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A Solution of the Cartesian $(2:1)$ Unbalanced Poisson-Boltzmann Equation

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Abstract

In this paper, we solve the cartesian Poisson-Boltzman (PB) equation in two-dimensions for a $2:1$ electric charge unbalanced configuration. We apply, the tanh, Ricatti functions and Jacobi elliptic solitary wave methods, getting several families of solutions.

Keywords: Unbalanced Poisson-Boltzman (PB) equation, Tanh method, Ricatti functions, Jacobi elliptic functions

1 Introduction

The Poisson-Boltzmann (PB) equation gives account of the electrostatic potential of an electrolyte solution [1]. PB is a highly diffcult differential equation to solve analytically. Basically, it is a Poisson equation with sources that are exponential field dependent [2]. Therefore, its most direct and powerful application is found in the framework of computational field, Delphi [3]-[4] and Charmm [5]. In this work, we use analytical methods, known as solitary wave solutions [6]-[9], in order to find solutions to electrical potential of an asymmetrical electrolyte $(2:1)$.

2 Two-dimensional Unbalanced Poisson-Boltzmann equation

The unbalanced (2:1) cartesian Poisson-Boltzmann equation is:

$$\frac{\partial^2\phi}{dx^2} + \frac{\partial^2\phi}{dy^2} = -\kappa^2(e^{(-2\phi)} - e^{(\phi)}) \tag{1}$$

Here, κ^{-1} is the Debye screening length [1]. Using the transformation $\xi = x+y$, and its second derivative, we get:

$$2\frac{d^2}{d\xi^2}\phi = \kappa^2(-e^{(-2\phi)} + e^{(\phi)}) \tag{2}$$

Now defining the variable and the first and second derivative

$$v = v_0 e^{\phi}, \quad \frac{d\phi}{d\xi} = \frac{1}{v}\frac{dv}{d\xi}, \quad \frac{d^2\phi}{d\xi^2} = -\frac{1}{v^2}(\frac{dv}{d\xi})^2 + \frac{1}{v}\frac{d^2v}{d\xi^2} \tag{3}$$

And replacing in eqs. (2)

$$-v_0(\frac{dv}{du})^2 + v_0 v\frac{d^2v}{du^2} + \frac{\kappa^2}{2}v_0^3 - \frac{\kappa^2}{2}v^3 = 0 \tag{4}$$

Now, we introduce a new independent variable [6]:

$$Y = \tanh(\mu u) \tag{5}$$

Then, the derivatives of u, are:

$$\frac{d}{du} = \mu(1-Y^2)\frac{d}{dY}, \quad \frac{d^2}{du^2} = -2Y\mu^2(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)^2\frac{d^2}{dY^2} \tag{6}$$

The solutions are postulated as [6]:

$$v = \sum_{i=1}^{m} a_i Y^i \tag{7}$$

Then replacing

$$\begin{aligned} &-2v_0 v\mu^2 Y(1-Y^2)\frac{dv}{dY} + v_0 v\mu^2(1-Y^2)^2\frac{d^2v}{dY^2} \\ &-v_0\mu^2(1-Y^2)^2(\frac{dv}{dY})^2 + \frac{\kappa^2}{2}v_0^3 - \frac{\kappa^2}{2}v^3 = 0 \end{aligned} \tag{8}$$

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (8). So, $vY^4\frac{d^2v}{dY^2} \to v^3 \to m = 2$. So, replacing in eq. (7), we obtain:

$$v = a_0 + a_1 Y + a_2 Y^2 \tag{9}$$

Replacing in eq. (8), we get a polynomial in Y^i and order by order in Y^i we obtain a set of equations. Doing some algebra, we get:

$$f_1 \to (a_0 = -\frac{v_0}{2}, a_1 = 0, a_2 = 3\frac{v_0}{2}, \kappa = -2\mu\sqrt{\frac{2}{3}}) \tag{10}$$

$$f_2 \to (a_0 = -\frac{v_0}{2}, a_1 = 0, a_2 = 3\frac{v_0}{2}, \kappa = 2\mu\sqrt{\frac{2}{3}}) \tag{11}$$

$$f_3 \to (a_0 = -(-1)^{2/3}\frac{v_0}{2}, a_1 = 0, a_2 = (-1)^{2/3}3\frac{v_0}{2}, \kappa = -2(-1)^{-1/3}\mu\sqrt{\frac{2}{3}}) \tag{12}$$

$$f_4 \to (a_0 = -(-1)^{2/3}\frac{v_0}{2}, a_1 = 0, a_2 = (-1)^{2/3}3\frac{v_0}{2}, \kappa = 2(-1)^{-1/3}\mu\sqrt{\frac{2}{3}}) \tag{13}$$

$$f_5 \to (a_0 = (-1)^{1/3}\frac{v_0}{2}, a_1 = 0, a_2 = -(-1)^{1/3}3\frac{v_0}{2}, \kappa = -2(-1)^{-2/3}\mu\sqrt{\frac{2}{3}}) \tag{14}$$

$$f_6 \to (a_0 = (-1)^{1/3}\frac{v_0}{2}, a_1 = 0, a_2 = -(-1)^{1/3}3\frac{v_0}{2}, \kappa = 2(-1)^{-1/3}\mu\sqrt{\frac{2v_0}{3}}) \tag{15}$$

Then, we get six families of solutions.

3 Solitary wave method 2, Solutions Riccati equation

We use using the method presented in [7], to get solutions for eqs. (4). So:

$$v = \sum_{i=1}^{n} a_i F^i \tag{16}$$

where F solves, table (1), the Riccati equation, then:

	A_1	C_1	F
1	1/2	-1/2	$coth(\xi) \pm cosh(\xi)$, $tanh(\xi) \pm isech(\xi)$
2	1/2	1/2	$sec(\xi) \pm itan(\xi)$
3	-1/2	- 1/2	$csc(\xi) \pm icot(\xi)$
4	1	- 1	$tanh(\xi)$, $coth(\xi)$
5	1	1	$tan(\xi)$
6	-1	-1	$cot(\xi)$

Table 1: Solutions for eqs. (17), [7] .

$$F^{'} = (C_1F^2 + A_1), \quad F^{''} = 2C_1F(C_1F^2 + A_1) \tag{17}$$

here A_1 and C_1 are constants, table (1). Replacing in eqs. (4), and balancing nonlinear terms, we have $n = 2$. Then, eq. (16) is, $v = (a_0 + a_1F + a_2F^2)$. Therefore, the derivatives are:

$$\begin{aligned} v^{'} &= (a_1 + 2a_2F)F^{'} = (a_1 + 2a_2F)(C_1F^2 + A_1), \;\; v^{''} = ((2a_2F^{'})F^{'} \\ &+(a_1 + 2a_2F)F^{''}) = (2a_2(C_1F^2 + A_1)^2 + (a_1 + 2a_2F)2C_1F(C_1F^2 + A_1)) \end{aligned} \tag{18}$$

Replacing in eq. (4), we obtain a group of algebraic equations, order by order in F^i. And doing some algebra, we get:

$$g_1 \leftarrow (a_1 = 0, a_2 = \frac{3C_1a_0}{A_1}, \kappa = -2i\sqrt{\frac{2A_1C_1}{3}}, v_0 = -2a_0) \tag{19}$$

$$g_2 \leftarrow (a_1 = 0, a_2 = \frac{3C_1a_0}{A_1}, \kappa = 2i\sqrt{\frac{2A_1C_1}{3}}, v_0 = -2a_0) \tag{20}$$

$$g_3 \leftarrow (a_1 = 0, a_2 = \frac{3C_1a_0}{A_1}, \kappa = -2(-1)^{1/6}\sqrt{\frac{2A_1C_1}{3}}, v_0 = 2(-1)^{1/3}a_0) \tag{21}$$

$$g_4 \leftarrow (a_1 = 0, a_2 = \frac{3C_1a_0}{A_1}, \kappa = 2(-1)^{1/6}\sqrt{\frac{2A_1C_1}{3}}, v_0 = 2(-1)^{1/3}a_0) \tag{22}$$

$$g_5 \leftarrow (a_1 = 0, a_2 = \frac{3C_1a_0}{A_1}, \kappa = -2(-1)^{5/6}\sqrt{\frac{2A_1C_1}{3}}, v_0 = -2(-1)^{2/3}a_0) \tag{23}$$

$$g_6 \leftarrow (a_1 = 0, a_2 = \frac{3C_1a_0}{A_1}, \kappa = 2(-1)^{5/6}\sqrt{\frac{2A_1C_1}{3}}, v_0 = -2(-1)^{2/3}a_0) \tag{24}$$

Then, we get thirty six families of solutions, g_i, using Ricatti method [7].

	ϵ	a	b	c	G
1	-1	$-m^2$	1	1	$sn(\xi)$
2	-1	m^2	$1-m^2$	1	$cn(\xi)$
3	-1	1	m^2-1	1	$dn(\xi)$
4	-1	$-m^2$	1	1	$cd(\xi)$
5	m^2-1	m^2	1	1	$sd(\xi)$
6	$1-m^2$	1	-1	1	$nd(\xi)$
7	1	1	$-m^2$	-1	$dc(\xi)$
8	1	$1-m^2$	m^2	-1	$nc(\xi)$
9	1	$1-m^2$	1	1	$sc(\xi)$
10	1	1	$-m^2$	-1	$ns(\xi)$
11	1	1	m^2-1	m^2	$ds(\xi)$
12	1	1	$1-m^2$	1	$cs(\xi)$

Table 2: The Solutions for eq. (25), [8] .

4 Solitary wave method 3, Jacobi solutions

We start with the solutions, table (2), given by the next differential equation:

$$(G^{'})^2 = (c + \epsilon G^2)(aG^2 + b) \tag{25}$$

Where a, b, c and ϵ are given in table (2). Also, they satisfy the next relations:

$$\begin{aligned} & sn(\xi,k)^2 + cn(\xi,k)^2 = k^2 sn(\xi,k)^2 + dn(\xi,k)^2 = 1 \\ & 1 + cs(\xi,k)^2 = k^2 + ds(\xi,k)^2 = ns(\xi,k)^2 \\ & (1-k^2)sd(\xi,k)^2 + 1 = dc(\xi,k)^2 = (1-k^2)nc(\xi,k)^2 + k^2 \\ & k^2(1-k^2)sd(\xi,k)^2 = k^2(cd(\xi,k)^2 - 1) = (1-k^2)(1 - nd(\xi,k)^2) \end{aligned} \tag{26}$$

and $k^{'} = sqrt(1-k^2)$

$$\begin{aligned} & sn(i\xi,k) = (i)sn(\xi,k^{'}), \quad dc(i\xi,k) = dn(\xi,k^{'}) \\ & cn(i\xi,k) = nc(\xi,k^{'}), \quad nc(i\xi,k) = cn(\xi,k^{'}) \\ & dn(i\xi,k) = dc(\xi,k^{'}), \quad sc(i\xi,k) = (i)sn(\xi,k^{'}) \\ & cd(i\xi,k) = nd(\xi,k^{'}), \quad ns(i\xi,k) = (-i)cs(\xi,k^{'}) \\ & sd(i\xi,k) = (i)nd(\xi,k^{'}), \quad ds(i\xi,k) = (-i)ds(\xi,k^{'}) \\ & nd(i\xi,k) = cd(\xi,k^{'}), \quad cs(i\xi,k) = (-i)ns(\xi,k^{'}) \end{aligned} \tag{27}$$

and the second derivative is:

$$G^{''} = 2a\epsilon^2 G^3 + (ac + b\epsilon)G \tag{28}$$

We use a version of the method given in [9], where the solutions to eq. (4), are given by;

$$v = \sum_{i=1}^{n} a_i G^i \tag{29}$$

Balancing nonlinear terms in eq. (4), we have $n = 2$, the solution is:

$$v = (a_0 + a_1 G + a_2 G^2) \tag{30}$$

Then, we obtain a group of equations, order by order in G^i. And doing algebra, we get:

$$g_1 \leftarrow \left\{ a_0 = 0, a_1 = \frac{1}{2e}, a_2 = 0, b = \frac{i\sqrt{a}\kappa v_0}{\sqrt{2}\sqrt{e}}, c = -\frac{i\sqrt{e}\kappa v_0}{\sqrt{2}\sqrt{a}} \right\} \tag{31}$$

$$g_2 \leftarrow \left\{ a_0 = 0, a_1 = \frac{1}{2e}, a_2 = 0, b = -\frac{i\sqrt{a}\kappa v_0}{\sqrt{2}\sqrt{e}}, c = \frac{i\sqrt{e}\kappa v_0}{\sqrt{2}\sqrt{a}} \right\} \tag{32}$$

Then, we get twenty four families of solutions, g_i, using Jacobi solutions [8]-[9].

5 Conclusions

In this paper, we solve the unbalanced (2 : 1) Cartesian Poisson-Boltzmann equation applying several solitary wave methods. Then, using tanh method we find six families of solutions. Also, using the Ricatti functions, we get thirty six families of solutions. At last, utilizing Jacobi elliptic functions, we obtain twenty four families of solutions. In general, the solutions are:

$$\phi = \ln\left(\frac{a_0 + a_2 \tanh^2(\mu u)}{v_0}\right), \phi = \ln\left(\frac{a_0 + a_2 F^2(u)}{v_0}\right), \phi = \ln\left(\frac{a_0 + a_1 G(u)}{v_0}\right) \tag{33}$$

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