N. We regress Y on a constant, X, and X^2. The resulting conditional expectation function is $\mathrm{E}[Y \mid X] = a_0 + a_1 X + a_2 X^2$. Then bring S_m^{N-1} into the function to calculate the expected return of holding the option.

Finally, we determine the exercise strategy. If the return of exercising is greater than the expected return of holding the option, the option should be exercised at $N-1$. Analogously, repeat the above steps at time $n = N-2, N-3, \cdots, 1$. We get the exercise strategy and the corresponding option price of each path. The option value equals to the average value of all paths.

5 Numerical example

We compute the price of an one year period American maximum option. The initial price of the underlying asset is S yuan. The minimum payment $L = 49$ yuan. The annual risk free interest rate $r = 8\%$. The annual dividend rate $D = 2\%$. The annual volatility of the underlying asset price $\sigma = 20\%$. We use two numerical methods to calculate the option price $P(S,t)$ in intervals $S \in [40,60]$ and $t \in [0,1]$.

Dividing the time interval into 100 segments and the price interval into 1000 segments we have $\Delta\tau = 0.01$ and $\Delta x = 0.02$ in each segment. Some results obtained by using finite difference method are as follows:

Price S/yuan	Time t/year										
	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
40.12	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00
41.76	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00
43.46	49.10	49.08	49.07	49.05	49.04	49.02	49.00	49.00	49.00	49.00	49.00
45.23	49.50	49.47	49.44	49.40	49.35	49.30	49.23	49.16	49.08	49.00	49.00
47.08	50.19	50.15	50.11	50.05	49.98	49.90	49.80	49.67	49.50	49.26	49.00
49.00	51.17	51.13	51.07	51.01	50.93	50.83	50.72	50.56	50.35	50.02	49.00
51.00	52.43	52.38	52.32	52.26	52.18	52.09	51.97	51.83	51.63	51.35	51.00
53.08	53.94	53.89	53.85	53.79	53.72	53.65	53.55	53.44	53.30	53.14	53.08
55.25	55.69	55.66	55.62	55.58	55.53	55.48	55.42	55.35	55.28	55.25	55.25
57.50	57.68	57.66	57.63	57.61	57.58	57.55	57.53	57.51	57.50	57.50	57.50
59.85	59.88	59.87	59.87	59.86	59.85	59.85	59.85	59.85	59.85	59.85	59.85

Table 1: Results of finite difference method

Dividing the time interval into 100 segments we have $\Delta\tau = 0.01$ in each segment. Some results obtained by using least-squares Monte Carlo method with 5000 paths are as follows:

Price S/yuan	Time t/year										
	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
40.12	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00
41.76	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00	49.00
43.46	49.09	49.07	49.06	49.04	49.04	49.01	49.00	49.00	49.00	49.00	49.00
45.23	49.49	49.45	49.42	49.38	49.35	49.32	49.25	49.17	49.08	49.00	49.00
47.08	50.14	50.10	50.05	50.02	49.98	49.88	49.78	49.66	49.50	49.27	49.00
49.00	51.10	51.04	50.98	50.93	50.86	50.77	50.67	50.52	50.33	50.02	49.00
51.00	52.38	52.34	52.30	52.25	52.15	52.06	51.93	51.80	51.61	51.34	51.00
53.08	53.91	53.85	53.77	53.73	53.67	53.60	53.51	53.40	53.28	53.14	53.08
55.25	55.70	55.68	55.64	55.58	55.52	55.46	55.40	55.33	55.32	55.27	55.25
57.50	57.68	57.65	57.62	57.60	57.61	57.57	57.59	57.52	57.52	57.53	57.50
59.85	59.89	59.91	59.88	59.88	59.91	59.85	59.85	59.86	59.85	59.87	59.85

Table.2: Results of least-squares Monte Carlo simulation method

According to the definition of the optimal exercise boundary we draw plane $P(S,t) = S$ and plane $P(S,t) = L$ to get their intersection lines with option price plane $P(S,t)$. The two intersection lines are the optimal exercise boundaries of American maximum option. The graphs are as follows:

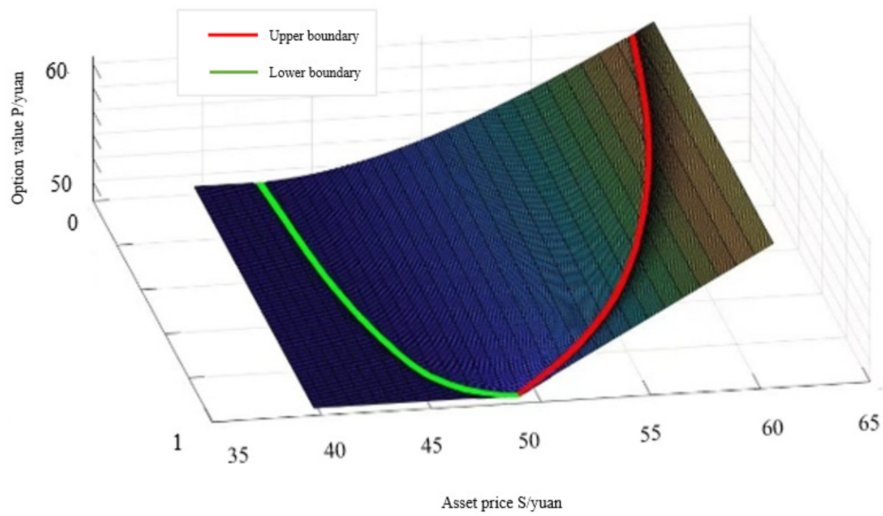

Figure 1: Exercise boundaries from finite difference method

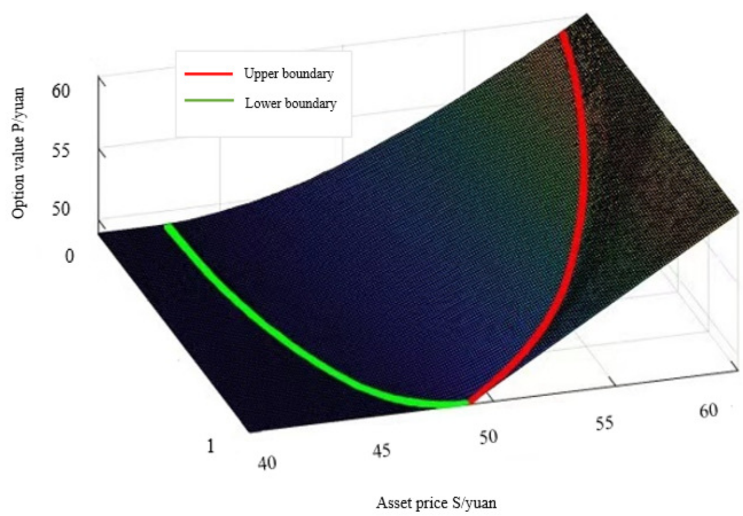

Figure 2: Exercise boundaries from least-squares Monte Carlo method

6 Conclusion

In this paper we build pricing model of American maximum option. Finite difference method and least-squares Monte Carlo method are derived and used in a numerical example. By comparing the results getting from two methods, the validity of finite difference method is verified. The graphs show that the American maximum option has two optimal exercise boundaries.

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